Analytical modelling of non-uniform deformation in thin-walled orthotropic tubes under pure bending

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- Model development
- Numerical solutions
  - Variation of shear modulus
  - Examples of mode jumping
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Instabilities: Tubes under bending

- Under lateral loading (Snap-through)
- deformation localized
Demonstration
Demonstration
Demonstration
Demonstration
Engineer’s bending theory

- Linear theory, small strain assumption.
- “Plane sections remain plane” and normal to the neutral surface.
- Euler–Bernoulli equation

\[
\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}.
\]

- Cross section remains undeformed.
The behaviour of real tubes

The Brazier effect (homogeneous ovalization)

- Initially circular section ovalizes under increasing $M$.
- First modelled by L. G. Brazier (*Proc. R. Soc. A* 1927)

Key model characteristics:

- Progressive destiffening of the response.
- Ovalization assumed to occur uniformly.
The behaviour of real tubes

Kink formation

- Reissner’s model: theoretical limiting moment.
- Formation of kinks is known to occur much sooner.
- This is a localization phenomenon: more severe post-buckling response (e.g. buckling of shells, sandwich panels and pipelines).
The behaviour of real tubes

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- **Aim 1**: to account for non-uniform cross-section ovalization
  - would lead to kink formation

- **Aim 2**: to present results for elastic orthotropic tubes
Progressive deformation

(a) Orientation of the undeformed tube and axes.
Progressive deformation (cont. . . )

(b) Small curvature

(c) Ovalization

(d) Larger curvature

(e) Localization
Model formulation

Consider a thin circular tube with thickness $t$, radius $r$, length $L$ made of a linear elastic material with Young’s modulus $E$ and Poisson’s ratio $\nu$.

- Loading: applied uniform moment $M$.
- Deformation: a prismatic beam bends into a circular arc (approximated by a parabola).
Assume separate *sway* and *tilt* modes (with non-dimensional amplitudes $q_s$ and $q_t$ respectively.)
Model formulation (cont. . . )

- Under constant applied moment the overall deformation is

\[ W(z) = q_s z(z - L)/L \]
\[ \theta(z) = q_t (2z - L)/L \]

- Euler–Bernoulli theory: \( q_s \equiv q_t \).
Definition of coordinates and functions

- Reissner deformation: \((x, y)\) moves to \((x + \zeta, y + \eta)\).
- Additional radial deflection \(w(z, \varphi)\).
- In-plane displacement \(\tilde{u} \equiv uy/r\).
Definition of constants and dimensionless quantities

- \( C = Et \)
- \( D = \frac{Et^3}{12(1 - \nu^2)} \)
- \( \Delta = \frac{r^4C}{L^2D} \)
- Curvature \( \alpha \equiv 2qt\sqrt{\Delta} \)
- Applied moment \( m \equiv \frac{M}{\pi r\sqrt{CD}} \)
Assumptions of the analysis

- Small strains but small nonlinear corrections.
- First order:

$$\varepsilon_z = \frac{y + \eta}{R} \equiv \frac{2q_t(y + \eta)}{L}.$$  

- von Kármán’s strain expression:

$$\varepsilon = \varepsilon_z + \frac{\partial \tilde{u}}{\partial z} + \frac{1}{2} \left( \frac{\partial w}{\partial z} \right)^2.$$
There are five components of energy to consider.

- Radial deformation is decomposed into the first three Fourier cosine components:

\[ w(z, \varphi) = w_0(z) + w_1(z) \cos \varphi + w_2(z) \cos 2\varphi. \]
Total potential energy components

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  \[ w(z, \varphi) = w_0(z) + w_1(z) \cos \varphi + w_2(z) \cos 2\varphi. \]

- Bending energy (a dot denotes differentiation wrt \( z \)):
  \[ U_b = \frac{1}{2} Et \int_0^L \int_0^{2\pi r} \left[ \ddot{W}^2 + \ddot{\varphi}^2 \right] (y + \eta)^2 \, ds \, dz. \]

- Membrane energy:
  \[ U_m = \frac{1}{2} Et \int_0^L \int_0^{2\pi r} \varepsilon^2 \, ds \, dz. \]
Shear strain energy:

\[ U_s = \frac{1}{2} G t \int_0^L \int_0^{2\pi r} \gamma_{yz}^2 \, ds \, dz \]

where \( G = \frac{E}{2(1+\nu)} \), \( \gamma_{yz} = (q_s - q_t) \left( \frac{2z}{L} - 1 \right) + \dot{w} + u/r \).
Total potential energy components

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- Circumferential bending energy:

\[
U_{cs} = \int_0^L \int_0^{2\pi r} \frac{1}{2} D \left( \frac{\partial^2 w}{\partial s^2} \right)^2 \, ds \, dz.
\]
Total potential energy components

■ Work done by load:

\[ M\Theta = \int_0^L \int_0^{2\pi r} E t \varepsilon_z \hat{u} \, ds \, dz + \int_0^L M \left( \dot{u} + \frac{2q_t}{L} \right) \, dz. \]

■ Total potential energy functional:

\[ V = U_b + U_m + U_s + U_{cs} - M \Theta. \]
Governing equations

$V$ is integrated around the section: single integral wrt $z$. 
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- Calculus of variations is used to derive the Euler–Lagrange equations for this system:
  - 3 linked 4th-order ODEs (one for each Fourier component of $w$) and a 2nd-order ODE in $u$ plus two integral constraints for $q_s$ and $q_t$. 
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- Solutions sought with solver AUTO97 on half-interval with simple-support conditions at $z = 0$ and appropriate symmetry conditions at $z = \frac{1}{2}L$. 

Buckling of orthotropic circular tubes in pure bending – p.19/30
Parametric study

- Properties of tubes investigated $E = 205$ kN/mm$^2$, $\nu = 0.3$, $L = 250.0$ mm and $r = 10.0$ mm.
- Shear modulus $G$ deviates from $G_i$ (isotropic case)
- Three slenderness ratios ($r/t$) taken:
  - 10 (○)
  - 33.3 (□)
  - 100 (+)
- Chosen examples for detailed study:
  - $r/t = 10$ with $G/G_i = 0.35$ and $G/G_i = 0.70$. 
Parametric study
Equilibrium diagram

Nondimensional quantities—Moment $m$ vs curvature $\alpha$:

- $G/G_i$ values: 0.35, 0.70
Deflected shapes

The following figures show tube displacements at:

1. Pre-buckling moment $m = 0.4$
2. Limit-point moment $m = m_l$
3. Post-buckling moment $m = 0.4$
Tube: \( G/G_i = 0.35 \)
Tube: $G/G_i = 0.70$
Cross-section profiles

Pre-limit point and post-buckling cross sections at $m = 0.4$ for $G/G_i = 0.35$ (Top) and $G/G_i = 0.70$ (bottom)
Mode jumping

Number of waves \((N)\) along the length for functions:

- \(w_0\) (○)
- \(w_1\) (□)
- \(w_2\) (+)

Simultaneous increase in two functions’ wave numbers:

- marks the boundary between strong and weak mode interactions
- mode jumps are shown by arrows on next graph
Mode jumping: $r/t = 10$
Conclusions

- Tubes modelled using elasticity and energy methods.
- Variation in shear modulus affects the deformation wavelength:
Conclusions

- Tubes modelled using elasticity and energy methods.
- Variation in shear modulus affects the deformation wavelength:
  - Mode jumping occurs
  - Large drop zones in the limit moment
  - Ovalization is non-uniform in drop-zone
  - Brazier/Reissner theory becomes non-conservative in drop-zone
Thank you!