The Skorokhod embedding problem and its financial applications
Pricing and hedging convex payoffs of the local time

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Based on a joint work with A.M.G. Cox and David Hobson

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Outline

1. Introduction: SEP from financial perspective
   - Motivation and the problem
   - Brief history of the problem

2. Some examples of solutions to SPE and their implications
   - Short biased guided tour of SEP
   - The same tour from financial perspective

3. Pricing and hedging convex payoffs of the local time
   - Motivation
   - Pathwise inequalities with local time
From Exotic’s prices to embeddings

**Aim:** price and hedge \( F = F(S_t : t \leq T) \), where \( (S_t : t \leq T) \) is the stock price.

**Assumptions:**
- \( S_t \) is a martingale under the risk-neutral measure
- we trust liquid market data

**Analysis:**

- \( S_t - S_0 = B_{\Gamma_t} \) is a time-changed Brownian Motion
- From call prices at maturity \( T \) we can read the distribution
  \( \mu \sim (S_T - S_0) = B_{\Gamma_T} \).
- ’The fair price’ for \( F \) lies in
  \[ \inf_{\tau} \mathbb{E}[F(B_t : t \leq \tau)], \sup_{\tau} \mathbb{E}[F(B_t : t \leq \tau)] \], \( \tau : B_\tau \sim \mu \)
  and the bounds attained by \( S_t - S_0 := B_{\tau^*} \wedge [t/(T-t)] \).
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- and the bounds attained by $S_t - S_0 := B_{\tau^* \wedge [t/(T-t)]}$. 
From Exotic’s prices to embeddings

Aim: price and hedge $F = F(S_t : t \leq T)$, where $(S_t : t \leq T)$ is the stock price.

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  and the bounds attained by $S_t - S_0 := B_{\tau^* \wedge [t/(T-t)]}$.

New aim(s): Given $\mu$, find all $\tau$ such that $B_{\tau} \sim \mu$.
Identify the optimal ones $\leadsto$ option’s price range.
Gain pathwise understanding $\leadsto$ hedging.
The Skorokhod Embedding Problem (SEP)

Given $B = (B_t : t \geq 0)$, a Brownian motion, and a probability measure $\mu$, find a stopping time $\tau$, such that $B_\tau \sim \mu$. 
Given \( B = (B_t : t \geq 0) \) a Brownian motion, and a probability measure \( \mu \), find a stopping time \( \tau \), such that \( B_\tau \sim \mu \).
The Skorokhod Embedding Problem (SEP)

Given $B = (B_t : t \geq 0)$ a Brownian motion, and a probability measure $\mu$, find a stopping time $\tau$, "small", such that $B_\tau \sim \mu$
The Skorokhod Embedding Problem (SEP)

Given $B = (B_t : t \geq 0)$ a Brownian motion, and a probability measure $\mu$, find a stopping time $\tau$, such that $B_\tau \sim \mu$ and $(B_{t \wedge \tau})$ is UI.
The Skorokhod Embedding Problem (SEP)

Given $B = (B_t : t \geq 0)$ a Brownian motion, and a probability measure $\mu$, find a stopping time $\tau$, $\sim \tau$ optimal: min or max $\mathbb{E}F_\tau$ such that $B_\tau \sim \mu$ and $(B_{t \wedge \tau})$ is UI.
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In this talk we focus on $\tau$ which are first hitting times because when $Y_t = (B_t, F_t)$ is Markov and $\tau = \inf\{t : Y_t \in D\}$ the value of $\mathbb{E}F_\tau$ is often minimal/maximal.
Other guided tours...

- Skorokhod 61
- Root 69
- Rost 71
- Monroe 72
- Chacon – Walsh 74
- Azéma – Yor 79
- Vallois 83
- Perkins 85
- Jacka 88
- Bertoin – Le Jan 92
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Discrete vs Continuous approach
Pb: Given \((B_t : t \geq 0)\) a BM and a centered prob. measure \(\mu\)
we look for a process \(Y_t\) and a region \(D\) such that

\[
\tau = \inf\{t : Y_t \in D\} \quad \text{s.t.} \quad B_\tau \sim \mu
\]

- Skorokhod: \(Y_t = B_t\)
- Root: \(Y_t = (B_t, t)\)
- Azéma and Yor: \(Y_t = (B_t, \max_{u \leq t} B_u)\)
- Vallois: \(Y_t = (B_t, L_t)\)
**Pb:** Given \((B_t : t \geq 0)\) a BM and a centered prob. measure \(\mu\) we look for a process \(Y_t\) and a region \(D\) such that

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\[
\tau = \tau_{a,b} = \inf\{t : B_t \notin [a, b]\} \quad \text{and} \quad B_\tau \sim \frac{b}{b-a} \delta_a + \frac{-a}{b-a} \delta_b.
\]

Write \(\mu = \frac{B}{B-A} \delta_A + \frac{-A}{B-A} \delta_B\) and enlarge filtration with indep \((A, B)\).

Then \(B_{\tau_{A,B}} \sim \mu\).
Pb: Given \((B_t : t \geq 0)\) a BM and a centered prob. measure \(\mu\) we look for a process \(Y_t\) and a region \(D\) such that

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Barrier = boarder and all points on the right. There exists a barrier \(D\) s.t. \(B_\tau \sim \mu\). Generally unique. Generally we don’t know it...
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we look for a process \(Y_t\) and a region \(D\) such that

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There exists a unique increasing \(\Psi_\mu\) such that

\[
D = \{(x, m) : m \geq \Psi_\mu(x)\}
\]

\[
\Psi_\mu(x) = \frac{1}{\mu([x, \infty))} \int_x^\infty u\mu(du).
\]
**Pb:** Given \((B_t : t \geq 0)\) a BM and a centered prob. measure \(\mu\)

we look for a process \(Y_t\) and a region \(D\) such that

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There exist unique increasing \(\varphi_+/-\) such that

\[
D = \{(x, l) : x \notin [\varphi_-(l), \varphi_(l)]\}
\]

Likewise, a decreasing couple.
**Pb:** Given \((B_t : t \geq 0)\) a BM and a centered prob. measure \(\mu\) we look for a process \(Y_t\) and a region \(D\) such that

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There exist unique increasing \(\varphi_+/-\) such that

\[
D = \{ (x, l) : x \notin [-\varphi_-(l), \varphi_+(l)] \}.
\]

Likewise, a decreasing couple.
**Root/Rost:** $Y_t = (B_t, t)$

$$\tau_R = \inf\{t : Y_t \in D\} = \inf\{t : (B_t, t) \in D\}$$

**Thm.** For any centered $\mu$ there exist (unique) barriers $D^-, D^+$ such $B_{\tau_{R-}} \sim B_{\tau_{R+}} \sim \mu$ and for any convex function $F$

$$\mathbb{E}F(\tau_{R-}) \leq \mathbb{E}F(\tau) \leq \mathbb{E}F(\tau_{R+}),$$

for all $\tau : B_{\tau} \sim \mu$, and $(B_{\tau \wedge t})$ is uniformly integrable.
Root/Rost: \( Y_t = (B_t, t) \)

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\tau_R = \inf\{t : Y_t \in D\} = \inf\{t : (B_t, t) \in D\}
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**Thm.** For any centered \( \mu \) there exist (unique) barriers \( D^-, D^+ \) such \( B_{\tau_R^-} \sim B_{\tau_R^+} \sim \mu \) and for any convex function \( F \)

\[
\mathbb{E}F(\tau_{R^-}) \leq \mathbb{E}F(\tau) \leq \mathbb{E}F(\tau_{R^+}),
\]

for all \( \tau : B_{\tau} \sim \mu \), and \((B_{\tau \wedge t})\) is uniformly integrable.

**Rephrase:** let \( \mu \sim (S_T - S_0) \) then \( S_t = S_0 + B_{\Gamma_t} \) and

\[
\mathbb{E}F(\tau_{R^-}) \leq \mathbb{E}F(\langle S \rangle_T) \leq \mathbb{E}F(\tau_{R^+}),
\]

describes the lower and upper bounds on the payoffs of the realized quadratic variation (of \( S \), which is assumed continuous here).

*Hard to get analytically, easier to simulate.*
Azéma-Yor: \( Y_t = (B_t, \sup_{u \leq t} B_u) = (B_t, \overline{B}_t) \)

\[
\tau_{AY} = \inf\{t : Y_t \in D\} = \inf\{t : \Psi_\mu(B_t) \leq \overline{B}_t\}
\]

**Thm.** There exists \( \Psi_\mu \) (barycentre function) such that \( B_{\tau_{AY}} \sim \mu \) and for any \( x > 0 \), \( K := \Psi_\mu^{-1}(x) \), for all \( \tau: B_\tau \sim \mu \)

\[
P(B_\tau \geq x) \leq P(\overline{B}_{\tau_{AY}} \geq x) = P(B_{\tau_{AY}} \geq K) = \mu([K, \infty)) =: \overline{\mu}(K).
\]
Azéma-Yor: \( Y_t = (B_t, \sup_{u \leq t} B_u) = (B_t, \overline{B}_t) \)

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\[ \mathbb{P}(\overline{B}_{\tau} \geq x) \leq \mathbb{P}(\overline{B}_{\tau_{AY}} \geq x) = \mathbb{P}(B_{\tau_{AY}} \geq K) = \mu([K, \infty)) =: \overline{\mu}(K). \]

**Rephrase:** Let \( \mu \sim S_T \) (with \( S_t := S_t - S_0 \)). Price of a digital barrier (lookback) option \( \mathbb{P}(\sup_{u \leq T} S_u \geq x) \) is bounded above by \( \overline{\mu}(K) \).

(Hobson ’98) We also have hedging: observe that

\[ 1_{\sup_{u \leq T} S_u \geq x} \leq \left( \frac{(S_T - K)^+}{x - K} \right)_{\text{calls portfolio}} + \left( \frac{x - S_T}{x - K} \right)_{\text{martingale}} 1_{\sup_{u \leq T} S_u \geq x} \]

and thus lookback option is superhedged by: *buy calls at \( K \) and if the stock price reaches \( x \) sell stocks*. The maximal loss using this strategy is option premium - \( \overline{\mu}(K) \).
Azéma-Yor: \( Y_t = (B_t, \sup_{u \leq t} B_u) = (B_t, \bar{B}_t) \)

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\[
\mathbb{P}(\bar{B}_\tau \geq x) \leq \mathbb{P}(\bar{B}_{\tau_{AY}} \geq x) = \mathbb{P}(B_{\tau_{AY}} \geq K) = \mu([K, \infty)) =: \mu(K).
\]

**Rephrase:** Let \( \mu \sim ST \) (with \( S_t := S_t - S_0 \)). Price of a digital barrier (lookback) option \( \mathbb{P}(\sup_{u \leq T} S_u \geq x) \) is bounded above by \( \mu(K) \).

*(Hobson ’98)* We also have hedging: observe that

\[
1_{\sup_{u \leq T} S_u \geq x} \leq \left(\frac{(S_T - K)^+}{x - K}\right)_{\text{calls portfolio}} + \left(\frac{(x - S_T)}{x - K}\right)_{\text{martingale}}\]

and thus lookback option is superhedged by: *buy calls at K and if the stock price reaches x sell stocks*. The maximal loss using this strategy is option premium - \( \mu(K) \).
Vallois: $Y_t = (B_t, L_t)$

$$\tau_V = \inf\{t : Y_t \in D\} = \inf\{t : B_t \notin (\varphi_-(L_t), \varphi_+(L_t))\}$$

**Thm.** There exist unique increasing functions $\varphi_\pm$ s.t. $B_{\tau_V} \sim \mu$ and

$$\mathbb{E}F(L_{\tau_V}) \leq \mathbb{E}F(L_{\tau_V}), \quad \text{for all} \quad \tau : B_{\tau} \sim \mu,$$

where $F$ is any increasing convex function.
Vallois: \( Y_t = (B_t, L_t) \)

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Thm. There exist unique increasing functions \( \varphi_\pm \) s.t. \( B_{\tau_V} \sim \mu \) and

\[
\mathbb{E} F(L_\tau) \leq \mathbb{E} F(L_{\tau_V}), \quad \text{for all} \quad \tau : B_\tau \sim \mu,
\]

where \( F \) is any increasing convex function.

Rephrase: Let \( S_t \) be continuous, \( \mu \sim S_T \) then \( S_t = B_{\Gamma_t} \) and \( L_t^S = L_{\Gamma_t}^B \).

The upper bound on the arbitrage-free price of an option paying \( F(L_T^S) \) is \( \mathbb{E} F(L_{\tau_V}) \).

- Can we express the last value in terms of \( \mu \)?
- Can we give a superhedge?
- Why do we care?
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\mathbb{E} F(L_{\tau}) \leq \mathbb{E} F(L_{\tau_V}), \quad \text{for all} \quad \tau : B_{\tau} \sim \mu,
\]

where \( F \) is any increasing convex function.

*Rephrase:* Let \( S_t \) be continuous, \( \mu \sim S_T \) then \( S_t = B_{\Gamma_t} \) and \( L^S_t = L^B_{\Gamma_t} \).

The upper bound on the arbitrage-free price of an option paying \( F(L^S_T) \) is \( \mathbb{E} F(L_{\tau_V}) \).

- Can we express the last value in terms of \( \mu \)?
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So why do we care?

- It’s nice mathematics!!!
  Applications to solving optimal stopping problems.
- Local time can be seen as an approximation of number of downcrossings of an interval or of a corridor variance swap:
  \[ \epsilon L_t^S \approx \int_0^t 1_{s_0 \leq s_u \leq s_0 + \epsilon} d\langle S \rangle_u. \]
- Local time calls can secure against big losses from using the naïve hedging such as the Stop-loss start-gain strategy (Seidenverg ’88): borrow $K$ and keep $K$ or one stock whichever is worth more. At $T$ pay back $K$ and end up with $(S_T - K)^+$ ... The paradox explained by accumulation of the local time at level $K$ (Carr and Jarrow ’90) - that is by the Itô-Tanaka formula.
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Designing pathwise inequalities I

We are given $F$ convex, increasing. We want to price (and hedge) $F(L_\tau)$ where $B_\tau \sim \mu \sim (S_T - S_0)$.

Suppose we can find $H$ convex and $\Delta_t$ such that:

$$F(L_t) \leq H(B_t) + \int_0^t \Delta_s dB_s, \quad a.s.$$  

Then $\mathbb{E}F(L_\tau) \leq \mathbb{E}H(B_\tau) = \int H(x) \mu(dx) =: \Theta$ and we have a robust model-free superhedge: buy portfolio $H$ of calls and hold dynamically $\Delta_t$ stocks.

**Aim:** Find $H, M_t = \int_0^t \Delta_s dB_s$ and $\tau$ which give the tight bound.
Suppose $\mu$ and $H$ are symmetric and let $\varphi$ be increasing. We have $H(b) \geq H'(a)(|b| - a) + H(a)$ so that

$$H(B_t) \geq H'(\varphi(L_t))|B_t| - H'(\varphi(L_t))\varphi(L_t) + H(\varphi(L_t)),$$

$$= H'(\varphi(L_t))|B_t| - \int_0^{L_t} H'(\varphi(x))dx + \theta_H(L_t)$$

The aim now is to find $H, \varphi$ such that $\theta_H = F$ and $\tau_\varphi = \inf\{t : |B_t| = \varphi(L_t)\}$ embeds $\mu$: $B_{\tau_\varphi} \sim \mu$. We then have

$$\mathbb{E}F(L_\tau) \leq \mathbb{E}F(L_{\tau_\varphi}) = \int H(x)\mu(dx), \quad \text{for all}^* \tau : B_{\tau_\varphi} \sim \mu.$$
This is achieved explicitly:

\[ \varphi^{-1}(x) = \int_0^x \frac{s}{\mu(s)} \mu(ds); \quad H'(b) = \frac{1}{\nu(\varphi^{-1}(b))} \int_{\varphi^{-1}(b)}^\infty F'(m) \nu(dm), \]

where \( \nu(l) = \exp \left(-\int_0^l \frac{dm}{\varphi(m)} \right). \)

One justifies the use of optional sampling theorem to the arising martingales.

We extend the reasoning to arbitrary centered measures \( \mu \) and convex increasing \( F \) (which yields asymmetric \( H \)).

Finally, we use the method also to solve optimal stopping problems of the form

\[ \sup_{\tau} \mathbb{E} \left[ F(L_\tau) - \int_0^\tau \beta(B_s) ds \right]. \]
Designing pathwise inequalities III

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where \( \nu(l) = \exp \left( - \int_0^l \frac{dm}{\varphi(m)} \right) \).

One justifies the use of optional sampling theorem to the arising martingales.

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Finally, we use the method also to solve optimal stopping problems of the form

\[ \sup_{\tau} \mathbb{E} \left[ F(L_\tau) - \int_0^\tau \beta(B_s) ds \right]. \]
Pricing argument revisited

The forward price $S_t := S_t - S_0$ is a martingale and thus $S_t = B_{\Gamma_t}$ with $S_T = B_{\Gamma_T} \sim \mu$. In consequence

$$\mathbb{E} F(L^S_T) \leq \int H(x) \mu(dx) =: \Theta$$

and the upper bound is attained for $S_t := B_{\tau \varphi \wedge t/(T-t)}$. $F(L^S_T)$ is hedged buying set of calls $H(S_T) = \int_0^\infty (S_T - K)^+ H''(dK)$ and portfolio $V_t$ with $dV_t = \Delta_t dS_t$,

$$\Delta_t = H'(\varphi \mu(L^S_t)) 1_{S_t < 0} - H'(\varphi \mu(-L^S_t)) 1_{S_t > 0}.$$

A selling price $\tilde{\Theta} < \Theta$ can only be justified if the forward price process is known to belong to some subclass of models. Even in this case the seller can still use the hedging mechanism described above and be certain that his potential loss is bounded below by $\Theta - \tilde{\Theta}$ regardless of all other factors.
Motivation
Pathwise inequalities with local time

Pricing argument revisited

The forward price \( S_t := S_t - S_0 \) is a martingale and thus \( S_t = B_{\Gamma_t} \) with \( S_T = B_{\Gamma_T} \sim \mu \). In consequence

\[
\mathbb{E}F(L_T^S) \leq \int H(x) \mu(dx) =: \Theta
\]

and the upper bound is attained for \( S_t := B_{\tau_\varphi \wedge t/(T-t)} \). \( F(L_T^S) \) is hedged buying set of calls \( H(S_T) = \int_0^\infty (S_T - K)^+ H''(dK) \) and portfolio \( V_t \) with \( dV_t = \Delta_t dS_t \),

\[
\Delta_t = H'(\varphi_{\mu}(L_t^S))1_{S_t < 0} - H'(\varphi_{\mu}(-L_t^S))1_{S_t > 0}.
\]

A selling price \( \tilde{\Theta} < \Theta \) can only be justified if the forward price process is known to belong to some subclass of models. Even in this case the seller can still use the hedging mechanism described above and be certain that his potential loss is bounded below by \( \Theta - \tilde{\Theta} \) regardless of all other factors.
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Solutions based on first-hitting times are often useful.
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Azéma-Yor barrier (lookback) prices & hedging
Vallois convex functions of the local time

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THE END