INCENTIVE EFFECTS OF SECOND PRIZES

S. SZYMANSKI, T.M. VALLETTI

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Incentive effects of second prizes

Stefan Szymanski\textsuperscript{a}, Tommaso M. Valletti\textsuperscript{a,b,*}
\textsuperscript{a} Tanaka Business School, Imperial College London, London SW7 2AZ, U.K.
\textsuperscript{b} CEPR, London, U.K.

Abstract: Most of the contest literature deals with first prizes; this paper deals with the optimality of second prizes. We show that in a three person contest where one contestant is very strong, a second prize can be optimal from the point of view of eliciting maximum effort from every contestant. Moreover, we consider the desirability of second prizes from the point of view of competitive balance, which matters for contests such as sports competitions.

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* Corresponding author. Tanaka Business School, Imperial College London, South Kensington Campus, London SW7 2AZ, UK. Tel. +44-20-7594 9215, fax +44-20-7823 7685, e-mail t.valletti@imperial.ac.uk, http://www.ms.ic.ac.uk/tommaso/
1. Introduction

However, when they had been running half an hour or so, and were quite dry again, the Dodo suddenly called out `The race is over!' and they all crowded round it, panting, and asking, `But who has won?' This question the Dodo could not answer without a great deal of thought, and it sat for a long time with one finger pressed upon its forehead (the position in which you usually see Shakespeare, in the pictures of him), while the rest waited in silence. At last the Dodo said, `Everybody has won, and all must have prizes.'

Alice in Wonderland, Chapter 3

Economists think of prizes as incentive mechanisms. Since Tullock's seminal work it has been possible to analyze the impact of a prize in a contest where agents choose how much effort or investment to supply. However, the issue of second prizes has been less widely addressed. What is the incentive effect of a second prize? On the face of it, a second prize rewards inferior effort, and therefore diminishes the incentive to compete. If the aim of the contest designer is to elicit the maximum possible effort, why offer a second prize?

Two possible answers suggest themselves. Firstly, the designer can value competitive balance as well as effort, and therefore will try to ensure that contestants who fall behind the leader do not give up. This is especially true in sporting contests (e.g. a Marathon or a soccer league) because greater competitive balance can increase demand for and therefore revenue from the contest itself. The organizers of the North American major leagues have gone to great lengths to reduce the impact of contest incentives in the name of competitive balance, and for this reason both the courts and legislature have been sympathetic to what would be perceived as illegal restraints in any other industry (e.g. revenue sharing, collective selling of broadcast rights, salary caps1). Competitive balance can be desirable in other contexts as well. For example, in the case of research tournaments organized by government, there may be political constraints, which makes it desirable to involve as many different suppliers as possible.

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1 In fact, this is the central issue addressed in the literature on sporting contests. See Szymanski (2003) for a review of both theory and evidence on competitive balance in sport.
Secondly, second prizes can increase total effort/investment when there is asymmetry in the ability of the contestants. When there is a single prize and a single strong contestant, the weaker contestants may give up, allowing the strong contestant to win with minimal effort. In this paper we show how a second prize can raise effort among weaker contestants, which then puts some pressure on the strong contestant, leading to an increase in effort in the aggregate.

Imagine a sprint race involving the World Record holder (at the time of writing Tim Montgomery, who set a time of 9.78 seconds for the 100 metres on September 14th 2002), and two relatively poor sprinters, call them Szymanski and Valletti (neither of the authors claims a personal best of under 15 seconds). The two slowcoaches have a very low probability of winning even if the star sprinter exerts only a small amount of effort. If all of the prize fund is allocated to the first prize, the incentive for the slowcoaches to exert any effort at all is negligible – and, if these contestants make no effort, there is no incentive for the star to make anything other than a minimal effort either. But once some value is attached to the second prize, our two weaker contestants will now have an incentive to exert some effort in hope of winning the second prize. Moreover, this increase in effort provides an incentive for the star to exert additional effort as well. However, this is only true because Szymanski and Valletti are so inferior - if they were better runners to begin with, then the motivation effect of the first prize would dominate, and it would be optimal to put all the weight on the first prize.

In general, asymmetry is a problem in contest design, since it reduces the influence of effort on the outcome of the contest and undermines the incentives of weaker contestants (since they do not expect to win) and stronger contestants (who believe they can win even if they put in little effort). One solution, proposed by Baye et al. (1993), is the “exclusion principle”. Since excluding the bidder with the highest

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2 This also suggests a novel interpretation of prize-giving and political correctness. Many decry the “Alice in Wonderland” aspect of school sports nowadays, where almost all contestants seem to be awarded a prize, as an excess of political correctness. But if, as seems plausible, the variance of children’s abilities is large, multiple prizes may be an optimal mechanism for eliciting maximum effort from everyone, including the most gifted.
valuation of the prize will evoke higher bids from the remaining contestants, total revenue can be increased in an all-pay auction. Clark and Riis (1998a) point out that the exclusion principle is less likely to hold when more than one prize is on offer, since the second prize can ensure that the weaker players have a large enough incentive to bid even when the strong player is competing. However, this leaves open the question of the optimal distribution of prizes. Clark and Riis (1998b) show that increasing the number of prizes on offer can reduce the incentive to supply effort, and conclude that “an income maximizing contest administrator obtains the most rent-seeking contributions when he makes available a single, large prize”, p. 623. However, in their model, contestants have both symmetric valuations of the prize and symmetric costs. Moldovanu and Sela (2001) show that it can sometimes be optimal to award a second prize to the second best contestant (second highest bidder) in an all-pay private value auction. However, the all-pay auction is a perfectly discriminating contest – the highest bidder always wins – while many, if not most, contests are not perfectly discriminating. For example, no matter how much the Yankees spend on hiring players, they can never be certain of winning the World Series. In this paper we show that second prizes can also be optimal in imperfectly discriminating contests.

3 It is perhaps true that excluding the New York Yankees from the World Series would increase the effort of the weaker teams in Major League Baseball, but it would also significantly diminish the quality of the contest in the eyes of many fans.

4 Or, equivalently the bidders whose valuations of the first prize are lowest.

5 Barut and Kovenock (1998) look at the same issue. See also Glazer and Hassin (1988) for an earlier attempt with an application to labor markets.

6 In Moldovanu and Sela’s framework it is certain that the highest effort wins, but contestants know only their own cost of supplying effort, which is taken as an indicator of ability. In sporting contests the reality would seem to be the reverse of this - each contestant has fairly good information about the ability of rivals, but no guarantee that exerting maximum effort will bring victory. This would also seem to characterize a number of other contest situations - e.g. competition among research labs to develop a new technology, or contests for promotion in the workplace.

7 See Hillman and Riley (1989) for a comparison of perfectly and imperfectly discriminating contests. Most imperfectly discriminating contest models adopt a logit formulation of the contest success function - e.g. Tullock (1980), Skaperdas (1996), Nti (1997), while a few papers have used the probit (e.g. Lazear and Rosen, 1981; Dixit, 1987). Perfectly discriminating contests and all pay auctions have been extensively studied and characterized by, inter alia, by Hillman and Samet (1987), Baye et al. (1996), and Krishna and Morgan (1997).
We begin by confirming that, in a two-person contest, a second prize is never optimal from the point of view of maximizing either total effort or the winning effort\(^8\). We also show that there is no trade-off with competitive balance in the two person case - a second prize never increases competitive balance.

Our main results focus on a three person contest. With two (equally) strong contestants and one weak contestant, shifting resources from the first prize to the second prize always reduces total effort, because, although a second prize may raise the effort of the weak contestant, this is more than offset by the diminution of incentives for the strong contestants to win the first prize. However, when there are two (equally) weak contestants and one strong contestant, then a second prize can be optimal, even if competitive balance is not a concern, as long as the difference in ability is large enough. Moreover, it is possible that shifting resources to a second prize will not only increase the effort of the two weak contestants but also the effort of the strong contestant.

The paper is structured as follows. Section 2 sets out the general problem and considers the symmetric case. Section 3 considers the asymmetric case first for two persons and then for three. Section 4 concludes.

### 2. The general problem

We begin by defining the probability of winning a prize in the contest. With \(n\) contestants for a single prize, the probability of winning is defined as:

\[
p_i = \frac{e_i^\gamma}{\sum_{k=1}^{n} e_k^\gamma}
\]

The logit formulation is widely employed in the contest literature\(^9\). In equation (1) \(e_i\) is the (simultaneous) effort contribution of each contestant and \(\gamma > 0\) is the

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\(^8\) Clark and Riis (1998b) proposition 2 shows that, for linear costs, any redistribution of the prize mass from lower prizes to higher prizes will reduce effort.

\(^9\) For a review of the general properties of logit contest success functions, see Nti (1997).
discriminatory power of the contest success function (so as \( \gamma \to \infty \) the success function becomes perfectly discriminating). We suppose that the entrants to a contest compete for a prize fund \( V \) that is divided with \( 1/2 \leq k \leq 1 \) allocated to the first prize and \( 1 - k \) to the second prize. If there are \( n \) contestants, the probability of winning the first prize is given by (1). We follow Clark and Riis (1998b) who analyze the probability of winning second prize thus: “If [contestant] \( i \) does not win this first rent, his contribution remains in the pool giving a chance of winning the second prize. The unconditional probability that \( i \) wins the second prize is the product of the probability that \( i \) does not win the first and the probability that he does win the second” (p. 609).\(^{10}\) Thus the probability of winning the second prize is simply the probability of winning a contest with \( n - 1 \) contestants conditional on not having won the first prize (which is the sum of probabilities that each of the other contestants won first prize)\(^{11}\). For contestant \( i \) this can be defined as

\[
\sum_{j \neq i} p_j p_{i\rightarrow j} \text{ where } p_{i\rightarrow j} = e_i^j / \sum_{h=1}^{n} e_h^j
\]

Note that in a symmetric contest the \( p_{i,j} \)'s are identical in equilibrium, considerably simplifying the analysis. However, for many contest situations asymmetric contests are more relevant\(^{12}\). Each contestant then maximizes an objective function

\[
p_i kV + \sum_{j \neq i} p_j p_{i\rightarrow j} (1-k)V - c_i e_i
\]

where \( c_i \) is the (constant) marginal cost of effort \( e_i \). The first order condition for the contestant’s payoff to be a maximum is

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\(^{10}\) This method generalizes earlier contributions on how to distribute several homogeneous rents in an imperfectly discriminating rent-seeking game studied by Berry (1993) and by Clark and Riis (1996).

\(^{11}\) This can be thought of as the probability of winning the first prize in a second contest from which the winner of the first prize in the full contest has been eliminated, even though ours is not a sequential model. See Gradstein and Konrad (1999) and Morgan (2003) for an analysis of sequential contests.

\(^{12}\) This is especially true of sporting contests: see Szymanski (2003). Asymmetry has been studied in the imperfectly discriminating contest literature, e.g. Dixit (1987), Baik (1994), Nti (1999) and Stein (2002).
which in the logit formulation reduces to

\begin{equation}
\gamma p_i (1 - p_i) kV + \sum_{j \neq i} p_j p_{i \rightarrow j} (1 - p_i - p_{i \rightarrow j})(1 - k)V - c_i e_i = 0
\end{equation}

We assume that the second order condition is satisfied at equilibrium, which in the logit case corresponds to

\begin{equation}
\gamma^2 V [k(1-2p_i)p_i(1-p_i) + (1-k)\sum_{j \neq i} p_j p_{i \rightarrow j}[(1 - p_i - p_{i \rightarrow j})^2 - p_i(1 - p_i) - p_{i \rightarrow j}(1 - p_{i \rightarrow j})]] - c_i e_i < 0.
\end{equation}

In a symmetric contest \((c_i = c\) for all \(i\)) in a symmetric equilibrium, each contestant exerts identical levels of effort \((e_i = e)\) and has an identical probability of winning the first prize \((p_i = p_j = 1/n)\). If \(k < 1\) each contestant also has an equal probability of failing to win the first prize equal to \((n - 1)/n\) and an equal probability of winning the second prize \((p_{i \rightarrow j} = 1/(n - 1))\). (5) and (6) reduce to

\begin{equation}
\gamma V \left[\frac{n-1}{n^2} - \frac{1-k}{n(n-1)}\right] = ce
\end{equation}

\begin{equation}
\frac{(n-2)\gamma^2 V}{n^3} \left[k(n-1) + (1-k)\frac{n^3 - 6n^2 + 4n - 1}{(n-1)^2}\right] - ce =
\end{equation}

\begin{equation}
\frac{\gamma V}{n} \left\{\frac{n-1}{n} \left[k\gamma(n-2) - 1\right] + \frac{1-k}{n-1} \left[\frac{\gamma(n-2)(n^3 - 6n^2 + 4n - 1)}{n^2(n-1)^2} - 1\right]\right\} < 0
\end{equation}

In equilibrium, the effort contribution of contestants is decreasing in the proportion of the prize fund allocated to the second prize \((1 - k)\) and maximum effort is extracted when only a first prize is awarded (see Clark and Riis, 1998b, proposition 2(d)). The result follows by differentiating (7) with respect to \(k\), since the RHS always increases in effort \(e\) when an equilibrium exists\(^{13}\).

\(^{13}\) In a previous version of the paper we showed that this result is not limited to constant marginal costs but extends to more general cost functions \(C_i(e_i) = (c_i e_i)\). We do not address the issue of existence, since it is extensively addressed by Clark and Riis (1998b). As it is well known, the discriminatory power \(\gamma\) has to be limited in order to ensure existence of pure strategies equilibria. Thus we require
3. Asymmetric costs

It is well known that in a symmetric contest effort per contestant is decreasing in the number of contestants, while total effort depends on the precise functional form of the contest success function (see Nti, 1997, propositions 1 and 4). If the organizer is interested in obtaining the maximum winning effort, then the optimal number of contestants is two. Fullerton and McAfee (1999) report a similar result for an asymmetric contest, as long as the best two contestants are selected (which they will be in their model). However, the organizer may also care about the distribution of effort in the contest - either because the contest is being promoted for more reasons than simply finding a winner (e.g. to promote academic research in Universities) or because there is a demand for the contest itself as in sports.

In a symmetric contest there is no trade-off between the effort level of the winner and the effort level of the other contestants. However, in an asymmetric contest there may be a complex interaction between winning effort, total effort, and effort per contestant.

3.1 Asymmetry with two contestants

With only two contestants the first order condition (5) simplifies to

\[ (9) \quad (2k - 1)\gamma V p_1 p_2 = e_i e_j. \]

Proposition 1. In a two person asymmetric contest, a second prize will:

(a) reduce the effort of each contestant

(b) have no impact on competitive balance.

\[ \gamma \leq \frac{n(n-1)}{(n-1)^2 - (n-4)} \] to ensure that (8) is strictly satisfied. If the discriminatory power is very high, a pure strategy equilibrium does not exist; however it is likely that a mixed strategy equilibrium will, but this is beyond the scope of this paper. The case of mixed strategies with a single prize is investigated by Baye et al. (1994), who identify equilibria for a discrete choice set but not for continuous choices.
We omit the proof of part (a) since it is already familiar from the literature (e.g. Glazer and Hassin, 1988). Intuitively, in a contest with two players the gain to winning the first prize becomes the difference between the value of the first and second prize. The second prize is the gain from losing - the minmax value for a player who bids zero - and this should be made as small as possible in order to make the gain of winning as large as possible.

We define competitive balance as the ratio of the probabilities of success for each player, $CB = p_1/p_2 = (e_1/e_2)^\gamma$, as seems natural in a two person contest. Part (b) follows from taking the ratio of efforts from eq. (9), which yields $e_2/e_1 = c_1/c_2$, which is independent of $k^{14}$.

3.2 Asymmetry with three contestants

Clearly the analysis for a three person contest will be highly sensitive to the cost functions and distributions assumed. Moreover, while the concept of aggregate effort remains clear, the meaning of competitive balance becomes more problematic. In an $n$-person asymmetric contest there is no unique measure of competitive balance. In the sports literature, the most widely used measure is the standard deviation of the percentage of matches won in a tournament (i.e. outcomes rather than inputs), which corresponds approximately to the success probabilities in our model.$^{15}$

However, the usefulness of second prizes can be appreciated using a very simple example. We restrict ourselves to cases where there are two identical contestants and one that is differentiated. This ensures that the concept of competitive balance will still be well defined in any pure strategy equilibrium (as the ratio of winning probabilities between the two identical contestants and the differentiated contestant, which in our model coincides with the ratio of efforts).

If the third contestant is weaker than the other two, there is no gain to the introduction of the second prize. Suppose the strong (and identical) contestants have a marginal

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$^{14}$ We are grateful to an anonymous referee for this point.

$^{15}$ Szymanski (2003) reviews the different measures of competitive balance that have been used in the sports literature.
cost $0 < c_s < 1$, while the weaker player has a higher marginal cost $c_w = 1$. Let $p_s$ be the probability of a strong contestant winning the first prize and $q_{ws}$ be the probability of a weak contestant winning the second prize conditional on not having won the first prize (this can be thought of as the probability of the weak contestant winning against the remaining strong contestant in the race for the second prize). Using equation (5) we can show that the profit maximizing efforts are

$$e_w = 2\gamma V \left[ kp_s (1 - 2p_s) + (1 - k) p_s q_{ws} (2p_s - q_{ws}) \right]$$

$$e_s = \frac{\gamma V}{c_s} \left[ kp_s (1 - p_s) + (1 - k) \left( (1 - q_{ws}) (p_s q_{ws} - p_s^2) + \frac{(1 - 2p_s)^2}{4} \right) \right].$$

If $c_s$ is close to unity, it is straightforward to show that the contest becomes symmetric and therefore, recalling Proposition 2 of Clark and Riis (1998b), effort is maximized when all the weight is placed on the first prize. As $c_s$ becomes smaller, the stronger contestants will exert more effort and the weaker contestant will become discouraged, having little chance of winning either the first or second prize. In the limit, as $c_s$ approaches zero, $e_s$ becomes very large and $e_w$ goes to zero. To see this, note that if $e_s$ approaches infinity, then $p_s$ approaches $\frac{1}{2}$ and $q_{ws}$ approaches zero, and therefore from (10) $e_w$ does indeed vanish. The value of $e_s$ is obtained from (11), which approaches the following limit:

$$e_s^{\text{lim}} = \lim_{c_s \to 0} e_s = \frac{\gamma V (2k - 1)}{4c_s}.$$

Hence only the strong players contribute effort in the limit, and the contest reduces to a two person contest where, as we have already shown with Proposition 1, it makes sense to place all the weight on the first prize.

The case of one strong contestant and two identical weak contestants is more interesting. From eq. (5) the profit maximizing effort choices for the weak and for the strong contestants are respectively

$$e_w$$

$$e_s$$
\[ e_w = \gamma V \left\{ k p_w (1 - p_w) + (1 - k) \left[ \frac{(1 - 2 p_w)^2}{4} + p_w q_{ws} (1 - q_{ws} - p_w) \right] \right\} \]

\[ e_s = \frac{2\gamma V}{c_s} \left[ k p_w (1 - 2 p_w) - (1 - k) p_w (1 - q_{ws})(1 - q_{ws} - 2 p_w) \right] \]

where \( p_w \) is the probability that one of the weak contestants wins the first prize.

**Proposition 2.** With two equally weak contestants and one strong contestant:

(a) both \( e_s \) and \( e_w \) are increasing in \( k \) for \( c_s \) close to unity

(b) \( e_w \) is decreasing in \( k \) for \( c_s \) close to zero

(c) \( e_s \) is decreasing in \( k \) for \( c_s \) close to zero and for sufficiently high \( k \).

**Proof:** For result (a) it suffices to recall Proposition 2 of Clark and Riis (1998b), since as \( c_s \to 1 \) the contest becomes symmetric. As the contest approaches symmetry, all the players’ efforts increase in \( k \).

Results (b) and (c) are more surprising and their proof is somewhat more involved. As \( c_s \to 0 \), let us conjecture that \( e_s \to \infty \). It is then apparent that \( p_w \) and \( q_{ws} \) must approach zero, and from (13) we obtain

\[ e_w^{\text{lim}} = \lim_{c_s \to 0} e_w = \frac{\gamma V (1 - k)}{4} . \]

Hence, if we could confirm our initial conjecture, then result (b) would follow immediately. Let us now substitute the previous value for \( e_w \) into (14), which can be rearranged to obtain:

\[ -c_s e_s \left\{ e_s^{2\gamma} + 3e_s^{\gamma} [(1 - k)\gamma V]^{\gamma} 4^{-\gamma} + 2^{1-4\gamma} [(1 - k)\gamma V]^{2\gamma} \right\}^{\frac{1}{2}} + 2^{1-2\gamma} e_s^{\gamma} [(1 - k)\gamma V]^{\gamma} e_s^{2\gamma} (2k - 1) + 2^{1-2\gamma} e_s^{\gamma} k [(1 - k)\gamma V]^{\gamma} 2^{-4\gamma} (2 - k) [(1 - k)\gamma V]^{2\gamma} = 0 . \]

The above is a complicated equation in \( e_s \), but we can characterize its asymptotic behavior by concentrating only on the higher-order terms (the order is denoted by \( O(\cdot) \)). We consider the case when \( k \) is strictly greater than \( \frac{1}{2} \). If \( e_s \) becomes large as \( c_s \)
→ 0, then the polynomial in the first curly bracket is $O(e_s^{2\gamma})$ and the second square curly bracket is $O(e_s^{2\gamma})$, while all the other lower-order terms can be disregarded. Hence, when $k > \frac{1}{2}$, in the limit (16) reduces to

$$- c_s e_s^{4\gamma} + 2^{1-2\gamma} e_s^{\gamma} \gamma V [(1 - k) \gamma V] e_s^{2\gamma} (2k - 1) =$$

$$e_s^{3\gamma} [-c_s e_s^{\gamma + 1} + 2^{1-2\gamma} (\gamma V)^{\gamma + 1} (1 - k) \gamma (2k - 1)] = 0$$

and the solution in the limit is

$$e_s^{\text{lim}} = \left( \frac{2^{1-2\gamma} (1 - k)^{\gamma} (2k - 1)}{c_s} \right)^{\frac{1}{1+\gamma}} \gamma V,$$

which indeed becomes arbitrarily large as $c_s \to 0$, so confirming our initial conjecture. When $\gamma = 1$, (17) simplifies to

$$e_s^{\text{lim}} = \sqrt{\frac{(1 - k)(2k - 1)}{2c_s}} V.$$

Result (c) then follows as

$$\text{sign}\left[ \frac{\partial e_s^{\text{lim}}}{\partial k} \right] = \text{sign}\left[ (2 + \gamma) - 2k(1 + \gamma) \right].$$

which is negative as long as $k > k^* = (2 + \gamma)/(2(1 + \gamma))$. Notice that $\frac{1}{2} < k^* < 1$. $QED$

Before commenting the result, it is worth making some technical points about the limiting solution that we have derived:

- Eq. (17) is valid in the limit as $c_s$ is very small but positive and $\frac{1}{2} < k < 1$;
- When $k = 1$ we cannot apply eq. (17), which presumes that the weak players are competing for the second prize. If $c_s$ is very small but positive, the strong player puts a large but finite effort, and the weak players put small but positive
effort (in fact, when $c_s > 0$ a corner solution where the weak players put a zero effort could not be Nash, since the strong player would respond by putting zero effort as well). This can be seen for $\gamma = 1$, when we can solve directly the system of FOCs to obtain: $e_w = 2Vc_s/(2 + c_s)^2$ and $e_s = 2V(2 - c_s)/(2 + c_s)^2$.

- When $c_s = 0$ and $k = 1$, there is a corner solution where the weak players put no effort, while the strong contestant puts as much effort as required to keep them out;
- If $c_s = 0$ and $k < 1$, such a corner solution would disappear since the weak players would put some positive effort as they compete for the second prize (still, the strong player puts in an infinite amount of costless effort and he wins the first prize with almost probability one).

Eq. (17) is also valid when $k$ is strictly greater than $\frac{1}{2}$. Since it is of interest to understand what happens if the two prizes are identical, imagine now $k = \frac{1}{2}$. If $e_s$ becomes large as $c_s \to 0$, then the first curly bracket eq. (16) is $O(e_s^{2\gamma})$ as before, while the second square bracket is $O(e_s^{\gamma})$, since the first term in the bracket vanishes. Hence when $k = \frac{1}{2}$, in the limit (16) gives the solution

$$e_s^{\text{lim}} \bigg|_{k=\frac{1}{2}} = \mathcal{N} \left( \frac{k(1-k)^{2\gamma}}{4^{\gamma - 1} c_s} \right)^{1/2} = \mathcal{N} \left( \frac{2^{1-6\gamma}}{c_s} \right)^{1/4}$$

which also becomes arbitrarily large as $c_s \to 0$\textsuperscript{16}. When $\gamma = 1$, it can be approximated by $e_s^{\text{lim}} \bigg|_{k=\frac{1}{2}} \cong 0.315V^{\frac{1}{4}}/c_s$.

We now return to the intuition behind our Proposition 2. The weak players become increasingly discouraged as the third player becomes stronger, so that eventually they become almost entirely focused on the second prize, and at this point increasing the second prize will raise their effort. This is the 'direct' effect of a change in $k$. This effect is not significant for the strong player, since this player’s chance of winning the

\textsuperscript{16} The sign of $\frac{\partial e_s^{\text{lim}}}{\partial k}$ is still given by the expression derived under Result 2b since the higher order terms are the same when comparative statics is done at equilibrium, hence the sign is positive at $k = 1/2$. 

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second prize is very low. On the contrary, the direct effect for the strong player is to exert less effort as the first prize becomes smaller. However, there is a 'strategic' effect that motivates the strong player to put more effort as well, since this player’s reaction function would be increasing in the rivals' effort. The effort of weak players is a strategic complement for the strong player in very asymmetric contests. But if $k$ becomes too small, then the direct effect of the first prize on the effort of the strong player dominates, and so the strong player's effort is once again increasing in the size of the first prize.

In more formal terms, the first-order condition for the strong player can be written as

$$\frac{\partial \pi^s}{\partial e_s} = 0.$$  

The effect of a change in $k$ can thus be decomposed as:

$$\frac{\partial^2 \pi^s}{\partial e_s^2} de_s + \frac{\partial^2 \pi^s}{\partial e_s \partial k} dk + \frac{\partial^2 \pi^s}{\partial e_w \partial k} \frac{\partial e_w}{\partial k} dk = 0,$$

where the negative sign of the first term comes from the SOC. Concentrating only on the higher-order terms as $c_s$ tends to zero and using eq. (15), we have when $\gamma = 1$:

$$\frac{\partial^2 \pi^s}{\partial e_s \partial k} = \frac{4e_s}{e_s^2} = \frac{V(1-k)}{e_s^2} > 0$$

$$\frac{\partial^2 \pi^s}{\partial e_s \partial e_w} \frac{\partial e_w}{\partial k} = \frac{2(2k-1)}{e_s^2} \left( -\frac{V}{4} \right) = -\frac{V(2k-1)}{2e_s^2} < 0$$

from which we see that the strategic effect is negative and prevails over the direct effect as $k > \frac{3}{4}$. This analysis leads us to an interesting result when the gap between the ability of the strong player and the two weak players is large enough.

**Proposition 3.** With two equally weak contestants and one strong contestant for $c_s$ close to zero:

(a) allocating a proportion of the prize fund to a second prize increases both total effort and competitive balance
(b) if the contest designer wants to maximize total effort, it is optimal to adopt a 3:1 rule for allocating the prize fund when $\gamma = 1$

(c) if the contest designer wants to maximize total effort, the portion allocated to the first prize increases as the contest becomes less discriminatory.

Proof: The first part of result (a) is a direct consequence of Proposition 2. Competitive balance is once again given by $CB = p_w/p_s = (e_w/e_s)^\gamma$. From equations (15) and (17) we can sign:

$$\text{sign}[\partial CB / \partial k]_{\gamma \to 0} = \text{sign}[e_s \partial e_w / \partial k - e_w \partial e_s / \partial k]_{\gamma \to 0} = \text{sign}[-((1-k)^\gamma (2k-1))^{1/\gamma}] < 0$$

so that competitive balance increases by allocating part of the fund to the second prize.\[^{17}\]

Part (b) comes from equation (17). As $c_s$ approaches zero, the effort exerted by the strong player has a maximum at $k = k^* = (2 + \gamma)/(2(1 + \gamma))$, which is 0.75 when $\gamma = 1$. In this limiting case, total effort is basically coincident with the effort supplied by the strong player.

Part (c) comes from $\partial k^* / \partial \gamma < 0$. QED

This result is surprising in that, when the value of $k$ lies between $k^*$ and unity there is no trade off between competitive balance and total effort, increasing the weight on the second prize will increase both.\[^{18}\] Moreover, when $\gamma = 1$ this 3:1 rule does not depend on number of weak contestants (as long as there are at least two). It is straightforward to show that, when $\gamma = 1$, equation (17) generalizes when there are $n$ weak contestants to:

\[^{17}\text{One might have reservations about the meaning of competitive balance in this case where } e_s \text{ is extremely large.}\]

\[^{18}\text{Since competitive balance is maximized when } k = \frac{1}{2}, \text{ the trade-off does exist for } \frac{1}{2} < k < k^*.\]
\[ e_s^{\text{lim}} = \lim_{c_s \to 0} e_s = \sqrt{\frac{(1-k)(2k-1)(n-1)}{nc_s}}. \]

Propositions 2 and 3 are valid for the limiting cases of very similar or very different players. We have also been able to derive some comparative static results valid for intermediate degrees of asymmetry, when \( k \) is in the region of unity. In other words, starting with a 100% of the fund allocated to the first prize, we ask what impact on effort and competitive balance arises from shifting a small percentage of the fund to the second prize. Thus in general with three players the equilibrium is characterized by three FOCs corresponding to equation (5):

\[ \gamma p_i (1-p_i) k V + \sum_{j\neq i} p_j p_{j-i} (1-p_i)(1-k) V - c_i e_i = 0 \quad i = 1, 2, 3 \]

When \( k = \gamma = 1 \) the system can be solved to obtain

\[ e_i \bigg|_{k=1} = 2V \left( \sum_{j\neq i} c_j - c_i \right) / \left( \sum_j c_j \right)^2 \quad i = 1, 2, 3. \]

In the Annex, we conduct comparative statics at equilibrium. When \( c_1 = c_2 = c_w = 1 \) and \( 0 < c_s < 1 \), we obtain

\[ \frac{\partial \sum_{i=1}^3 e_i}{\partial k} \bigg|_{k=1} = \frac{V(7c_s - 2c_s^2 - 2)}{2c_s(c_s + 2)}. \]

The numerator of equation (19) has only one root for \( 0 < c_s < 1 \) (the expression is negative if \( c_s < (7 - \sqrt{33})/4 \approx 0.31 \)). Similar expressions can be obtained for the effect on each individual’s marginal effort.\(^{19}\)

\(^{19}\) Once the identity of the “strong” and “weak” players is reversed, our analysis is also valid when \( c_s > 1 \), i.e. when there are two strong players competing against a weak player. In this case, the numerator of eq. (19) has a root for \( c_s = (7 + \sqrt{33})/4 \approx 3.19 \) and is positive for lower values of \( c_s \). However, it can also be shown how an interior solution in this case with three active contestants exists only if \( c_s < 2 \). In other words, when there are two strong players and a weak one, total effort is maximized for \( k = 1 \).
The effect on individual and aggregate effort incentives resulting from a change in the weight on the first prize is illustrated in figure 1 where $V = 1$. The figure plots the effect on marginal effort incentives for the various contestants of increasing the weight on the first prize. The thicker line plots the effect on total marginal effort. Notice how figure 1 confirms the limiting cases of Proposition 2 and adds some novel results. Proposition 2 is valid for any value of $k$ as $c_s$ approaches either 1 or 0, while figure 1 gives the picture only for values of $k$ around 1. However, figure 1 describes the marginal effects for any value taken by the cost parameter $c_s$.

In this case we are also able to say something quite general about competitive balance. In this example for values of $k$ around 1, competitive balance increases with the introduction of a second prize for any value of $c_s$ since:

$$\text{sign}[\partial CB / \partial k]_{k \to 1} = \text{sign}[e_s \partial e_w / \partial k - e_w \partial e_s / \partial k]_{k \to 1} = -(1 - c_s^2)/(2 + c_s)^2 < 0$$

Hence, by awarding a small second prize, the percentage difference between top to bottom effort decreases, leading to increased competitive balance.

![Figure 1: Marginal effect on effort incentives evaluated at $k = 1$](attachment:image.png)
4. Conclusions

Clark and Riis (1998b) conclude (p. 624): “We would wish to allow for the fact that players have asymmetric valuations of winning prizes. This […] remains a challenge in the imperfectly discriminating framework.” Barut and Kovenock (1998) similarly conclude (p. 643): “As has occurred in the evolution of single-prize theory, the next step is to deal with asymmetries among players in a manner that is sufficiently general to include interesting rank order effects.” This paper is a step in filling this gap. We have shown how effort incentives and competitive balance can be affected by the introduction of second prizes in imperfectly discriminating (logit) contests with three contestants and where two contestants are (equally) weak and one strong. In some cases our results can be extended to the case of one strong and an unlimited number of (equally) weak contestants. Second prizes are never effort or balance increasing in either a symmetric or an asymmetric two person contest. However, in a contest with three or more players, second prizes can increase both effort and competitive balance. In particular we have shown, for the case where many weak contestants face a strong competitor, that if the weak contestants are sufficiently weak, then shifting rewards from the first prize to the second prize increases both individual and total effort contribution. In fact, we show that at least 25% of the prize fund should be allocated to the second prize when $\gamma = 1$ and $c_s$ tends to zero (hence we call it the 3:1 rule). This happens because weak contestants have little incentive to exert effort when the gap in abilities is very large if the only prize is for first place, while introducing a second prize raises their effort significantly, enough to even put some little pressure on high ability contestant, who is thus provoked into exerting more effort.

Our results also complement those of Moldovanu and Sela (2001) who have analyzed the problem of second prizes in the case of an all-pay auction. While they found that the desirability of second prizes depended on the concavity or convexity of the cost function, we find that it depends on the difference in the cost functions of weak and strong players. In particular, where they find that a second prize is never optimal when costs are concave or linear in a perfectly discriminating contest, we have shown that it can be in an imperfectly discriminating contest.
This suggests that optimal contest design may be quite sensitive to the technology of winning. In a competition to win the World Series, for example, even weak contestants stand a (small) chance. This means that strong players are less able to hold back on investment in a logit contest compared to an auction where contestants can be confident that the highest bidder wins. Thus, in a logit contest, the shifting of incentives toward second prizes has less of a negative effect on effort incentives. This point is reinforced by the observation that the contest organizer may care about the variance of effort as well as the mean, a point that is stressed in the literature on the economics of sports but has not been widely considered in the contest literature.

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Appendix

We conduct standard comparative statics at the equilibrium described by three FOCs given by eq. (5). We then use Cramer's rule and calculate the expressions when \( k = 1 \) and eq. (18) applies. Some manipulations yield:

\[
\frac{\partial e_i}{\partial k} \bigg|_{k=1} = V c_j \sum_{j \neq i} (c_i c_j)^2 (e_i + c_j)^2 (e_j - c_j) + (c_i c_j c_k)^2 [9 \sum_{j \neq i} c_j - 11c_i - \frac{\left( \sum_{j \neq i} c_j \right)^2}{c_i}]
\]

\[
\frac{\partial e_i}{\partial k} \bigg|_{k=1} = \frac{2(c_i c_j c_k)^2 (c_1 + c_2 + c_3)^2}{2(c_i c_j c_k)^2 (c_1 + c_2 + c_3)^2}
\]

for \( i = 1, 2, 3 \). Adding up these three expressions, the marginal effect on total effort is:

\[
(A1) \quad \frac{\partial \sum_{i=1}^3 e_i}{\partial k} \bigg|_{k=1} = V c_i c_2 c_3 \left[ c_i^2 (c_2 + c_3) + c_i^2 (c_1 + c_3) + c_i^2 (c_1 + c_2) \right] \frac{[9 \sum_{j \neq i} c_j - 11c_i - \frac{\left( \sum_{j \neq i} c_j \right)^2}{c_i}]}{2c_i c_2 c_3 (c_1 + c_2 + c_3)}
\]
In the special case $c_1 = c_2 = c_w = 1$ and $0 < c_s < 1$, (A1) simplifies into eq. (19). The expressions for individual changes are

$$
\frac{\partial c_s}{\partial k} \bigg|_{k \to 1} = \frac{V(c_s^3 + 6c_s - 4)}{2(2 + c_s)^2}
$$

$$
\frac{\partial c_s}{\partial k} \bigg|_{k \to 1} = \frac{V(-2c_s^4 - 2c_s^3 - 9c_s^2 + 20c_s - 4)}{2c_s(2 + c_s)^2}
$$

These expressions are plotted in Figure 1.

References


