A robustly stabilising adaptive controller for systems in feedback form

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Introduction

- A considerable amount of the adaptive control literature has been devoted to nonlinear systems with parametric uncertainties.
- The most widely studied is the class of feedback linearisable systems that depend linearly on the unknown parameters.
- The standard methodology for dealing with this class of systems relies on finding a parameter update law such that a quadratic function of the states and the parameter estimation error becomes a Lyapunov function for the closed-loop system.
- This is achieved by cancelling out the parameter-dependent terms from the derivative of the Lyapunov function.
Introduction (cont’d)

- A typical example where this cancellation is straightforward is the class of systems in normal form

\[
\begin{align*}
\dot{x}_1 &= x_2, & \cdots & \dot{x}_{n-1} &= x_n, & \dot{x}_n &= u + \phi(x)^T \theta, \\
\end{align*}
\]

which satisfy the so-called matching condition. In this case, the (certainty equivalence) control law is given by

\[
u = -K^T x - \phi(x)^T \hat{\theta},\]

where \(\hat{\theta}\) is the estimate of \(\theta\) and the vector \(K\) is such that the matrix

\[
A = \begin{bmatrix}
0 & I_{n-1} \\
-K^T &
\end{bmatrix}
\]

satisfies the Lyapunov equation \(A^T P + PA = -Q\), for some \(P, Q > 0\).
Consider the candidate Lyapunov function

\[ V = x^T P x + \gamma^{-1} |\hat{\theta} - \theta|^2 \]

with \( \gamma > 0 \) and the update law

\[ \dot{\hat{\theta}} = \gamma \phi(x) e_n^T P x, \]

where \( e_n = [0, \ldots, 0, 1] \), which is such that

\[ \dot{V} = -x^T Q x - 2x^T P e_n \phi(x)^T (\hat{\theta} - \theta) + 2\gamma^{-1} (\hat{\theta} - \theta)^T \dot{\hat{\theta}} \]

\[ = -x^T Q x \leq 0. \]

For more general systems in feedback form that do not satisfy the matching condition, the above cancellation must be carried out in steps, each time dealing with a first-order problem – a procedure known as adaptive backstepping.
The estimation error $\tilde{\theta} = \hat{\theta} - \theta$ is only guaranteed to be bounded (and converging to an unknown constant). However, its dynamical behaviour may be unacceptable.

Due to the strong coupling between the plant and estimator dynamics, increasing the adaptation gain $\gamma$ will not necessarily speed-up the response of the system.

Another issue is whether the corresponding non-adaptive asymptotic controller is stabilising. The counter-example of Townley (TAC’99) has shown that, in general, this is not so.
• In this work an algorithm for the stabilisation via state feedback of a class of linearly parameterised systems in feedback form is proposed, which addresses the two foregoing issues.

• The method allows for prescribed (uniformly stable) dynamics to be assigned to the estimation error, thus leading to a modular scheme which is easier to tune.

• In addition, the respective non-adaptive controller robustly globally stabilises the system for any fixed estimate of the unknown parameter vector.

• The algorithm is also applicable in the presence of unknown control coefficients and can be extended to deal with actuator dynamics.
Adaptive control design

• Consider the class of single-input nonlinear systems

\[
\begin{align*}
\dot{x}_1 &= f(x_1) + g(x_1)x_2 \\
\dot{x}_2 &= u + \phi(x)^T \theta,
\end{align*}
\]

(Σ1)

where \( x = [x_1^T, x_2]^T \in \mathbb{R}^n \times \mathbb{R} \) is the state, \( u \in \mathbb{R} \) is the control input and \( \theta \in \mathbb{R}^p \) is an unknown constant vector.

• Assumption: The system \( \dot{x}_1 = f(x_1) \) is globally asymptotically stable with a proper, positive definite Lyapunov function \( V_1(x_1) \) satisfying \( \partial V_1/\partial x_1 f(x_1) \leq -\kappa(x_1) \), where \( \kappa(\cdot) \in \mathcal{K}_\infty \).

• The control objective is to find a control law of the form

\[
\begin{align*}
\dot{\theta} &= \alpha(x, \hat{\theta}), \\
u &= \nu(x, \hat{\theta})
\end{align*}
\]

such that all closed-loop trajectories are bounded and \( \lim_{t \to \infty} x(t) = 0 \).
Adaptive control design (cont’d)

• Proposition 1: Consider the system (Σ1) and the adaptive state feedback control law

\[
\dot{\theta} = -\frac{\partial \beta}{\partial x_1} (f(x_1) + g(x_1)x_2) - \frac{\partial \beta}{\partial x_2} \left( u + \phi(x)^T \left( \dot{\theta} + \beta(x) \right) \right)
\]

\[
u = -kx_2 - \epsilon \frac{\partial V_1}{\partial x_1} g(x_1) - \phi(x)^T \left( \dot{\theta} + \beta(x) \right),
\]

where

\[
\beta(x) = \gamma \int_0^{x_2} \phi(x_1, \chi) d\chi
\]

and \( k > 0, \epsilon > 0, \gamma > 0 \) are constants.

Then all closed-loop trajectories are bounded and \( \lim_{t \to \infty} x(t) = 0 \).
Adaptive control design (cont’d)

- **Proof:** Define the “estimation error” \( z = \hat{\theta} - \theta + \beta(x) \) and note that the “error dynamics” are given by

\[
\dot{z} = \hat{\theta} + \frac{\partial \beta}{\partial x_1} (f(x_1) + g(x_1)x_2) + \frac{\partial \beta}{\partial x_2} (u + \phi(x)^T (\hat{\theta} + \beta(x) - z))
\]

\[
= -\frac{\partial \beta}{\partial x_2} \phi(x)^T z
\]

\[
= -\gamma \phi(x) \phi(x)^T z.
\]

Consider now the function \( V_2(z) = \gamma^{-1} z^T z \) whose time derivative satisfies \( \dot{V}_2(z) = -2(\phi(x)^T z)^2 \leq 0 \), which implies that

\[
z(t) \in \mathcal{L}_\infty \quad \text{and} \quad \phi(x(t))^T z(t) \in \mathcal{L}_2
\]

for any \( u \).
Adaptive control design (cont’d)

- **Proof (cont’d):** The system (Σ1) can be rewritten as

\[
\begin{align*}
\dot{x}_1 &= f(x_1) + g(x_1)x_2 \\
\dot{x}_2 &= -k x_2 - \epsilon \frac{\partial V_1}{\partial x_1} g(x_1) - \phi(x)^T z.
\end{align*}
\]

Consider the function \( W(x, z) = 2\epsilon V_1(x_1) + x_2^2 + k^{-1} V_2(z) \) and note that

\[
\begin{align*}
\dot{W}(x, z) &= 2\epsilon \frac{\partial V_1}{\partial x_1} f(x_1) - 2k x_2^2 - 2x_2\phi(x)^T z \\
&\quad - 2k^{-1} \left( \phi(x)^T z \right)^2 \\
&\leq -2\epsilon \kappa(x_1) - k x_2^2 - k^{-1} \left( \phi(x)^T z \right)^2 \leq 0.
\end{align*}
\]

It follows that \( x(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty \) and \( \dot{x}(t) \in \mathcal{L}_\infty \), hence, by Barbalat’s Lemma, \( \lim_{t \to \infty} x(t) = 0 \).
Adaptive control design (cont’d)

• Integrating $\dot{z} = -\gamma \phi(x)\phi(x)^T z$, when $z \in \mathbb{R}$, yields the solution

$$z(t) = z(0) \exp(-\gamma \int_0^t \phi(x(\tau))^2 d\tau),$$

which implies that the convergence of the estimation error to its limit value can be arbitrarily increased by increasing the parameter $\gamma$.

• The control law $u$ differs from the certainty equivalence control law in that it contains an additional “feedback” expressed by the term $\phi(x)^T \beta(x)$.

• When $\phi(x)^T \theta$ is a polynomial function in $x_2$, this term renders the closed-loop system input-to-state stable (ISS) with respect to the error $\hat{\theta} - \theta$.

• The linear feedback $-kx_2$ ensures that the system is $L_2$-stable with respect to the perturbation $\phi(x)^T z$. However, a nonlinear feedback would accommodate a wider class of perturbations.
Robustness of the non-adaptive system

• **Proposition 2:** Consider the system (\(\Sigma 1\)) and the control law

\[
u = -kx_2 - \epsilon \frac{\partial V_1}{\partial x_1} g(x_1) - \phi(x)^T \left(\bar{\theta} + \beta(x)\right),
\]

where \(\bar{\theta}\) is a constant. Suppose that \(\phi(x)\) can be expressed as

\[
\phi(x) = \left[\psi_1(x_1)x_2^{q_1}, \cdots, \psi_p(x_1)x_2^{q_p}\right]^T,
\]

where \(q_1, \ldots, q_p\) are non-negative constants. Then all closed-loop trajectories are bounded.

• **Proof:** Integrating \(\phi(x)\) yields

\[
\beta(x) = \gamma x_2 \left[\psi_1(x_1)\frac{x_2^{q_1}}{q_1 + 1}, \cdots, \psi_p(x_1)\frac{x_2^{q_p}}{q_p + 1}\right]^T
\]

\[
= \gamma x_2 \Gamma \phi(x),
\]
Robustness of the non-adaptive system (cont’d)

where $\Gamma = \text{diag}(\frac{1}{q_1 + 1}, \ldots, \frac{1}{q_p + 1})$.

Consider now the function $V(x) = 2\epsilon V_1(x_1) + x_2^2$ and note that

$$
\dot{V}(x) = 2\epsilon \frac{\partial V_1}{\partial x_1} f(x_1) - 2kx_2^2 - 2x_2\phi(x)^T (\bar{\theta} - \theta) - 2\gamma x_2^2\phi(x)^T \Gamma \phi(x)
$$

$$
\leq -2\epsilon \kappa(x_1) - 2kx_2^2 + \frac{2\gamma}{\bar{q} + 1} x_2^2 |\phi(x)|^2 + \frac{\bar{q} + 1}{2\gamma} |\bar{\theta} - \theta|^2
$$

$$
- \frac{2\gamma}{\bar{q} + 1} x_2^2 |\phi(x)|^2
$$

$$
= -2\epsilon \kappa(x_1) - 2kx_2^2 + \frac{\bar{q} + 1}{2\gamma} |\bar{\theta} - \theta|^2,
$$

where $\bar{q} = \max(q_1, \ldots, q_p)$. As a result, $x(t) \in \mathcal{L}_\infty$ and the zero equilibrium is *practically stable*. 
Robustness of the non-adaptive system (cont’d)

• The same result will hold for any vector $\phi(x)$ with elements of the form

$$\phi_i(x) = \psi_i(x_1)\rho_i(x_2),$$

provided the function $\rho_i(\cdot)$ satisfies the differential inequality

$$-cx_2 \frac{\partial \rho_i}{\partial x_2} \rho_i(x_2) + x_2^2 \left( \frac{\partial \rho_i}{\partial x_2} \right)^2 \leq 0$$

for some $c > 0$. A solution to the above inequality is given by

$$\rho_i(x_2) = \rho_0 \exp\left( \int \frac{cx_2}{x_2^2 + h(x_2)} \, dx_2 \right), \quad h(x_2) \geq 0.$$
Extensions

- The result in Proposition 1 can be extended in two directions.
  1. Systems where the control input is multiplied by an *unknown* parameter.
  2. Systems that do not necessarily satisfy the matching condition. (This will allow for actuator dynamics to be *appended* to the plant dynamics.)
Unknown control coefficients

• Consider the class of systems

\[
\begin{align*}
\dot{x}_1 &= f(x_1) + g(x_1)x_2 \\
\dot{x}_2 &= \theta_2 u + \phi(x)^T \theta_1,
\end{align*}
\]  

(Σ2)

where \( \theta_1 \in \mathbb{R}^p \), \( \theta_2 \in \mathbb{R} \), and suppose that the sign of \( \theta_2 \) (the “control direction”) is known. Without loss of generality, suppose that \( \theta_2 > 0 \).

• Proposition 3: Consider the system (Σ2) and the adaptive state feedback control law

\[
\begin{align*}
\dot{\hat{\theta}} &= -\left(I_{p+1} + \frac{\partial \beta}{\partial \hat{\theta}}\right)^{-1} \left( \frac{\partial \beta}{\partial x_1} (f(x_1) + g(x_1)x_2) + \frac{\partial \beta}{\partial x_2} \left(-kx_2 - \epsilon \frac{\partial V_1}{\partial x_1} g(x_1)\right) \right) \\
\hat{u} &= \left(\hat{\theta}_2 + \beta_2(x, \hat{\theta}_1)\right) \left(-kx_2 - \epsilon \frac{\partial V_1}{\partial x_1} g(x_1) - \phi(x)^T \left(\hat{\theta}_1 + \beta_1(x)\right) \right),
\end{align*}
\]
where

\[
\begin{align*}
\beta_1(x) &= \gamma_1 \int_{0}^{x_2} \phi(x_1, \chi) d\chi \\
\beta_2(x, \hat{\theta}_1) &= \gamma_2 \left( k \frac{x_2^2}{2} + \epsilon \frac{\partial V_1}{\partial x_1} g(x_1) x_2 \right) + \gamma_2 \int_{0}^{x_2} \phi(x_1, \chi)^T \left( \hat{\theta}_1 + \beta_1(x_1, \chi) \right) d\chi,
\end{align*}
\]

and \( k > 0, \epsilon > 0, \gamma_1 > 0, \gamma_2 > 0 \) are constants. Then all closed-loop trajectories are bounded and \( \lim_{t \to \infty} x(t) = 0 \).

• Proof: Define the errors

\[
\begin{bmatrix}
z_1 \\
z_2 \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
\hat{\theta}_1 \\
\hat{\theta}_2
\end{bmatrix} -
\begin{bmatrix}
\theta_1 \\
\theta_2\end{bmatrix}^{-1} +
\begin{bmatrix}
\beta_1(x) \\
\beta_2(x, \hat{\theta}_1) \\
\beta(\cdot)
\end{bmatrix}.
\]
Unknown control coefficients (cont’d)

• Proof (cont’d): The error dynamics are given by

\[
\dot{z} = \left( I_{p+1} + \frac{\partial \beta}{\partial \hat{\theta}} \right) \hat{\theta} + \frac{\partial \beta}{\partial x_1} \left( f(x_1) + g(x_1)x_2 \right) \\
+ \frac{\partial \beta}{\partial x_2} \left( \theta_2 u + \phi(x)^T \left( \hat{\theta}_1 + \beta_1(x) - z_1 \right) \right) \\
= -\Gamma \Phi(x, \hat{\theta}_1) \Phi(x, \hat{\theta}_1)^T z,
\]

where

\[
\Phi(x, \hat{\theta}_1) = \begin{bmatrix}
\phi(x) \\
\theta_2 \left( kx_2 + \epsilon \frac{\partial V_1}{\partial x_1} g(x_1) + \phi(x)^T \left( \hat{\theta}_1 + \beta_1(x) \right) \right)
\end{bmatrix}
\]

\[
\Gamma = \begin{bmatrix}
\gamma_1 & 0 \\
0 & \gamma_2 \theta_2^{-1}
\end{bmatrix}.
\]
Unknown control coefficients (cont’d)

- **Proof (cont’d):** Consider the function $V_2(z) = z^T \Gamma^{-1} z$, whose time derivative satisfies $\dot{V}_2(z) = -2 \left( \Phi(x, \hat{\theta}_1)^T z \right)^2 \leq 0$, hence $z(t) \in L_\infty$ and $\Phi(x(t), \hat{\theta}_1(t))^T z(t) \in L_2$. The system $(\Sigma 2)$ can be rewritten as

\[
\begin{align*}
\dot{x}_1 &= f(x_1) + g(x_1)x_2 \\
\dot{x}_2 &= -kx_2 - \epsilon \frac{\partial V_1}{\partial x_1} g(x_1) - \Phi(x, \hat{\theta}_1)^T z,
\end{align*}
\]

i.e. as an asymptotically stable system perturbed by an $L_2$ signal. The proof is completed by considering the Lyapunov function $W(x, z) = 2\epsilon V_1(x_1) + x_2^2 + k^{-1} V_2(z)$ and invoking similar arguments as in the proof of Proposition 1.
Consider again the system (Σ2) in *cascade* with a system in normal form representing the dynamics of the actuator, namely

\[
\begin{align*}
\dot{x}_1 &= f(x_1) + g(x_1)x_2 \\
\dot{x}_2 &= \theta_2 \nu_1 + \phi(x)^T \theta_1 \\
\dot{\nu}_1 &= \nu_2 \\
&\quad\cdots\\n\dot{\nu}_{m-1} &= \nu_m \\
\dot{\nu}_m &= \nu_{m+1} = h(\nu) + u_a,
\end{align*}
\]

(Σ3)

where \( \nu \in \mathbb{R}^m \) is the state of the actuator with input \( u_a \) and output \( \nu_1 \) and \( h(\cdot) \) is a known function.

Note that, if \( \nu_1 \) was the control input, the design would be identical to the previous one with \( \nu_1 = u \). Therefore, the objective is to find a control law \( u_a \) such that \( \nu_1 \to \nu_1^* = u \) asymptotically, with all signals bounded.
Appended dynamics (cont’d)

- **Proposition 4:** Consider the system \((\Sigma 3)\) and the adaptive state feedback control law

\[
\dot{\theta} = -\left(I_{p+2} + \frac{\partial \beta}{\partial \theta}\right)^{-1}\left(\frac{\partial \beta}{\partial x_1} (f(x_1) + g(x_1)x_2) + \frac{\partial \beta}{\partial x_2}\left(-kx_2 - \epsilon \frac{\partial V_1}{\partial x_1} g(x_1)\right)\right)
\]

\[
+ \left(\hat{\theta}_3 + \beta_3(x, \hat{\theta}, v_1)\right)(v_1 - v_1^*) + \frac{\partial \beta}{\partial v_1} v_2
\]

\[
v_1^* = \left(\hat{\theta}_2 + \beta_2(x, \hat{\theta}_1)\right)\left(-kx_2 - \epsilon \frac{\partial V_1}{\partial x_1} g(x_1) - \phi(x)^T (\hat{\theta}_1 + \beta_1(x))\right)
\]

\[
v_{i+1}^* = -\alpha_i(x, \hat{\theta}, v_1, \ldots, v_i) + \frac{\partial v_i^*}{\partial \hat{\theta}} \dot{\hat{\theta}} + \frac{\partial v_i^*}{\partial x_1} (f(x_1) + g(x_1)x_2)
\]

\[
+ \frac{\partial v_i^*}{\partial x_2}\left(-kx_2 - \epsilon \frac{\partial V_1}{\partial x_1} g(x_1) + \left(\hat{\theta}_3 + \beta_3(x, \hat{\theta}, v_1)\right)(v_1 - v_1^*)\right)
\]

\[
+ \sum_{j=1}^{i-1} \frac{\partial v_i^*}{\partial v_j} v_{j+1}, \quad i = 1, \ldots, m
\]

\[
u_a = v_{m+1}^* - h(v),
\]
where

\[
\begin{align*}
\beta_1(x) &= \gamma_1 \int_0^{x_2} \phi(x_1, \chi) d\chi \\
\beta_2(x, \hat{\theta}_1) &= \gamma_2 \left( k \frac{x_2^2}{2} + \varepsilon \frac{\partial V_1}{\partial x_1} g(x_1)x_2 \right) + \gamma_2 \int_0^{x_2} \phi(x_1, \chi)^T \left( \hat{\theta}_1 + \beta_1(x_1, \chi) \right) d\chi \\
\beta_3(x, \hat{\theta}, \nu_1) &= \gamma_3 \nu_1 x_2 - \gamma_3 \int_0^{x_2} \nu_1^*(x_1, \chi, \hat{\theta}) d\chi
\end{align*}
\]

with \( k > 0, \varepsilon > 0, \gamma_1 > 0, \gamma_2 > 0, \gamma_3 > 0 \) constants and

\[
\begin{align*}
\alpha_1 &= \left( c_1 + \frac{\varepsilon}{2} \left( \frac{\partial \nu_1^*}{\partial x_2} \right)^2 \right) (\nu_1 - \nu_1^*) + \left( \hat{\theta}_3 + \beta_3(x, \hat{\theta}, \nu_1) \right) x_2 \\
\alpha_i &= \left( c_i + \frac{\varepsilon}{2} \left( \frac{\partial \nu_i^*}{\partial x_2} \right)^2 \right) (\nu_i - \nu_i^*) + (\nu_{i-1} - \nu_{i-1}^*)
\end{align*}
\]

for \( i = 2, \ldots, m \), where \( c_i > 0 \) and \( \varepsilon > 0 \) are constants. Then all closed-loop trajectories are bounded and \( \lim_{t \to \infty} x(t) = 0 \).
Application to an autonomous aircraft

- Under the standard assumptions of a flat earth and a rigid body of constant mass, the equations of \textit{longitudinal} motion are given by

\begin{align*}
\dot{h} &= u \sin \theta - w \cos \theta \\
\dot{u} &= -qw - g \sin \theta + F_x/m + T/m \\
\dot{w} &= qu + g \cos \theta + F_z/m \\
\dot{\theta} &= q \\
\dot{q} &= \frac{M}{I_{yy}},
\end{align*}

where $h$ is the altitude, $u, w$ are the components of the airspeed along the $x, z$ body axes, $\theta$ is the pitch angle, $q$ is the pitch rate, $m$ is the aircraft mass, $I_{yy}$ is the moment of inertia, $F_x, F_z$ are the aerodynamic forces along the $x, z$ body axes and $T$ is the thrust.
Application to an autonomous aircraft (cont’d)

- The pitching moment $M$ is defined as

\[ M = C_m Q S \bar{c}, \]

where $Q$ is the dynamic pressure, $S$ is the total wing area, $\bar{c}$ is the mean aerodynamic chord and $C_m$ is the aerodynamic coefficient which is expressed in the (simplified) form

\[ C_m = C_{m0} + C_{m1} q + C_{m2} \delta_e, \]

where $C_{m0}, C_{m1}, C_{m2}$ are unknown constants and $\delta_e$ is the elevator deflection angle and acts as the control input.
Application to an autonomous aircraft (cont’d)

• Defining the variables

\[ x_1 = \theta - \theta_*, \quad x_2 = q + \lambda (\theta - \theta_*) \], \quad u = \delta_e \]

with \( \lambda > 0 \), and the parameters

\[
\theta_1 = \begin{bmatrix}
C_{m0}QS \bar{c}/I_{yy} \\
-\lambda \left(C_{m1}QS \bar{c}/I_{yy} + \lambda \right) \\
C_{m1}QS \bar{c}/I_{yy} + \lambda
\end{bmatrix}, \quad \theta_2 = C_{m2}QS \bar{c}/I_{yy}
\]

yields a system of the form \((\Sigma 2)\), namely

\[
\begin{align*}
\dot{x}_1 &= -\lambda x_1 + x_2 \\
\dot{x}_2 &= \theta_2 u + \phi(x)^T \theta_1,
\end{align*}
\]

where \( \phi(x) = [1, x_1, x_2]. \)
Simulations have been carried out using a 6DoF model of the XRAE-140 unmanned air vehicle which is driven by a single propeller and has a wing span of 2.6 m.
Time histories of the pitch angle $\theta$, pitch rate $q$ and elevator deflection angle $\delta_e$. 
Time histories of the altitude $h$, airspeed $V = (u^2 + w^2)^{1/2}$ and thrust $T$. 
Conclusions

- An adaptive controller for the stabilisation via state feedback of systems in feedback form has been presented.
- The main advantage of the proposed algorithm is that it allows for prescribed (stable) dynamics to be assigned to the parameter estimation error and these can be easily tuned to achieve the desired performance.
- The corresponding non-adaptive controller is globally practically stabilising for any polynomial regressor.
- The proposed methodology has been used to design a longitudinal controller for an autonomous aircraft with unknown aerodynamic and inertial properties.
- The results can be extended to the problem of tracking arbitrary (sufficiently smooth) signals.