On discretization and discrete-time stabilization of Hamiltonian systems

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The work is done in collaboration with Dr. Alessandro Astolfi
under EPSRC Portfolio Grant

ICM, EEE-CAP, 16 December 2004
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**Introduction**

- Discretization of Hamiltonian (conservative) systems is studied.
- Algorithm for constructing a discrete-time model for Hamiltonian systems is proposed.
- Applications to port-controlled Hamiltonian systems are presented.
- Stabilization design and numerical examples are presented.
  - Design based on discrete-time model, conserving the Hamiltonian in discrete-time.
  - Design based on Euler model, almost conserving the Hamiltonian in discrete-time via feedback.
What happens if we discretize a Hamiltonian system?

A Hamiltonian system \( \dot{q} = p, \quad \dot{p} = -q, \) with \( H = \frac{1}{2}(p^2 + q^2). \)
Objectives and Motivations

We develop a theory and propose tools for discrete-time stability design of Hamiltonian conservative systems.

(a) Discretization issue of Hamiltonian systems is different from dissipative systems.

(b) Hamiltonian systems are important in engineering practice, e.g. to model mechanical systems (pendulum, ball and beam, aircraft, UWV, etc), electromagnetic systems (magnetic levitation, etc).

(c) Prevalence use of computer controlled systems.

(d) In many cases, emulation controllers applied to Hamiltonian systems are too sensitive to sampling.
Objective and Motivations (cont’d)

(d) Available results on:
- Continuous-time design of Hamiltonian systems (Ortega, Marsden, Leonard, van der Schaft etc.).
- Approximate based sampled-data design for dissipative systems (Nešić, Teel & Kokotović, Nešić & Laila).
- Numerical analysis for Hamiltonian systems (Stuart, Humphries, Gonzales, etc.).
- Geometric integration and discrete gradient (McLachlan, Budd, etc.)
Consider port-controlled Hamiltonian (PCH) systems:

\[
\dot{x} = J(x) \frac{\partial H}{\partial x} + G(x)u
\]

\[
y = G'(x) \frac{\partial H}{\partial x}(x)
\]

(1)

where \( J(x) \) is the skew symmetric structure matrix, and \( H(x) \) is the Hamiltonian, which often can be written as

\[
H(x) = K(x) + V(x)
\]

(2)

PCH system with \( u = 0 \) is conservative, i.e.

\[
\frac{dH(x)}{dt} = \frac{\partial H}{\partial x} \dot{x} = 0
\]
More special case: (mechanical systems)

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} = \begin{bmatrix}
0 & I_n \\
-I_n & 0
\end{bmatrix} \begin{bmatrix}
\nabla_q H \\
\nabla_p H
\end{bmatrix} + \begin{bmatrix}
0 \\
G(q)
\end{bmatrix} u, \tag{3}
\]

\[
H(q, p) = \frac{1}{2} p' M^{-1} q' p + V(q), \quad M > 0. \tag{4}
\]

Separable PCH systems: (linear systems)

\[
H(q, p) = \frac{1}{2} p' M^{-1} p + V(q), \quad M > 0, \text{constant}. \tag{5}
\]
Numerical result

1. Forward Euler: \( (u = 0) \)

\[
q(k + 1) = q(k) + T \dot{q}(k) = q + T \nabla_p H(q, p) \\
p(k + 1) = p(k) + T \dot{p}(k) = p - T \nabla_q H(q, p) .
\]

2. q-Euler:

for separable Hamiltonian systems (separable PCH with \( u = 0 \))

\[
q(k + 1) = q + T \dot{q}(k) = q + TM^{-1} p \\
p(k + 1) = p + T \dot{p}(k + 1) = p - T \nabla_q V(q)(k + 1) .
\]
Numerical result (cont’d)

3. Automatic Conserving Algorithm:

The algorithm is proven to be Hamiltonian conserving.

\[
\frac{x_{k+1} - x_k}{T} = J\left(x^{1/2}_k\right) \nabla H\left(x^{1/2}_k\right) + J\left(x^{1/2}_k\right) \left[H\left(x_{k+1}\right) - H\left(x_k\right)\right] \\
- \left< \nabla H\left(x^{1/2}_k\right), S\left(x^{1/2}_k\right) (x_{k+1} - x_k) \right> \frac{x_{k+1} - x_k}{\left\| x_{k+1} - x_k \right\|_S^2},
\]

where the matrix \( S(\cdot) := J^+(\cdot) J(\cdot) \) (\( J \) may be singular),

\[
x^{1/2}_k = \frac{1}{2} (x_{k+1} + x_k),
\]

\[
\left\| x_{k+1} - x_k \right\|_S^2 := (x_{k+1} - x_k)' S\left(x^{1/2}_k\right) (x_{k+1} - x_k).
\]
Application to quadratic Hamiltonian

PCH system with quadratic Hamiltonian, i.e.

\[ H(x) = \frac{1}{2} x'Ax \quad \text{and} \quad \frac{\partial}{\partial x} H(x) = Ax, \] (7)

with \( A \) is a positive definite symmetric matrix, and \( J(x) \) satisfies

\[ J(ax) := aJ(x), \quad \forall a \in \mathbb{R}. \] (8)

The algorithm reduces to an Implicit Midpoint Algorithm:

\[ \frac{\Delta x}{T} = J(x_{\frac{1}{2}}) \nabla H(x_{\frac{1}{2}}) \]

\[ = J\left(\frac{1}{2}x_{k+1} + \frac{1}{2}x_k\right)A\left[\frac{1}{2}x_{k+1} + \frac{1}{2}x_k\right] \]

\[ = \frac{1}{4} J(x_{k+1} + x_k) A[x_{k+1} + x_k]. \] (9)
Stabilization problem 1

Result 1

A discrete-time approximate model obtained by Algorithm (6), of a PCH system (1) with the structure matrix $J(x)$ satisfying (8) and the Hamiltonian $H(x)$ satisfying (7), with $A > 0$, is semiglobally practically stabilized by the discrete-time feedback

$$u = u_k^T = -cG(x_{\frac{1}{2}})'Ax_{k+1}^\circ,$$

where $c > 0$ and $x_{k+1}^\circ$ denotes $x_{k+1}$ for the system with $u = 0$. 
Example 1

Rigid Body (General PCH with J singular)

\[
\begin{bmatrix}
I_\alpha \dot{\alpha} \\
I_\beta \dot{\beta} \\
I_\gamma \dot{\gamma}
\end{bmatrix} =
\begin{bmatrix}
0 & -\gamma & \beta \\
\gamma & 0 & -\alpha \\
-\beta & \alpha & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial \alpha} \\
\frac{\partial H}{\partial \beta} \\
\frac{\partial H}{\partial \gamma}
\end{bmatrix}
+ \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} u . \tag{10}
\]

Principal moments of inertia: \( I_\alpha = 3, I_\beta = 2, I_\gamma = 1 \),
Hamiltonian: \( H = \frac{1}{2} (I_\alpha \alpha^2 + I_\beta \beta^2 + I_\gamma \gamma^2) \).
The dynamic of the system can be written as

\[
\begin{aligned}
\dot{\alpha} &= \frac{1}{3} \beta \gamma + \frac{1}{3} u \\
\dot{\beta} &= -\alpha \gamma + \frac{1}{2} u \\
\dot{\gamma} &= \alpha \beta + u . \tag{11}
\end{aligned}
\]
Example 1 (cont’d)
**Example 1 (cont’d)**

**Continuous-time control**

\[
u^c = -k G' \frac{\partial}{\partial x} H(x) = -k(\alpha + \beta + \gamma), \; k > 0
\]

is a stabilizing controller for the system.

**Discrete-time control**

Applying mid point algorithm,

\[
\Delta H = H(x_{k+1}) - H(x_k)
\]

\[
= \Delta_H^o + T u^T_k (\alpha + \beta + \gamma) + T^2 u^T_k \left( \frac{1}{12} \Sigma_\beta \Sigma_\gamma - \frac{1}{4} \Sigma_\gamma \Sigma_\alpha + \frac{1}{4} \Sigma_\alpha \Sigma_\beta \right) + \frac{11}{12} T^2 (u^T_k)^2.
\]
Example 1 (cont’d)

Applying

\[ u_k^T = -cG(x^{\frac{1}{2}})'\nabla H(x(k + 1)) \]

\[ = -2\left(\alpha + \beta + \gamma + 4T\left(\frac{1}{12}\beta\gamma - \frac{1}{4}\gamma\alpha + \frac{1}{4}\alpha\beta\right)\right), \]

we obtain that

\[ \Delta_H = -2T\left(\alpha + \beta + \gamma + 4T\left(\frac{1}{12}\beta\gamma - \frac{1}{4}\gamma\alpha + \frac{1}{4}\alpha\beta\right)\right)^2 + O(T^2). \]
Simulations 1

Response with midpoint (blue line) and emulation (red line) controllers.
Simulations 1 (cont’d)

Hamiltonian with midpoint (blue line) and emulation (red line) controllers.

Parameters: $T=0.1\text{s}$, $x_0 = (25\ 25\ 25)'$. 
**Stabilization problem 2**

**IDA-PBC design**

Construct a controller for system (3) so that the stabilization is achieved assigning a desired energy function

\[
H_d(q, p) = K_d(q, p) + V_d(q) = \frac{1}{2} p' M_d^{-1}(q)p + V_d(q),
\]

such that \( q^* = \arg \min V_d(q) \), where \( (q^*, 0) \) is the equilibrium point.

An IDA-PBC controller is of the form

\[
u = u_{es} + u_{di}.
\]

The desired dynamics

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} = \begin{bmatrix} J_d(q, p) - R_d(q, p) \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\
\nabla_p H_d
\end{bmatrix}
\]

(14)
Stabilization problem 2 (cont’d)

where

\[ J_d = -J_d^T = \begin{bmatrix} \ 0 & M^{-1}M_d \\ -M_dM^{-1} & J_2(q,p) \end{bmatrix} \]  \hspace{1cm} (15)

\[ R_d = R_d^T = \begin{bmatrix} \ 0 & 0 \\ 0 & G_{kv}G^T \end{bmatrix} \]  \hspace{1cm} (16)

Note that \( M_d, V_d, J_2 \) and \( R_d \) are free parameters to design.

The energy shaping controller \( u_{es} \) is obtained by solving the equation

\[
\begin{bmatrix} \ 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_qH \\ \nabla_pH \end{bmatrix} + \begin{bmatrix} \ 0 \\ G(q) \end{bmatrix} u_{es} = \begin{bmatrix} \ 0 & M^{-1}M_d \\ -M_dM^{-1} & J_2(q,p) \end{bmatrix} \begin{bmatrix} \nabla_qH_d \\ \nabla_pH_d \end{bmatrix} . \]  \hspace{1cm} (17)
Stabilization problem 2 (cont’d)

Solving some PDEs related to (17), then $u_{es}$ is obtained as

$$u_{es} = (G'G)^{-1}G'\left(\nabla_q H - M_dM^{-1}\nabla_q H_d + J_2M_d^{-1}p\right),$$  \hspace{1cm} (18)$$

with which the desired Hamiltonian $H_d$ is conserved, and this implies that (critical) stability of the closed-loop system is preserved.

Moreover, the damping injection controller $u_{di}$ is constructed as

$$u_{di} = -kv G'\nabla_p H_d, \hspace{0.5cm} kv > 0$$  \hspace{1cm} (19)$$

to add the damping to the closed-loop system to yield asymptotic stability.
Stabilization problem 2 (cont’d)

Result 2: Discrete-time IDA-PBC

Discrete-time design, using forward Euler model,

\[
q(k + 1) = q(k) + T \nabla_p H
\]

\[
p(k + 1) = p(k) - T \left( \nabla_q H(q(k)) - G u(k) \right).
\]

yields the discrete-time IDA-PBC controller

\[
u^T_{es} = (G' G)^{-1} G' \left( \nabla_q H - M_d M^{-1} \left[ \frac{\Delta H_d}{\Delta q} \right] + J_2 M_d^{-1} p \right)
\]

\[
u^T_{di} = -k_v G'' \nabla_p H_d, \quad k_v > 0
\]
Stabilization problem 2 (cont’d)

where

\[
\begin{bmatrix}
\frac{\Delta H_d}{\Delta q}
\end{bmatrix} := \nabla_q H_d + T \kappa L_V(q) M^{-1} p ,
\]

with \( L_V \) is a semidefinite positive matrix such that

\[
\overline{M} := M_d^{-1} I_G M_d M^{-1} L_V M^{-1}
\]

is semidefinite positive and \( \kappa > 0 \).

It can be shown that

\[
\Delta H_d^{u_{es}} - \Delta H_{d}^{u_{es}} = -T^2 \kappa p' M_d^{-1} I_G M_d M^{-1} L_V M^{-1} p + O(T^3)
\]

\[
= -T^2 \kappa p' \overline{M} p + O(T^3) .
\]
Example 2

Inertia Wheel Pendulum, with

\[ M = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}, \]

\[ G = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \]

\[ V(q) = mgL(\cos q_1 - 1) = m_3(\cos q_1 - 1). \]

This system is a separable Hamiltonian system.
Example 2 (cont’d)

Setting the parameters $I_1 = 0.1$, $I_2 = 0.2$, $m_3 = 10$, and following the controller design procedure, we choose

$$M_d(q, p) = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}, \quad J_d = 0.$$

and the desired potential energy

$$V_d(q) = -\cos q + \frac{k_1}{2} z(q)^2,$$

$$z(q) = q_2 + q_1.$$
Example 2 (cont’d)

Hence, we obtain the controllers:

**Continuous-time controller:**

**Energy shaping controller:**

\[ u_{es} = (G'G)^{-1}G'(\nabla_q V - M_d M^{-1} \nabla_q V_d) \]
\[ = 30 \sin(q_1) + 5k_1(q_2 + q_1), \]

(22)

**Damping injection controller:**

\[ u_{di} = -k_v G' \nabla_p K_d = -k_v(-2p_1 - p_2). \]

(23)
Example 2 (cont’d)

Discrete-time controller:

Starting with the same steps as the continuous-time design, and continue with finding $L_V$, we have a choice of $L_V$ is

$$L_V = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix},$$

Hence we obtain the energy shaping controller:

$$u_{es}^T = (G'G)^{-1}G' \left( \nabla q V - M_d M^{-1} \left[ \frac{\Delta V_d}{\Delta q} \right] \right)$$

$$= 30 \sin(q_1) + 5k_1 (q_2 + q_1) + T \kappa (p_1 + 0.5p_2),$$

(24)
Example 2: IDA-PBC Design (cont’d)

and the damping injection controller:

\[ u_{di} = -k_v G' \nabla_p K_d = -k_v (-2p_1 - p_2). \]  \hfill (25)

Setting the simulation parameters:

- initial state \((q_\circ, p_\circ)' = (2, 0, 0, 0)',\)
- \(k_1 = 1, k_v = 5, \kappa = 7, T = 0.033,\)

we observe:
Simulations 2

![Graph showing continuous and discrete simulations over time](image)

- Time (second)
- $q_1$ (rad)
- Continuous
- Emulation
- Discrete

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Simulations 2 (cont’d)
Simulations 2 (cont’d)

![Graph showing H_d with u_es + u_di over time](image)

- **continuous**
- **emulation**
- **discrete**
Summary

- We have presented issues on Hamiltonian discretization.
- We have presented a discretization algorithm that guarantees Hamiltonian conservation.
- We have used the algorithm in stability design of port-controlled Hamiltonian systems with quadratic Hamiltonian and apply to a rigid body system.
- We have also presented result on discrete-time IDA-PBC design, based on Euler model.
- We have shown by simulations that for both proposed approaches, the approximate based controllers outperform their counterpart controllers designed using emulation.
- This study has motivated further research on approximate model based discrete-time and sampled-data design for Hamiltonian systems.