Topological Defects in the Universe and in Condensed Matter Systems

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Abstract

The subject of topological defects has become a very attractive field of study given its apparent relevance to as diverse systems as the early universe and condensed matter. According to the Kibble Mechanism, the fundamental group of the minima of the relevant to our physical system effective potential, provides enough information for a safe deduction about the capacity of that system to accommodate topological defects. The latter will correspond to some solution to the relevant field equations. At late times, when the field configuration has saturated at a particular solution, topological methods can help identify the defects. Up to what level, though, topological considerations can be relevant to their creation is a crucial issue requiring further investigation. According to the Ginzburg postulation, the stability of defects will be related to some topological charge which, in turn, could be dependent on thermal fluctuations in the order parameter over some length scale. In case defects are forming, then the question is to find their primary density. The Zurek scenario provides a plausible answer to that, by means of a series of causal constraints which lead to some scaling laws for quantities like the correlation length and the relaxation time relevant to the defect density at the time of stabilization. There have been many experiments trying to test the ideas put forward but none, so far, has provided watertight arguments in favor or against them. We apply those theories to the system of Josephson Tunnel Junctions which we propose as a possible candidate for further tests of the current picture.
The work presented here is the result of research carried out between August 1998 and August 2001 in collaboration with my supervisor Dr. R. J. Rivers. The research in Chapter 1 was greatly helped by numerous people including Dr. R.J. Rivers, Professor T.W.B. Kibble and Professor Y.M. Bunkov. A summary of the ideas presented there can be found at hep-ph/0111161. Chapter 2 and 3 incorporate the research presented in “The Onset of Phase Transitions in Condensed Matter and Relativistic QFT” which was published in Condensed Matter Physics 3, 133 (2000). Chapter 4 can be found at cond-mat/9901348. Chapter 5 incorporates the result of the endeavor found in Phys. Rev. Lett. 81, 3452 (2000) and in the Journal of Low Temperature Physics (June/July 2001) as part of the refereed proceedings of the ULTI meeting, Finland, Jan 2001.

According to Schiller, joy should be associated with work, if man is to come closer to himself again. During the writing of this thesis, there were moments when I felt I was on that path. And even though I do not know up to what level I have managed to put my stamp on mankind, instead of accepting the latter on me, I just want to believe that I can hope...

I am truly grateful to Dr. R.J. Rivers for being so very helpful and kind to me. If it weren’t for his support I would have been locked in some vicious cycle, orbiting around some kind of selfish void, dreaming of some kind of life. Perhaps I have been locked up in some vicious cycle around some selfish void, but I know I am living some kind of dream.
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Chapter 1

Introduction

1.1 Background

According to the “Big Bang” Theory, as the Universe expanded and cooled, it could have undergone a series of phase transitions[1]. The presently successful standard model of particle physics is a gauge theory, based on the symmetry group $SU(3) \times SU(2) \times U(1)$. The strong interactions are being described by the $SU(3)$ factor representing the color symmetry group. The $SU(2) \times U(1)$ is the electro-weak symmetry visible at energies well above 100 GeV. At lower energies it is being hidden by the process usually described as spontaneous symmetry breaking. only the $U(1)$ of electromagnetism survives. Symmetry breaking often signals a phase transition, and there may indeed be an electro-weak phase transition at a temperature of the order of 100 GeV. However, it is far from certain that true phase transitions indeed occur in reality.

At first sight, that possibility may be surprising. As usually envisaged, the breaking of a symmetry implies a sudden change from a symmetric state to an ordered state, in which some symmetry-breaking order parameter acquires a non-zero value. The latter will no longer have the same symmetry properties as the original field. In global theories symmetry breaking can be easily accommodated. However, strictly speaking, a local gauge symmetry can never be broken according to Elitzur’s theorem. Assuming a perturbative, continuous
gauge theory, we consider a field $\varphi$ to transform under elements of a group $G$. Both $\varphi$ and $g\varphi$ with $g \in G$, leave the Lagrangian of the system invariant. We then expand $\varphi$ to $\varphi_0 + \eta$ and check whether $G$ is also the symmetry group of $\eta$. In global theories, often, it is not. Then the original symmetry has been broken and the system is sensitive on the origin. In local theories the presence of the gauge field can absorb any arising terms due to the change of origin. Moreover, its transformations can maintain the gauge symmetry. Only when we fix the gauge can we talk about breaking a local symmetry.

It is nevertheless often useful to talk in terms of a symmetry-breaking order parameter. This is done routinely in superconductivity, and, as mentioned earlier, it is quite legitimate if one introduces a gauge-fixing term - in which case Elitzur’s theorem does not apply - that breaks the symmetry explicitly. Like, for example, choosing the Coulomb gauge. However, this possibility does not guarantee that a phase transition must exist. Indeed, it is unclear whether the breaking of the symmetry should, solely, be a result of some phase transition. The precise conditions for the existence of a phase transition in these circumstances are not fully known.

In broad terms, a phase transition is a process in which the order parameter balances the external influences to the physical system with the internal tendency for the lowest energy configuration. For continuous, or second order, phase transitions one has in mind a smoothly changing free energy density with an associated order parameter that transforms from a symmetric to a non symmetric configuration. If the changing free energy signals a phase transition, then symmetry breaking alone does not have to guarantee that indeed a phase transition took place. That is because the transition from a symmetric to a non symmetric state should not necessarily be a consequence of free energy modifications. Rather, it could simply be the result of the natural evolution of the relevant order parameter. In our discussion here, though, we are not going to be very concerned about this. On the contrary, we will assume that phase
transitions do occur in the models we will study. Whether, though, we can draw any definite conclusions about the realistic nature of our postulations is another matter.

Symmetry breaking phase transitions are important as they could provide a mechanism for the generation of topological defects. The latter configurations, however, will be identifiable well below the point when the transition takes place and we are in the ordered phase. In the first instance, we will attempt to define topological defects and to do that we need only talk about systems that could accommodate them, neglecting any dynamical processes such as phase transitions. Then, we will come to the matter of their creation where phase transitions will play a major role. Finally, we will focus on the way topological defects can occur in a specific system, which, here will be a Josephson Tunnel Junction.

1.2 About Ordered Media and Defects

Defects are usually thought of as configurations that arise due to the topological properties of the manifold of the minima of the relevant effective potential of our theory. The viewpoint we will try to advocate in favor in this section is that this is not enough. Instead we need to focus on the topological attributes of the physical space in relation to that of the minima.

The definition of a topological defect we will adopt is that it corresponds to some discontinuity in the map \( \varphi \) from a manifold \( A \), our physical space, to \( S \), the space of the minima. The reason for this discontinuity can be either some pathological behavior of \( \varphi \) that has nothing to do with the topology of either \( A \) or \( S \), in which case the defect will be topologically unstable, or some mismatch in the topological properties of the associated spatial areas, in which case the defect will be stable.

We will postulate that the stable topological defect will be the consequence of the difference in the topological attributes of the correlated areas rather than
the individual properties of those areas on their own.

Consider $A$, our physical manifold, to be $\mathbb{R}^2$ and $S$, the space of the minima, to be $S^1$. Any closed contour $l$ embedded in $\mathbb{R}^2 = A$ can be mapped to $S = S^1$ in various ways. In general, any map $\varphi : l \to S^1$ will yield a closed contour on $S$, which we will call $L$. The loop $l$ can continuously contract to a point. However, not all $\varphi[l] = L$, for all continuous maps $\varphi$, will have that characteristic. In fact, all loops $L$ that span $S = S^1$ at least once will not be able to shrink to a point while being closed contours on $S$. Thus, $l$, to any such closed path, will be topologically different since the former will be null homotopic in contrast to the latter. In such a situation there will be a closed loop in $A$, homotopic to $l$, where the map to $S$ will be discontinuous. That discontinuity will be the stable topological defect.

Discontinuities, however, are, usually, quite unfriendly physical phenomena and here we will try to minimize their impact by introducing some appropriate continuous structure over them in order to conceal them. This will be done by allowing $\varphi$ to take values outside the space of minima $S$, over some spatial area in $A$. This, therefore, brings about more energetic configurations. To be sure, apart from hiding the discontinuities, one can allow $\varphi$ to change to a continuous map from $A$ to $S$. In this case, if $\varphi$ must be a continuous function over the crucial contour, we, once again, have to expand the range of $\varphi$. However, instead of introducing that expansion over some collection of loops in $A$, we do it over a single loop in our physical space. The latter situation will concern us elsewhere.

1.2.1 The nature of defects

An ordered medium corresponds to some real smooth manifold $A$ where a function $\varphi$ is defined which assigns to every point of the medium an order parameter. In our discussion, we will refer to that function $\varphi$ by the name order parameter, even though this title might be somehow confusing\(^1\). Excluding the trivial case

\(^1\)One, for example, might consider the space of functions as that relevant to an ordered medium and thus start all sorts of investigations on a different sphere.
where the order parameter is constant throughout the medium (which is called, thus, uniform), we will focus our attention to non uniform media where the function, through connected space, varies continuously apart perhaps (depending on the specific configuration) at isolated regions\cite{3}.

Suppose $\varphi$ to be an order parameter. Take $\varphi : R^n \rightarrow M$ to be everywhere continuous apart from some region $C \subset R^n$. $A = R^n$ here. Clearly, if $\varphi$ exhibits a discontinuity somewhere within $C$, then the manifold $R^n \otimes \varphi[R^n]$ cannot be continuous. Surely, the fact that $\varphi[R^n]$ is discontinuous does not mean that $M$ is discontinuous too. In fact, $M$ can be a trivial, simply connected manifold and still have $C \neq \emptyset$. The discontinuity will be simply due to the definition of $\varphi$. We will be interested in situations where the occurrence of discontinuities does not rely on the explicit formula of $\varphi$ we will choose. Rather, we would like to have that even the most trivial $\varphi$ can generate areas of discontinuities. Such a situation happens when, in view of our domain $R^n$ being a smooth manifold, $M$ is not connected. That $M$ will be our order parameter manifold which will be relevant to the $R^n - C$ space where $\varphi$ can be applied unambiguously. The $R^n - C$ space will be our medium.

Our contemplation to consider order parameters with discontinuities might seem quite far fetched to someone having in mind the situation in physical systems where the order parameter is always continuous. Its relevance will become evident if we introduced the notion of a defect. We will postulate that a defect corresponds to an appropriate change in the order parameter, applied only to those regions where originally it was discontinuous, having as a main aim to restore the continuity of that function throughout the original medium. Thus, the discontinuities will indicate where the introduction of a defect is needed. Associating, however, those configurations with a particular order parameter behavior is another matter which we will look into later.

As we will see, for a certain order parameter behavior, the defect induces an invertible change to that function. Thus, if we were to lift the defect we
would surely recover the underlying discontinuity. However, there are situations for which the discontinuity happens artificially, and thus, unless we remember the exact formula for the order parameter, any change intended to make that function continuous can not be reversed. That will be linked with a topologically unstable defect.

Assume that the order parameter is given by

$$\varphi(x) = \begin{cases} 
f(x) & x > x_0 \\
c & x = x_0 \\
f(x) & x < x_0 
\end{cases}$$

for some constant $c$. For example, $f(x)$ could be $c_1 \Theta(x - x_0) + c_2 \Theta(x_0 - x)$ with $\Theta(x) = 1$ for $x > 0$ and zero otherwise. Take $f(x)$ to be a continuous function on its domain of definition. Thus, we can calculate the

$$\lim_{x \to x_0^-} f(x)$$

which we will call as $f^+(x_0)$ and $f^-(x_0)$ accordingly. If $f^+(x_0) \neq f^-(x_0)$ then, regardless of the actual value of $c = \varphi(x_0)$, $\varphi$ will be discontinuous at $x = x_0$ and that discontinuity will be topologically stable. Otherwise, for $f^+(x_0) = f^-(x_0)$ then, if there is a discontinuity of $\varphi$ at $x = x_0$, it will be due to the value of $c$ and thus, will be topologically unstable.

Therefore, $C$ will be the manifold which, in the absence of a defect, cannot be assigned order parameter values without the occurrence of discontinuities somewhere within it. In physical systems where the order parameter is everywhere continuous, defects are, possibly, already there and therefore the continuity has been appropriately restored. If, though, we were to remove those defects and recover the order parameter configuration, we would surely see those discontinuities for which this procedure is invertible.

It is, thus, inevitable to ask, how one identifies defects in continuous con-
figurations. Moreover, how one knows how the order parameter should behave if one were to take them away. Here, we, thus, have to introduce the notion of a preferred order parameter. Since our medium is a physical system, it should be described by certain equations the solution to which is our order parameter. For various systems there exist many solutions to the relevant equations\(^2\). Each such order parameter corresponds to a particular energy distribution over our medium. We will assume that the solution preferred will be the least energetic one. There could be situations where several order parameters will be equally good in terms of their associated energy distribution. We will postulate that our physical system does not distinguish between them.

Suppose that for a system in equilibrium one finds that the corresponding order parameter does not, over all, have a preferred configuration. Thus, there are areas where \( \varphi \) is a lowest energy order parameter and areas where it is not. Suppose that the regions where the function deviates from preferred configurations are surrounded by energetically beneficial values for \( \varphi \). We will denote those regions as \( D \). Assume that one takes for granted the environment of each \( D \) and tries to seek a good replacement for its associated order parameter among those having some physically justifiable priority. In case such an investigation turns out to be in vain, one, then, can safely link \( D \) with a defect. Here we have assumed that the way one applies preferred order parameters over the given region has to do with whether those functions satisfy the continuity requirement for the total order parameter defined over all the manifold. Thus, order parameters which occur in contrast to the natural tendency of the system for a lowest energy situation, should underline the existence of defects.

The idea of invertible can be clearly understood now. If for a given \( \varphi \) defined on a given \( \mathbb{R}^n \) we find that on some \( C \) there are discontinuities of that function being exhibited then, our first step is to call \( \mathbb{R}^n - C \) our relevant domain. Then, we will try and see if we can apply any preferred order parameter configuration

\(^{2}\)And here one should consider all possible solutions including those having the order parameter constant
on $\mathbb{C}$ so that $\mathbb{R}^n - C \to \mathbb{R}^n$ in the sense that the new order parameter will be a continuous function throughout $\mathbb{R}^n$. If we are unable to find any such preferred order parameter that will smooth-out our $\varphi$ completely then the introduction of a new order parameter configuration will be an invertible process. Otherwise, the additional alterations can not be recovered\(^3\).

Assume a symmetry breaking effective potential $V$ with $S$ the space of its minima. The points on $S$ will be our preferred order parameters. We will assume that $M = S$. Since we will be concerned with whether topological defects could be possible, we need to study the topological properties of $M$. To do that we will use only the fundamental group $\pi_1(M)$. Obviously, according to the nature of $S$, $\pi_1(M)$ can be trivial or not. As Kibble first outlined, for defects to be feasible, $\pi_1(M)$ and equivalently $\pi_1(S)$ must be non trivial.

Considering symmetry breaking potentials is the way forward if one is to create the necessary framework for the occurrence of topological defects. The reason is that the space of their minima has a non trivial fundamental group. However, it is unclear whether symmetry breaking is a necessary and sufficient condition for defects. One could, thus, wonder whether defects can happen in systems where the non trivial space of minima is not associated with some kind of symmetry breaking. This is, though, a matter that could concern us later.

The focus on the fundamental group immediately restricts us in considering homotopy classes of maps from a circle to a circle. Indeed, if $S = S^1$ the only non trivial homotopy group can be the fundamental group. However, this need not be a general consideration.

### 1.2.2 The fundamental group of Abelian defects

We shall examine the general structure of classes of homotopic loops on the preferred order parameter space $S$. Throughout the discussion we will restrict ourselves to the Abelian case. Non-Abelian media will just be a victim of our

\(^3\)In our deduction, we assumed that the order parameter outside $C$ does not sustain any modifications
inevitably imposed cut-offs. Our compensation is that they will not be relevant to our discussion further ahead.

Suppose we have a theory with a symmetry group $G$, and that $\varphi_0$ is a point on the surface $S$ of the minima of the effective potential $V[1]$. Even if this discussion is rather general, we will provide the reader with a pedagogical and indeed relevant to our whole approach, form for $V$. Consider, thus, $V$ to be:

$$V = \frac{1}{2}\lambda(\varphi^2 - |\varphi_0|^2)^2$$

Our order parameter is a complex function and the space of minima is just the circle $\varphi_0 e^{i\theta}$ in the complex plane, with $\theta$ the polar angle. Then all the points $\varphi$ obtained from $\varphi_0$ by the action of elements of $G$, $\varphi = g\varphi_0$, with $g \in G$, also lie on $S$. But not all $g\varphi_0$ are distinct, because there is a subgroup of elements that leave $\varphi_0$ unchanged, the isotropy subgroup $H \subset G$:

$$H = \{g \in G : g\varphi_0 = \varphi_0\}$$

Then the points of the vacuum manifold $S$ are in one-to-one correspondence with the left cosets $gH$ of $H$ in $G$. In other words, $S$ is homeomorphic to the quotient space $G/H$.

The fundamental group $\pi_1(S)$ is defined in terms of closed loops in $S$, continuous curves starting and ending at $\varphi_0$. Two such curves are homotopic if one can be continuously deformed into the other. The elements of $\pi_1(S)$ are homotopy classes of loops. The equivalence classes, or homotopy classes of loops based at $\varphi_0$, are the elements of $\pi_1(S, \varphi_0)$. This defines a product of homotopy classes, so that $\pi_1(S, \varphi_0)$ becomes a group. The identity is the class containing the trivial loop $t \mapsto \varphi(t) = \varphi_0$. The inverse is the class comprising the same loops traversed in the reverse sense and the product $ba$ of two loops $a$ and $b$ is defined as the loop obtained by following $a$ and then $b$.

It is intuitively clear that, if $S$ is connected, then $\pi_1(S, \varphi_0)$ does not depend
on the base point $\varphi_0$, and so the first homotopy group is often denoted simply by $\pi_1(S)$. In other words, all based fundamental groups, $\pi_1(S, \varphi)$, in Abelian media, are isomorphic copies. If $\pi_1(S)$ is trivial, comprising the identity element only, then $S$ is said to be simply connected. A necessary, but not sufficient, condition for the existence of stable defects is that $\pi_1(S)$ be non-trivial, or multiply connected. Hence $\pi_1(S)$ has more than one element, and is non-trivial.

For example, for $G = U(1)$, the isotropy group is $H = 1$, the group consisting of the identity element alone, so the vacuum manifold is a circle: $S \sim G/H \sim S^1$. Here the different homotopy classes of loops are distinguished by the number of times they wind around the circle. The fundamental group is simply the additive group of integers, $\pi_1(S^1) = \mathbb{Z}$.

In Abelian groups the product of two loops will correspond to a closed contour with a winding number being equal to the sum of the winding numbers of its individual components. The winding number is not independent from the direction we transverse a loop. Clearly, if $c$ had a winding number of 1 and $c^{-1}$ corresponded to 1 as well, then the $c \circ c^{-1}$ would have to have a winding number equal to 2. Obviously, that cannot happen for $c \circ c^{-1}$ is homotopic to the identity. Thus the $c^{-1}$ path must have a winding number of $-1$.

The idea of continuously deforming a loop to another is rather general and includes all possible ways the one contour could drop into the other. That is quite crucial for, sometimes, in order for the discontinuity to be exhibited somewhere, one needs to apply multiple transformations to the changing loop in order to deform it. Thus, instead of the usually considered translation, one might need to add, as the following example will demonstrate, rotations as well.

Suppose $\varphi$ to be equal to

$$\varphi(r, \vartheta) = \begin{cases} \cos \vartheta \overrightarrow{i} + \sin \vartheta \overrightarrow{j} & \text{if } r > 2\pi - \vartheta \\ \cos \frac{\vartheta r}{2\pi - \vartheta} \overrightarrow{i} + \sin \frac{\vartheta r}{2\pi - \vartheta} \overrightarrow{j} & \text{if } r < 2\pi - \vartheta \end{cases}$$

with $\vartheta$ the polar angle taking values from the $[0, 2\pi]$ interval. Assume $C$ to
be a disk of radius $2\pi$. As one can see, outside $C$, $\varphi = \cos \vartheta \mathbf{i} + \sin \vartheta \mathbf{j}$. If one tries to contract the loop $r_0 = 2\pi$ by simply translating it on $R^2$ up to $r_1 = 0$, nothing clearly ambiguous will happen. However, the discontinuity at $r = 0$ will be visible if each point $c^4$ on $r_0$ is forced to follow the path $\frac{(2\pi - \vartheta_t)c}{\vartheta_t}$ with $2\pi \geq \vartheta_t > c$. That configuration, even if it is not very attractive in terms of its energetic properties in Euclidean space, it could be relevant to a curved space-time where field discontinuities could happen due to non trivial parallel transport around some closed loop.

1.3 Considering Higher Homotopy groups

The condition that the theory supports topologically stable defects of dimensionality $N = 1$ when $A = \mathbb{R}^3$ and $N = 0$ when $A = \mathbb{R}^2$, is that the fundamental group be non trivial: $\pi_1(S) \neq 1$. Consider $S = S^l$ some $l$-sphere. $\pi_1(S^l)$ is non trivial when $l = 1$. Generally, the situation where non preferred order parameters must exist due to the inability of $\varphi$ to continuously cover the whole of $A$ with points from $S$ can be made feasible by requiring $\pi_k(S) \neq 1$. This means that the group of homotopy classes of maps from a $k$-sphere to an $l$-sphere must be non trivial. Thus, there exist at least two maps that cannot deform to one another continuously. If $A$ is of dimensionality $n$ then $k < n$. That requirement ensures that the $S^k$ will be embedded into $A$ so that $\pi_k(A)$ is trivial. It is clear that the only way $\pi_k(S^l) \neq 1$ is if $l \leq k$. The discontinuity will arise when a link between two maps belonging to different homotopy classes must be found. That is why one requires the homotopy group to be non trivial since one can find different homotopy classes and can create a stable defect by appropriately associating them.

Our theoretical understanding of defects is that they correspond to singularities in the order parameter that arise because two different homotopy classes of the same homotopy group have been associated. Our consequent work pivots

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4Corresponding to some phase $\vartheta_c$
The incredible charm of topology lies, up to a very high degree, on the fact that many of its most important aspects have a intuitive nature and they teach us directly about the nature of the space where some continuous processes take place. In accordance to that, we will sketch a way one can think about the defects in our framework. Consider a closed loop $l_1$ in space which encircles a hole. Take that there is another loop $l_2$ inside the area bounded by $l_1$ which is trivial. We can find paths from each point on $l_2$ to each point on $l_1$ in an $1-1$ fashion. However, due to the presence of the hole, we will always be able to find at least two paths between the $l_1$, $l_2$ contours, that will not be able to drop into one another continuously by keeping their points on $l_2$ and $l_1$ fixed. The hole will be identified as a defect when this type of association between the considered contours has taken place.

The homotopy group $\pi_k(S^l)$ in general, can be computed by finding the different homotopy classes of maps from a $k$-sphere to an $l$-sphere. So far we considered $l = k = 1$. However, one might like to investigate the effects of higher homotopy groups to the classification of different types of order parameter configurations.

Consider the case where $k = 2$ and $l = 2$. Our physical space is some not necessarily Euclidean 3D manifold where any 2-sphere can shrink to a point, whereas the order parameter space is $S^2$. It is clear that $\pi_2(S^2)$ is non trivial, in fact $\pi_2(S^2) = \mathbb{Z}$, since one can wrap a 2-sphere around another $n$ number of times. To see this consider the map:

$$f(\phi, \theta) = (\phi, n\theta)$$

with $\phi$ the angle with respect to the $z$-axis and $\theta$ the polar angle on the $x - y$ plane. The integer $n$ is the winding number. Any map from $S^2$ to itself is exactly homotopic to one of these functions. In general, $\pi_k(S^k) = \mathbb{Z}$.

Defects will occur when the order parameter in the 3D space of our medium
wraps a null homotopic 2-sphere around the order parameter manifold \((S^2)\) \(n > 0\) number of times. Things get more complicated when \(k > l\) and whereas \(\pi_k(S^l)\) can still be non trivial provided \(l > 1\), the discontinuities that could arise by the association of different homotopy classes are not easy to determine. Consider, \(k = 3\) and \(l = 2\). It is not difficult to calculate \(\pi_3(S^2)\) which turns out to be \(\mathbb{Z}\). The way forward is to compute an integer called the “Hopf invariant” that keeps track of the homotopy class of a map from the 3-sphere to the 2-sphere. There are lots of nice ways to compute it, but we will briefly sketch one. Suppose that the map \(f : S^3 \to S^2\) is smooth (otherwise we can always smooth it up). Then most points \(p\) in \(S^2\) have the property that the points \(x\) in \(S^3\) with \(f(x) = p\) form a “link”: a collection of knots in \(S^3\). If we take two different points in \(S^2\) with this property, we get two links. From these two links we can compute an integer called the “linking number”: for example, we can just draw these two links and count the times one crosses over or under the other (with appropriate plus or minus signs for each crossing). This number turns out not to depend on how we picked the two points. Moreover, it only depends on the homotopy class of \(f\). It’s called the Hopf invariant of \(f\).

Consider our physical space \(A\) to be 4 dimensional. In case a null homotopic 3-sphere in \(A\) gets mapped to the 2-sphere of our order parameter space by a map belonging to any homotopy class other than \([0]\), we would expect to have a defect. Still, those configurations are not very well studied and we hope to be able to shed some light into that direction in the future.

To demonstrate that indeed no defect is possible even if \(\pi_1(S)\) is non trivial, suppose, for instance, \(A\) to be an one dimensional closed contour just like \(S\). Thus, \(A = S = S^1\). If \(\varphi\) is continuously changing over \(A\) then there is no need for it to either alter its formula or introduce any additional energetic configurations. It will stay at the homotopy class of \(S\) it originally belonged. Therefore, even if \(S\) is non trivial, defects cannot arise due to the lack of the additional dimension in \(A\).
Assume the domain of $\varphi$ to be $A$. Defects can still happen even if $\pi_1(A) = \pi_1(S) \neq 1$ so long as the trivial homotopy class of $\pi_1(A)$ corresponds to a non-trivial one of $\pi_1(S)$. The picture here is that $\pi_1(A)$ and $\pi_1(S)$ either are the same group or the former is a subgroup of the latter. This realization will be useful later on when we consider annular geometries. If $A = S = S^1$ then no defect will occur. However, if $A$ is a 2D plane with some hole, then the defect will happen inside any loop that does not encircle the hole.

1.3.1 The topological condition for defects

Suppose $\varphi$ to be an order parameter. Take $\varphi : W \rightarrow M$. Usually, the condition for the feasibility of topological defects is

$$\pi_k(M) \neq 1$$

with $\pi_k(M)$ the $k$-homotopy group of $M$ and $k$ some positive integer.

The homotopy group $\pi_k(M)$ in general, can be computed by finding the different homotopy classes of maps from a $k$-sphere to $M$. Suppose that $M$ is some $m$-sphere with $m$ a natural number. In general, the $\pi_k(S^m)$ can be non-trivial only when $k \geq m > 1$. In the case when $m = 1$, $\pi_k(S^1)$ is non-trivial when $k = 1$. For $k > 1$, $\pi_k(S^1)$ is trivial.

Our topological condition is also very simple. We postulate thus, that for topological defects to form we must have that

$$\pi_k(M) \neq 1$$

and

$$\pi_k(W) \text{ a subgroup of } \pi_k(M)$$

Consider the case when $W$ is some $w$-sphere with $w$ a positive integer. The above conditions will yield the relation

$$w > k \geq m$$
where we have assumed that, for example, the $S^1$ sphere is different from the $S^2$ with two holes on it since on the surface of the latter we can shrink a loop to a point continuously, while maintaining it a closed contour, something we cannot do to the former ($S^1$), unless the loop we talk about is already a point. Therefore, we need to take care of the dimensionality of the manifolds we consider, something that the knowledge of the homotopy group does not necessarily prescribe.

When $m = 1$, the only way we can satisfy the first part of condition (1.2) is if we make $k = 1$ as well. In case $W$ is some $w$-sphere, then $w$ can be any $w > 1$ or $W$ can be a 2-sphere with two holes on it, if we want to satisfy the condition $\pi_1(W) = \pi_1(M)$. In general, $W$ can be any connected manifold where we can find a continuous homotopy of a closed contour to a point while the contour remains closed throughout the homotopy. Thus, we arrive at the usual situation where $\pi_1(M)$ is non trivial, since $M = S^1$ and $W$ is $\mathbb{R}^2$ or $\mathbb{R}^3$.

We see that in case $W = \mathbb{R}^2$, the discontinuity will happen at a single point inside a loop that in $W$ is null homotopic whereas in $M$ is not. In case $W = S^2$, we will generate two points of discontinuities since each loop can shrink to two different points on $S^2$. In case $W = \mathbb{R}^3$ we find that we can have a line of discontinuities coming from the $\mathbb{R}^2$ planes stuck on top of one another along the $z$ axis. In that situation, each point of discontinuity is generated by a single loop. Consider, however, $W = \mathbb{R}^3$ to be formed by a point that becomes a 2-sphere of varying radius. In that situation, if a line of discontinuities can be formed by the appropriate $\varphi$ defined on each 2-sphere of $W$, it is generated in such a way that two points on that line should be associated with the same loop on the appropriate 2-sphere that shrunk to create them.

The picture here is the following: Take a point on the defect line and call it the origin. Consider 2-spheres that are centered on the origin. For each different radius of the 2-sphere considered we will have, at least, two points of intersection with the defect line. Each such pair of points will be associated
with the same loop on the relevant 2-sphere that contracted to create them. This way of forming the defect is different from usual as the points belong to some $S^2$ rather than to an $\mathbb{R}^2$ plane.

The parameter that differentiates between homotopy classes, the winding number, for example, could be, in general, any discrete quantity relevant to the system. One such parameter can be the dimensionality of the system and therefore, one can pursue the subject of whether a defect can happen when one has to find a way to change the dimensionality of the associated by $\varphi$ topological configurations. One can speculate that it may be that during its expansion, the Universe underwent a dimensionality reduction which, however, did not affect the feasibility and density of defects. What it may have affected is the topological attributes of defects. In case, $W = S^w$ and $M = S^m$, then if each different choice of $(w, k, m)$ can be associated with a unique space of defects, can one deduce the kind of dimensional reduction that may have taken place? Could it happen that the defects directed the dimensional evolution of the Universe?

The topological condition, as mentioned already, is not sufficient to ensure the occurrence of defects. The crucial point for that is to find ways for the appropriate $\varphi$ to occur that will realize the non trivial association of homotopy classes between $W$ and $M$ necessary for defect creation. Thus, it talks about the how defects can be generated. One must pursue also the why.

**Short summary and further motivation**

Suppose an effective potential $V$ with a non trivial space of minima $S$. The order parameter configuration, $\varphi$, will have to assign points outside $S$ to points on our manifold $A$, when the loop in $A$ and its associated contour in $S$ do not belong to the same homotopy class. The acquisition of an order parameter that will have that attribute has to do with the actual laws $\varphi$ should be considered to obey. Thus, the topological properties of $S$ do not have to prescribe the
behavior of $\varphi$.

If our system would like to obtain the order parameter with the lowest possible energy always, then a defect will occur when there is a failure in recognizing the potential energetic cost of a certain $\varphi$ association over some part of our medium. Moreover, even if the appreciation of the consequences of some defect happening, takes place, it should come about late enough for any drastic changes to occur that could eliminate the otherwise inevitable defect. Alternatively, if $\varphi$ should correspond to some externally driven energy profile, then the defect will be there to assimilate the available energy and form a stable order parameter configuration over our medium. These are the topics we will be concerned with in the next section.
Chapter 2

The Kibble Mechanism

2.1 How topological defects can be created

The idea we have been discussing so far, was originally put forward in 1976[4] by Kibble. The way we phrased the Kibble argument is important for it draws the attention on the way two manifolds are associated. Usually, one considers that the topology of the manifold of the minima of $V$ is enough to tell us whether defects are, in general, possible. Clearly, this assumes that we have good knowledge of the dimensionality, for example, of our physical space. As we saw, topological defects can never happen in a situation where, for example, $S$ is one dimensional and the domain of $\varphi$ is also one dimensional. Thus, we believe, that, for a safe deduction about the feasibility of topological defects one needs to appreciate both manifolds rather than focus on just one. Nevertheless, the main criterion, though, is whether $\pi_1(S) \neq 1$.

As we saw, it basically provides an answer to the question of which systems can accommodate defects. The basic ingredient being that of a fixed potential $V$ with a non trivial space of minima $S$ and an unchanged medium $A$. Topological methods are useful to the identification of defects when the order parameter is at some equilibrium configuration. The question is whether they could be useful to the creation of a framework for defect generation.
2.1.1 Preliminaries

Since one wants to know whether topological defects could happen in nature, one needs to resolve the issue of whether topological processes have any physical significance or whether they are there to describe the state of our system after all physical processes have completed. If the former case is true, there will be areas of our medium $R^n$ that acquire an order parameter configuration later than others. If all areas where to be assigned a preferred order parameter value simultaneously, then it is hard to think of any other outcome than $\varphi = c$ throughout $R^n$, with $c \in S$, assuming an always continuous order parameter. If the necessary premises for defect creation are to be generated, while the order parameter is kept a continuous function taking values only from $S$, then one needs to find a loop on $R^n$ being associated with a non trivial homotopy class of $\pi_1(S)$. That can happen solely when $\varphi$ does not extend to the whole of $R^n$ since otherwise a discontinuity would have occurred.

Alternatively, $\varphi$ should always be a function of the whole of $R^n$ but some energetic or other constraints would not allow it to evolve to the lowest energy configuration. One can consider that $\varphi$ should assimilate all the energy available to the system. In doing so, it could form the defects as the most stable configurations that can correspond to the energy resources available. Thus, their deformation to lowest energy configurations should require energy in order to get fulfilled. In that situation, topological methods will be there to identify the defects, having nothing to do with their generation.

In the original paper[4], Kibble postulated that indeed topological methods had their physical equivalent and one could consider disconnected domains where the order parameter corresponded to a point on $S$. That idea alludes to a situation where two uncorrelated areas of $R^n$ are associated with the same effective potential $V$ and the order parameter in each one of them is, at equilibrium, a minimal energy solution to the appropriate Euler-Lagrange equations. Given the non trivial nature of $\pi_1(S)$, $S$ must not be a single point. Therefore,
there is the probability that in different domains, the order parameter does not take the same value. That can happen because of some random process which assigns $S$ values on different regions of $R^n$.

For those areas to exist, the theory considered must exhibit short range interactions between the gauge fields. Moreover, the domains should have no intrinsic boundary conditions. Thus, the order parameter within them should randomly take a value from $S$ rather than be subject to boundary constraints. It could be possible that the Universe as a whole may be non-neutral, if it is open and expanding. But then the direction of symmetry breaking would be fixed by the initial conditions rather than by random choice, and would be everywhere the same. Thus no domain structure would arise.

Because domains are most familiar in the context of ferromagnetism it may be well to point out a crucial difference between that case and the Universe. The long-range dipole-dipole interaction between spins ensures that it is energetically favorable for a large ferromagnet to break up into domains with different magnetization directions. This means that the true ground state does not correspond to a spatially uniform order parameter. However, in the models we will be considering, the source of the gauge field always involves the derivative of the order parameter, and vanishes for constant $\varphi$. Thus there is no long range force between differently oriented domains, and the true ground state necessarily has a spatially uniform $\varphi$. The phenomena we will be concerned with are non-equilibrium effects.

It is unclear whether the domains are identified after the order parameter has taken a certain configuration over our medium or whether what we see at the end is the late time result of the domain evolution. If the latter is to be considered, then our medium should not be a continuous space at the beginning. Rather it is easier to think of it as a collection of, initially independent, commensurate domains where the order parameter is allowed to evolve to the relevant lowest energy configuration. Those areas would, then, start expanding so as to cover
all available space. The latter will be our medium and any defects will be just the result of the interaction of the fields between the different areas as they come closer.

Even though the above viewpoint seems to us to be interesting\[28\], we are, though, going to adopt here the usual way forward. Thus, the order parameter is a continuous function throughout our medium and the domains are identified as being the areas where \( \varphi \) takes the relevant lowest energy configuration. Indeed there is a problem here with spontaneity and up to what level within each domain \( \varphi \) takes a random value. We are going to overcome it by assuming that in that framework, randomness has more to do with the actual association of order parameter values with points of our medium, rather than the choice of the order parameter itself.

Whereas the original idea by Kibble\[4\] clearly alluded to a situation where the order parameter configuration over our medium is determined by the random process of applying values to different areas in space, one can, alternatively, consider that the orientation of a certain order parameter is what really is random. At least, the latter is relevant to the whole of our manifold whereas the former is not.

It is clear that randomness is heavily constrained by continuity. Therefore, if the domains are to cover the whole of our medium, then they cannot be commensurate in terms of their random selection of order parameter values from a fixed range. Therefore, randomness and continuity cannot be, at the same time, maintained for the whole smooth manifold of our medium. In our discussion we are not concerned with the actual way a certain order parameter configuration occurs. Instead, we will be interested in the result of any random process and how it evolves.
2.2 Spontaneous symmetry breaking (SSB)

The concept of symmetry in particle physics runs very deep[5]. The idea that physical laws should be invariant under groups of transformations is a powerful one. In the context of quantum field theories, symmetries group particles together and relate their scattering probability amplitudes. The search for a single underlying symmetry behind the interactions of particles and fields is basically what particle physics is all about. That can give a quite strong indication of the importance of symmetry in the current understanding of the fundamental structure of nature.

The symmetries of quantum field theory come in two classes: space-time and internal. The former are those of the space-time through which the field propagates. For example, empty space has the Poincaré group of symmetries, consisting of translations, rotations and Lorentz boosts. Internal symmetries relate fields to one another. To implement these symmetries, a field must have some well-defined transformation properties under the action of the symmetry group. That is, it must form a representation of the group. The labels of different representations are often called quantum numbers. For example, the representations of the Poincaré group are labelled by their mass and their spin. Internal symmetries transform fields into one another, and, usually, there are only finite numbers of fields. The corresponding groups are finite-dimensional and compact. If they are not discrete groups then they are Lie groups. Thus, the study of Lie groups and their representations assumes fundamental importance in particle physics. Generally, a relativistic quantum field theory is specified by the representations of the fields which comprise it (i.e, their masses, spins and internal quantum numbers), which are then assembled into an invariant Lagrangian.

The most powerful type of symmetry is the gauge (or ‘local’) symmetry, where the Lagrangian is invariant under a symmetry transformation which may be different at every space-time point. If the invariance exists only for transfor-
mations which are constant in space and time, the symmetries are distinguished by being called ‘global’ or ‘rigid’. For internal symmetries this requires a spin 1 field, the gauge field, of which the electromagnetic field is the prototypical example. Gravity can also be formulated as a gauge theory of local Lorentz transformations: the gauge field is the gravitational field itself, which has spin 2.

The problem with invoking all these symmetries is how to hide or ‘break’ them. If one proposes a symmetric Lagrangian, as well as a symmetric ground state for the theory, then there is a deep theorem which states that the existence of a gauge symmetry implies that the associated spin 1 particles are massless. Only one such particle is known: the photon. The resolution of the problem is the breaking of the symmetry of the ground state by what is known as the Higgs mechanism. This works by introducing spin 0 fields (denoted $\varphi$) transforming non-trivially under the symmetry group $G$, and constructing for them an energy density which is minimized at some non-zero value $\varphi_0$. The theory then has a ground state which is invariant only under the subgroup of $G$ that leaves $\varphi_0$ unchanged. We say that the symmetry $G$ is broken to $H$.

It is important to keep in mind that the pattern for symmetry breaking is not arbitrary but depends on the structure of the theory in particular the (group) representation content of the scalar field (Kibble 1967, Li 1974). For example, if we have a triplet of real scalar fields $\phi$ instead of the complex doublet, the gauge symmetry $SU(2)$ will be broken down to a residual $U(1)$ gauge symmetry with one massless vector boson remaining. To see that this is the case, start with the scalar potential

$$V(\phi) = -\mu^2 \phi^2 + \lambda (\phi^2)^2$$

Again, minimization of $V(\phi)$ only determined the magnitude

$$| < \phi_0 > | = \frac{\mu}{\sqrt{2\lambda}}$$
We are free to choose the vacuum state so that

\[ | < \phi_0 > | = \frac{1}{\sqrt{2}} \hat{z} \]

with \( \hat{z} \) the 3 dimensional unitary vector in the z direction. The ground state \( \phi \) points in the 4-direction, the symmetry is spontaneously broken. The covariant derivative term

\[ [D_\mu \phi]^2 = [\partial_\mu \phi' - ig(\phi' + < \phi >_0) \times A_\mu]^2 \]

will not contain any term quadratic in \( A_\mu^3 \). Thus the \( A_\mu^3 \) field continues to describe a massless vector boson. We note that this pattern of symmetry breaking is related to the fact that \( < \phi >_0 \) is still invariant under a \( U(1) \) rotation in (1, 2) space.

### 2.2.1 The Higgs mechanism

The Lagrangian which interests us for now is of the form:

\[
L = \frac{1}{2} (\partial_\mu - \frac{iq}{\hbar c} A_\mu) \varphi^* (\partial^\mu + \frac{iq}{\hbar c} A^\mu) \varphi \\
+ \frac{1}{2} \mu^2 (\varphi^* \varphi) - \frac{1}{4} \lambda^2 (\varphi^* \varphi)^2 - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \quad (2.1)
\]

If we put straight away \( \varphi \) as \( \rho e^{i\omega} \) in the Lagrangian, we get the following:

\[
L = \frac{1}{2} (\partial_\mu \rho - i\rho (\partial_\mu \omega + \frac{q}{\hbar c} A_\mu))(\partial^\mu \rho + i\rho (\partial^\mu \omega + \frac{q}{\hbar c} A^\mu)) \\
+ \frac{1}{2} \mu^2 (\rho)^2 - \frac{1}{4} \lambda^2 (\rho)^4 - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \quad (2.2)
\]
Let’s now introduce a new variable $B_\mu$ which is equal to

$$B_\mu = \frac{\hbar c}{q} \partial_\mu \omega + A_\mu$$  \hspace{1cm} (2.3)

Thus, the Lagrangian in terms of this new variable and $\partial_\mu \rho$ is:

$$L = \frac{1}{2} (\partial_\mu \rho \partial^\mu \rho) + \frac{1}{2} \left( \frac{q \rho}{\hbar c} \right)^2 B^\mu B_\mu + \frac{1}{2} \mu^2 \rho^2 - \frac{1}{4} \lambda^2 \rho^4 - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}$$  \hspace{1cm} (2.4)

We can factorize the $\frac{1}{2} \mu^2 \rho^2 - \frac{1}{4} \lambda^2 \rho^4$ term by adding and subtracting $\frac{\mu^4}{4\lambda^2}$ to it and thus we get the form:

$$-\frac{\lambda^2}{4} (\rho - \frac{\mu}{\lambda})^2 (\rho + \frac{\mu}{\lambda})^2 + \frac{\mu^4}{4\lambda^2}$$

We will call

$$\eta = \rho - \frac{\mu}{\lambda}.$$

Thus the Lagrangian takes the form:

$$L = \frac{1}{2} (\partial_\mu \eta \partial^\mu \eta) + \frac{1}{2} \left( \frac{q (\eta + \frac{\mu}{\lambda})}{\hbar c} \right)^2 B^\mu B_\mu$$

$$-\frac{1}{4} \lambda^2 \eta^2 (\eta + \frac{2\mu}{\lambda})^2 - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{\mu^4}{4\lambda^2}$$  \hspace{1cm} (2.5)

The $\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}$ term, one can easily find out in case $\partial_\mu \partial_\nu \omega = \partial_\nu \partial_\mu \omega$, that remains the same in terms of the new variable $B_\mu$ as with $A_\mu$.

Then we see that we get the expected form for $L$, i.e

$$L = \left[ \frac{1}{2} (\partial_\mu \eta \partial^\mu \eta) - \mu^2 \eta^2 \right] + \frac{1}{2} \left( \frac{q \mu}{\hbar c \lambda} \right)^2 B^\mu B_\mu - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}$$

$$-\frac{1}{4} \lambda^2 \eta^4 - \lambda \mu \eta^3 + \frac{1}{2} \left( \frac{q \eta}{\hbar c} \right)^2 B^\mu B_\mu$$

$$+ \frac{\mu^4 q^2}{\lambda \hbar^2 c^2} B^\mu B_\mu + \frac{\mu^4}{4\lambda^2}$$  \hspace{1cm} (2.6)
The end result is that the Higgs technique groups the different variables \((\rho, \partial_\mu \omega, A_\mu)\) into new ones, in such a way that it can reveal a physical meaning for these new variables. \(B_\mu\) was taken to be \(B_\mu = \frac{\hbar}{q} \partial_\mu \omega + A_\mu\) and \(\rho\) was left unaltered so as to be able to consider \(B_\mu\) and \(\rho\) the original fields in a new gauge.

Any gauge transformation of the form \(\varphi \to \varphi \exp(i(-\theta + \alpha))\) with \(\partial_\mu \alpha = 0\) and \(\theta\) the angle of \(\varphi\) in the original gauge, will be good to reproduce the Higgs technique. This means, that \(\varphi\) is taken onto a certain axis in the complex plane. One could wonder whether any other valid choices for new variables exist and even though a more thorough investigation is required, one can see that in case \(\alpha\) is taken to be only time dependent and we choose the Coulomb gauge for \(B_\mu\) we can consider \(\rho \exp(i\alpha)\) to be our relevant scalar and \(B_\mu\) our gauge field.

Obviously, the original Lagrangian (2.1) in terms of \(\varphi\) was symmetric under the \(U(1)\) global gauge transformations. The latter (2.6) is not, for, clearly, if we apply any \(U(1)\) transformation on \(\eta\) alone then \(L\) is not going to stay invariant. Of course, our Lagrangian has not lost its original symmetry for if we transformed \(\eta\) and \(\frac{\rho}{\chi}\) at the same time, we would have preserved the form of \(L\).

As we already said, the symmetry breaking will only be used as a means to check whether a particular theory could be associated with defects. In fact, our relevant order parameter will always be a solution to the equations coming from the original Lagrangian (2.1). Hence, the defect solution will be a \(\varphi\) and not a \(\eta\) configuration where the symmetry breaking explicitly implements itself. We can, of course, assume, that if the order parameter takes a preferred configuration within a certain domain of the total manifold, then it has saturated at some symmetry breaking point. Here, we will consider that the defect will be there to restore the symmetry over all the manifold, if certain criteria are met.
2.2.2 Implementing the Symmetry Breaking

The mechanism described above shows how an appropriate transformation of the original field $\varphi$ and a proper choice of gauge can bring to light different symmetry properties associated with the new scalar field. The field $\varphi$ will be relevant to the disordered phase whereas $\eta$ will correspond to the ordered phase. If symmetry breaking can be a process happening in reality, then the original fields will have to naturally evolve to the ordered ones.

A good description of the transition between the disordered ($\Phi = 0$) and the ordered ($\Phi \neq 0$) phases can be obtained by postulating that the free-energy density for constant fields takes the form

$$f(\Phi) = \zeta + \alpha Tr[\Phi^2] + \beta Tr[\Phi^3] + \gamma Tr[\Phi^2]^2 + \ldots$$  \hspace{1cm} (2.7)

The value of $\Phi$ at the minimum of $f$ depends on the coefficients $\zeta, \alpha - \gamma$, which are temperature and pressure dependent. In particular, the transition occurs when the sign of $\alpha$ changes. The form of the free-energy function can tell us whether the transition is first or second-order. If $\beta \neq 0$, the equilibrium free energy changes discontinuously from one minimum to another, so that we have a first-order transition. Only if $\beta = 0$ is this transition second-order.

In the quantum field theory of scalar fields at non-zero (often called finite) temperature the free energy density for constant fields is known as the finite-temperature effective potential $V_T(\varphi)^1$. This is in principle calculable from the zero-temperature classical potential in a perturbative expansion in powers of $\hbar$, the loop expansion. One finds that broken symmetries in quantum field theories are almost always restored at high enough temperatures.

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1Here it will be simply written as $V(\varphi)$. 

2.3 SSB in Field Theories and Condensed Matter

The ability to draw parallels between the high energy and condensed matter phenomena rests on the similarity of the behavior of the free energy density, especially in the vicinity of the phase transition[8]. In particular, for the second-order phase transitions, potential contribution to the free energy is of the Landau-Ginzburg form:

\[ V(\varphi) = a|\varphi|^2 + \frac{1}{2}\beta|\varphi|^4 \]  

(2.8)

where \( \varphi \) is the order parameter. The coefficients \( a \) and \( \beta \) have a well-prescribed dependence on the relative temperature:

\[ \epsilon = -1 + \frac{T}{T_c} \]  

(2.9)

Thus:

\[ a \simeq a' \epsilon \]  

(2.10)

and

\[ a', \beta = \text{constants} \]

In field theories the form of the effective potential \( V(\varphi) \) is often simply postulated. In the condensed matter context, it is, on the other hand, usually derived in the mean field approximation from the underlying microscopic theory of the system in question. For instance, in the case for superconductors, the so-called Gorkov equations provide a link between the microscopic BCS theory and the Landau-Ginzburg theory. In any case, the parallels between the equilibrium aspects of the phase transitions in the field theories relevant to high-energy
physics and in the effective field theories emerging in the mean field description of condensed matter systems have been appreciated for quite some time.

Symmetry breaking arises when $T < T_c$, that is when the coefficient $a$ above (2.10) becomes negative. Then the global minimum of the effective potential changes from $\varphi = 0$ to the finite absolute value given by

$$\sigma = \sqrt{\frac{|a|}{\beta}}$$ (2.11)

When $\varphi$ is two dimensional, degenerate minima form a circle of radius $\sigma$. The depth of the minimum of $\sigma$ is given by

$$\Delta V = V(0) - V(\sigma) = \frac{a^2}{2\beta}$$ (2.12)

The tension between the long range order which is supposed to set in below the critical temperature and the relatively short range order over which the choice of the broken symmetry vacuum can be communicated, is responsible for the creation of the topological defects. In a finite time, the short range order is determined by the rate at which the phase transition is taking place. The tension between the long range and locality constitutes therefore the primary focus of the dynamics of non equilibrium phase transitions.

### 2.4 Strings in Field Theories

So far we talked about the conditions that need to be fulfilled for a topological defect to occur. First, in order for our theory to support defects we need to have that the space of minima for the relevant effective potential is multiply connected. Then, if defects are to be created then we need to appropriately evolve to a multiply connected space, and that is being done by choosing a proper free-energy density (or effective potential). In our picture, the driving force for this change (which we assume happens smoothly) of $S$ manifolds, will be the lowering temperature. However, we haven’t yet talked about the defects

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themselves. Clearly, if our theory supports them, then they should have some field representation. The latter must be a solution to the relevant equations. Our study so far, might have alluded to the type of defects we would focus on. They are one dimensional objects\(^2\) called, in the context of the early Universe, cosmic strings.

### 2.4.1 Global strings

The simplest theory exhibiting string solutions is that of a complex scalar field \(\varphi(x)\), described by the Lagrangian density

\[
L = \partial_\mu \varphi^* \partial^\mu \varphi - V(\varphi), \quad V = \frac{1}{2} \lambda (|\varphi|^2 - \frac{1}{2} \eta^2)^2
\]  

(2.13)

which has a global \(U(1)\) symmetry, under the transformation \(\varphi \to \varphi e^{i\alpha}\), with \(\alpha\) constant.

The Euler-Lagrange equations that follow from the above Lagrangian density are

\[
[\partial^2 + \lambda (|\varphi|^2 - \frac{1}{2} \eta^2)] \varphi = 0. 
\]  

(2.14)

The ground state, or vacuum, solution is

\[
\varphi = (\eta/\sqrt{2}) \exp(i\alpha_0)
\]

with \(\alpha_0\) constant, which has zero energy. Since the energy is bounded below by zero this solution (unlike \(\varphi = 0\)) is clearly stable. Around any \(\varphi = (\eta/\sqrt{2}) \exp(i\alpha_0)\), the potential \(V\) (2.13) is not invariant under the \(U(1)\) symmetry transformation: the symmetry is said to be broken by the vacuum. The mass \(m_s\) of the scalar particle in the symmetry-breaking vacuum is given by \(m_s^2 = \lambda \eta^2\). There is also a massless particle, the Nambu-Goldstone boson, which is associated with the broken global symmetry. It corresponds to

\(^2\)This assumes that no string exists within another. If that could happen then the dimensionality of our defect configurations could be higher than one.
space-dependent oscillations in the phase of $\varphi$.

Besides the vacuum, there are also static solutions with non-zero energy density. Let us make the following cylindrically symmetric ansatz:

$$\varphi = \frac{\eta}{\sqrt{2}} f(\rho) e^{in\theta}$$  \hspace{1cm} (2.15)

where $\{\rho, \varphi, z\}$ are cylindrical polar coordinates, and $n$ is an integer. The field equations then reduce to a single non-linear ordinary differential equation

$$f'' + \frac{1}{\varepsilon} f' - \frac{n^2}{\varepsilon^2} f - \frac{1}{2} f = 0,$$ \hspace{1cm} (2.16)

where $\varepsilon = m_s \rho$. As $\varepsilon \to 0$, continuity of $\varphi$ requires that $f \to 0$. At infinity, $f \to 1$, so that the field approaches its ground state $|\varphi| = \eta/\sqrt{2}$. Writing $f = 1 - \delta f$, it is not hard to show that at large $\varepsilon$, $\delta f \sim n^2/\varepsilon^2$. The energy density of that configuration is

$$E = \left|\frac{\partial \varphi}{\partial t}\right|^2 + |\nabla \varphi|^2 + V(\varphi).$$  \hspace{1cm} (2.17)

Although $E$ is well localized near the origin, it has a $\varepsilon^{-2}$ tail at large $\varepsilon$, which comes from the angular part of the gradient term. This means that the energy per unit length of this solution is infinite: inside a cylinder of radius $R \gg m_s^{-1}$ it is approximately $\pi n^2 \eta^2 \ln(m_s R)$. These solutions are known as global strings or vortices.

2.4.2 Vortices in Condensed Matter

In view of condensed matter applications, one can write the following Lagrangian

$$L = \frac{1}{2} \partial_{\mu} \varphi^* \partial^{\mu} \varphi - \frac{1}{2} \lambda (|\varphi|^2 - \frac{1}{2} \eta^2)^2$$  \hspace{1cm} (2.18)

in a dimensionless form by adopting the following variables:
\begin{align*}
\varphi & \rightarrow \psi = \frac{\varphi}{\sigma} \quad (2.19) \\
r & \rightarrow R = \frac{r}{\xi} \quad (2.20) \\
t & \rightarrow \tilde{t} = \frac{t}{\tau} \quad (2.21)
\end{align*}

where \( \sigma = \sqrt{\frac{|w|}{\beta}} \) (2.11), while \( \xi \) and \( \tau \) are the correlation length and relaxation time and are given by

\[ \xi = \tau = \frac{1}{\sqrt{a}} \quad (2.22) \]

with \( a = \frac{1}{2} \lambda c^2 \) and \( \beta = \lambda \). We are working in natural units where \( c = 1 \) and \( \hbar = 1 \).

The equation \( \varphi = \frac{n}{\sqrt{2}}f(\rho)e^{im\theta} \) or in the new variables \( \psi = f(\rho)e^{im\theta} \) represents a \textit{cosmic string} or a \textit{vortex} with the original symmetric vacuum locked out along the axis of symmetry by the broken symmetry phase. The dimensionless equation for \( \psi \) is the following:

\[ \nabla^2 \psi = (|\psi|^2 - 1)\psi \quad (2.23) \]

which for

\[ \psi = \rho e^{i\theta} \quad (2.24) \]

becomes

\[ \nabla^2 \rho + 2i \nabla \rho \nabla \theta + i(\nabla^2 \theta)\rho - (\nabla \theta)^2 \rho = \rho^3 - \rho \quad (2.25) \]

If we expand the \( \nabla^2 \psi \) in (2.23) taking under consideration the \( \psi \) representation of (2.24) we get:
\[
\frac{\partial^2 \rho}{\partial r^2} + \frac{1}{r} \frac{\partial \rho}{\partial r} - \left( \frac{\partial \theta}{\partial r} \right)^2 \rho - \frac{1}{r^2} \left( \frac{\partial \phi}{\partial \phi} \right)^2 \rho + 2i \frac{1}{r^2} \frac{\partial \rho}{\partial \phi} \frac{\partial \theta}{\partial \phi} + \frac{\partial^2 \theta}{\partial r^2} + \frac{i \theta}{r^2} \frac{\partial^2 \phi}{\partial r \partial \phi} + \frac{\partial \theta}{r} \frac{\partial \phi}{\partial r}
\]

\[
= \rho^3 - \rho
\]  

(2.26)

When \( \theta = \phi \) with \( \phi \) the polar angle and \( \rho \) a function of \( r \) we get:

\[
\frac{\partial^2 \rho}{\partial r^2} + \frac{1}{r} \frac{\partial \rho}{\partial r} - \frac{1}{r^2} \rho = \rho^3 - \rho
\]

(2.27)

which is a valid equation giving \( \rho \) as a function of only \( r \) as anticipated.

Another straight-forward solution to the equation (2.25) has

\[
\rho(x, y) = f(x)
\]

and

\[
\theta(x, y) = y
\]

If we put these forms of \( \rho \) and \( \theta \) into (2.25) above we get

\[
\frac{d^2 f}{dx^2} = f^3
\]

which is a legitimate result.

That particular order parameter configuration accepts a loop around which it realizes a full \( 2\pi \) rotation in the complex plane. The four corners of the rectangular contour correspond to the points: \((x_0, 0)\) with \( x_0 \) satisfying the equation \( f(x_0) = 0 \), \((x = x_0, y = 2\pi)\), \((x = x_1, y = 2\pi)\) with \( f(x_1) \neq 0 \), and \((x = x_1, y = 0)\). On that particular loop, we force the complex vector to fulfill a hidden rotation by \( 2\pi \) on the one part of it, since \( \rho = 0 \) there, then we increase the magnitude of \( \varphi \) and redo the rotation which now is not hidden and then we
reduce the magnitude of \( \varphi \) to zero to return to our original starting point.

It is common place that one, having in mind the polar cosmic string solution, focuses only at the behavior of the angle \( \theta \) (or \( n \theta \)) around a closed loop in \( R^2 \). However, as the above example confirms, one should also make sure that the magnitude of \( \varphi \) remains unchanged if one is to be sure about the occurrence of a true cosmic string.

However, the above solution has some interesting energetic properties. Calculating the energy density for that configuration we find that it corresponds to

\[
E = \frac{1}{4}(1 + 2\lambda)f^4 + \frac{1}{2}f^2 + \frac{1}{8}\lambda f^4
\]

assuming that the equation for \( f \) drops to \((f')^2 = \frac{1}{2}f^4\) with the arbitrary constant neglected. We see that unless \( f \) diverges somewhere, \( E \) does not have to be singular as in the polar situation where the singularity was the result of the \( \nabla \varphi \) being expressed in polar coordinates. Neglecting any constants we see that indeed \( f \) goes like \( \frac{1}{x} \), however it is not clear that the constants should be zero.

### 2.5 Determining the primary string density

Let us consider a ‘hot big -bang’ Universe and examine what happens as it expands and cools through the transition temperature \( T_c \). In unified models of weak and electromagnetic interactions \( T_c \) is of the order of the square root of the Fermi coupling constant, \( \sqrt{G_F} \), i.e a few hundred \( GeV \). Thus the transition occurs when the Universe is aged between \( 10^{-10} \) and \( 10^{-12} \) seconds and far above nuclear densities. In other models, however, \( T_c \) might be considerably smaller and the transition would occur correspondingly later.

The effective Lagrangian density for our scalar field \( \varphi \) will be:

\[
L = \partial_{\mu} \varphi^* \partial^{\mu} \varphi + a'(1 - \frac{T}{T_c})|\varphi|^2 - \frac{1}{2}\lambda|\varphi|^4
\]

(2.28)

corresponding to the effective potential of
\[ V(\varphi) = \frac{1}{2} \lambda (\varphi^2 - \frac{a'}{\lambda} (1 - \frac{T}{T_c}))^2 \]  

(2.29)

For \( T \) near \( T_c \) there will be large fluctuations in \( \varphi \). Once \( T \) has fallen well below \( T_c \), we may expect \( \varphi \) to have settled down with a non-zero expectation value corresponding to some point on \( S_T \). The latter will be the circle of radius 
\[ |\varphi_0| = \sqrt{\frac{a'}{\lambda} \sqrt{1 - \frac{T}{T_c}}} = \frac{1}{\sqrt{2}} \eta^3. \]

According to the Kibble Mechanism\([4]\) one can anticipate the formation of an initial domain structure with the expectation value of \( \varphi \) varying from region to region in a more or less random way. This is a picture that actually assimilates quite a lot of ideas that are not clear at first sight. Let us explore them.

First one has to wonder whether \( T \) is really an externally imposed condition on our system or whether it should be a result of the natural evolution of our \( \varphi \) configuration. If the latter is the case then we find it difficult to answer the question of why we need to introduce a temperature dependent effective potential in order to simulate a field evolution to some lower energy configuration\(^4\), since we could accommodate such field changes within a single and fixed effective potential. It seems more natural to us to assume that \( T \) is an external driving force which \( \varphi \) has to catch up with. The \( K_B T \) should correspond to a valid estimation of the amount of thermal energy our order parameter is allowed to absorb at the temperature \( T \). Assume \( E_\varphi(T) \) to correspond to the amount of energy the order parameter has assimilated. The idea is that \( E_\varphi(T) + O(K_B T) \) is the total amount of energy the order parameter could correspond to, at the temperature \( T \). The second term will determine the scale of the thermal fluctuations in our order parameter at \( T \). The picture here is that, at each temperature, there is some energy available, say \( E_a(T) \), to be assimilated by the order parameter. At

\(^3\)Adopting our previous notation with respect to the potential: \( \frac{1}{2} \lambda (|\varphi|^2 - \frac{\eta^2}{2})^2 \). There are models when the \( S_T \) is the circle with radius of \( \sqrt{\frac{a'}{\lambda} \sqrt{1 - \frac{T}{T_c}}} \sqrt{1 + \frac{T}{T_c}} \). The latter is the one loop diagram approximation for the effective potential assuming a small coupling constant. However, for low temperature situations the term \( \sqrt{1 + \frac{T}{T_c}} \) can be considered to be equal to 1.

\(^4\)Since \( T \) falls, the energy of \( \varphi \) must drop.
equilibrium, $E_\varphi(T)$ must correspond to that amount of energy.

Moreover, in order for the goal of a lowest energy configuration to be reached, $E_a(T)$ must fall. That is possible so long as the order parameter can adjust itself to an energetically appropriate configuration. As we will see, for temperatures close to the transition temperature $T_c$, $K_B T$ can be enough to take an equilibrium order parameter configuration corresponding to a defect, to a topologically trivial situation.

Suppose that we have a particular temperature dependent order parameter evolution. If the thermal fluctuations are the only processes that could influence the order parameter configuration at any $T$, then our $\varphi$ will be stabilized when $K_B T$ is not enough to induce any significant changes to our function. That is, when the topological attributes of our function are not dependent on the scale of thermal fluctuations anymore. We can say that, at that point, the externally driven phase transition has stopped and the order parameter has saturated at some configuration which will then tend to reach the closest available equilibrium state.

Since the expectation value of the order parameter at each disconnected domain should correspond to a solution of the Euler-Lagrange equations coming from the Lagrangian (2.28), we can postulate that, at equilibrium, $\varphi$ should be a solution to those equations over any simply connected manifold. The disconnected domains should not form such a space. The relevant Euler-Lagrange equations are:

\[
\partial_\mu \partial^\mu \varphi = -\lambda \varphi \varphi^* - \frac{a'}{\lambda} \left(1 - \frac{T}{T_c}\right)
\]

(2.30)

with

\[
|\varphi| = \sqrt{\frac{a'}{\lambda}} \sqrt{1 - \frac{T}{T_c}}
\]

being a direct solution to those equations\(^5\). Due to the non linear nature of

\(^5\)Assuming no relation between temperature and time here
(2.30), a field configuration of

\[ \varphi = \left( \sqrt{\frac{a'}{\lambda}} \sqrt{1 - \frac{T}{T_c}} \right) \sum_j c_j e^{i \theta_j} \]

with \( \theta_j \) a certain number for each distinct \( j \) is not a solution. However, if the domains join up appropriately, then a

\[ \varphi = \left( \sqrt{\frac{a'}{\lambda}} \sqrt{1 - \frac{T}{T_c}} \right) e^{i \theta} \]

(2.31)

with \( \theta \) the polar angle is allowed. This solution corresponds to a homotopy class of \( S_T \) being different from the identity. Thus, it cannot be deformed to a point and therefore, if we are constrained on \( S_T \) for the choice of values of our order parameter, we cannot form a continuous order parameter space over \( R^2 \). Thus, the space the domains create where the order parameter takes values from \( S_T \) alone, should not be simply connected.

According to Kibble[4], the stability of that field configuration (2.31) can be significantly suppressed for temperatures close to \( T_c \) and \( |\varphi| \) close to zero. The idea was that the field of (2.31) could transform itself to a trivial configuration by absorbing thermal energy. It is clear that if we want that \( \varphi \) to drop to a single point on \( S_T \) we have to put energy into the system. Even if one could prove that by introducing a simple homotopy between \( \varphi = \left( \sqrt{\frac{a'}{\lambda}} \sqrt{1 - \frac{T}{T_c}} \right) e^{i \theta} \) and a certain \( \varphi_0 = \left( \sqrt{\frac{a'}{\lambda}} \sqrt{1 - \frac{T}{T_c}} \right) e^{i \theta_0} \) assuming our relevant manifold to be the result of the subtraction of two disks, we will present the original postulation which has an answer ready.

### 2.5.1 The Ginzburg regime

The Ginzburg temperature \( T_G \) is the temperature below which thermal fluctuations of the order parameter on the scale of some correlation length \( \xi_G \) corresponding to a particular energy exchange, become improbable. For temperatures above \( T_G \), our relevant field (order parameter) can absorb thermal
energy and transform itself in a way that could eliminate any arising defect. Kibble postulated that the order parameter will be stable at temperatures below $T_G$. At that stage, therefore, one can safely identify defects in our field configuration.

The relevant thermal energy will be the potential energy density required for this field transformation, denoted as $\Delta V$, multiplied by the magnitude of the fixed domain over which this is taking place, denoted as $r^n$. In view of the potential in (2.29), we find that $\Delta V$ is equal to

$$\Delta V = \frac{(a')}^2}{2\lambda} (1 - \frac{T}{T_c})^2 = \frac{1}{8} \lambda \eta^4$$

(2.32)

with

$$\eta = \sqrt{\frac{2a'}{\lambda}} \sqrt{1 - \frac{T}{T_c}}.$$ 

(2.33)

Ignoring factors of order unity and taking $n = 2$ we find that the required amount of energy should be

$$E_G \sim \Delta V \xi^2 = \frac{(a')}^2}{2\lambda} (1 - \frac{T}{T_c})^2 r^2 = \frac{1}{8} \lambda \eta^4 r^2.$$ 

(2.34)

The Ginzburg condition then says that

$$\frac{1}{2} K_B T_G \sim E_G$$

(2.35)

if thermal fluctuations are to be important to the field configuration. For $T < T_G$,

$$\frac{1}{2} K_B T < E_G$$

(2.36)

making the process just described improbable. However, checking the (2.35) relation is something that requires knowledge about the domain structure, the way the temperature drops and indeed the order of magnitude of the various parameters and coupling constants involved.
Nevertheless, if the principle is correct, and it seems plausible, then, thereafter, fluctuations back to $\varphi = 0$ rapidly become less likely, so that the distinction between normal and ordered phases is well established as is that between ordered phases corresponding to well separated point on $S$. In natural units, we can write $E_G$ as $\frac{1}{2}(\frac{1}{\xi^n})$, for any dimension $n$. If we take that $\Delta V$ remains the same, then $\xi^n = \frac{4}{\lambda^2 n^4}$. Since, though, $\eta^2 = \frac{2a'(1 - \frac{T}{T_c})}{\lambda}$ we find that

$$\xi = \left(\frac{1}{a'}\right)^{2/n}(1 - \frac{T}{T_c})^{-2/n} \quad (2.37)$$

For $n = 4$ we can recover the previously mentioned correlation length (2.22). That correlation length, at $T = T_G$ determines the initial scale of the domains. Beyond this point it continues to fall, but the fluctuations are no longer large enough to disturb the established long-range order.

The assumption we made was that any thermal activations of the order parameter happen within a fixed domain of magnitude $r^n$. However, if we consider that $r \sim \xi(T_G)$ then we can calculate the relevant length $\xi$ at $T = T_G$. Thus,

$$\xi^n(T_G) \sim \frac{\lambda K_B T_G}{a'^2(1 - \frac{T_G}{T_c})^2} \quad (2.38)$$

We see that for $n = 4$ we can recover $\xi(T_G)$ to be $\left(\lambda K_B T_G\right)^{1/4} \frac{1}{\left(<a(T_G)\right)}$ which, for $\lambda \sim \frac{1}{K_B T_G}$, is the correlation length in (2.22) at $T = T_G$. Thus, for $n = 4$ the Ginzburg Temperature can be determined by the parameter $\lambda$. At $n = 3$, however, the equation for $T_G$ is quite more complicated. In particular

$$a'(1 - \frac{T_G}{T_c}) = \lambda^2 K_B^2 T_G^2 \quad (2.39)$$

which gives $T_G$ as a function of $T_c$, $\lambda$ and $a'$.

The definition of the Ginzburg regime has some caveats, since, for instance, the effective potential of (2.29) is only valid for the mean field and not on smaller scales. Thus, there could be additional factors in (2.39) due to the
accounting of more scales. A more rigorous definition arises from the range of temperatures below $T_c$ for which fluctuations are large and consequently where perturbative finite temperature field theory fails to be useful. In order to set up a perturbative scheme at finite temperature from an initial $3 + 1$ dimensional quantum field theory one implements dimensional reduction which is valid provided the temperature is high enough compared to all mass scales. As a result, the Ginzburg condition is being applied on the dimensionally reduced $3D$ field theory which, in the vicinity of the critical point gives (2.39).

**How many strings are there?**

The important realization from the string solution (2.31) is that no matter how far from the defect we are, we can always predict its presence by investigating the field changes on the appropriate contour. The usual criterion is that while the phase of our order parameter winds $n > 0$ times around the circle of minima of the effective potential, on our closed contour, we can be sure of the presence of a *net* number of defects of $n$, in the encircled area. Thus, the number of defects *minus* the number of anti-defects should be equal to $n$ in Abelian media and taking that a single defect of winding number $n$ is unstable. That is the lower bound in the number of defects within a certain contour. However, one would like to know if there is any upper bound to that number. This is a question that has to do with the energy of the system and how many defects, which are energetic configurations, it could in principle accommodate. In simple terms, if one could produce $n$ defects, over disconnected areas, then, if $E$ is the energy of each defect at the time of production, then $nE$ should be less or equal to the energy $E_a$ the total order parameter should aim for. As we will see, $E$ is temperature dependent and could provide the means of calculating the time when the initial defect density has been produced.
Short summary and further motivation

The Kibble Mechanism as originally proposed, alluded to a situation where the occurrence of a certain order parameter was the result of some random process of assigning values from $S$, the space of minima of our relevant effective potential, to regions of our medium $A$. This does not treat all regions of $A$ equivalently. In the context of the early Universe, the arising defects, in case $S$ is a closed one dimensional contour, are called cosmic strings. The latter have their equivalent in a number of different systems. In low temperature physics they are called vortices. The Kibble Mechanism provides a framework for defect generation. The evolution of those configurations can be affected by a number of different processes, one of which is thermal activation.

The topological stability of a certain defect is significantly suppressed close to the transition temperature $T_c$ when $T$ is within the Ginzburg regime. The latter is model dependent and can be fully estimated if one appreciates the microscopics of the system investigated. In the Ginzburg regime a defect fluctuates in and out of existence because there is enough thermal energy to force the trivialization of the topological features of our order parameter.

It is a crucial issue to discover how, during a phase transition, some defect density will survive. The Zurek picture is intended to ensure that defects will remain because of the dynamics of the phase transition imposed on our system. It also provides the means of estimating the initial defect density by considering that the energy of the system should be assimilated by the order parameter. The next section talks about those postulations.
Chapter 3

The Zurek Scenario

In view of the similarity in the topological features between cosmological and condensed matter models, Zurek[10] suggested that if one is to determine the initial defect density, one has to focus on the non equilibrium aspect of the phase transition. The prediction is that the characteristic domain size should be set by the critical slowing down. That is, the order parameter can adjust only on a relaxation time scale which diverges at $T = T_c$. Thus, as the critical temperature is approached from above, at a certain instant $t$, the evolution of the perturbations of the order parameter will become so sluggish that the time spent by the system in the vicinity of $T_c$ will be comparable with the relaxation time scale itself. The correlation length corresponding to such a “freezeout” instant will set the size of the regions over which the same vacuum can be selected. Hence, it will set the resulting density of the topological defects. We shall focus on second-order phase transitions in superfluids and Josephson Junctions which constitute two experimentally attractive alternative implementations of symmetry breaking phase transitions in systems with global gauge symmetries.

There are, therefore, two length scales: the equilibrium correlation length $\xi$ (2.22) that diverges at the transition point $T = T_c$ and the real correlation length, denoted as $\xi_r$, that does not. The expression for $\xi_r$ will be found by the
critical slowing down of the causal horizon written as:

\[ h(t) = 2 \int_0^t dt' c(t') \]  \hspace{2cm} (3.1)

with \( c(t) \) the speed at which the short range perturbations propagate inside the medium. At the “freezeout” instant, \( \bar{t}, \xi_\nu(\bar{t}) \sim h(\bar{t}) \).

### 3.1 Superfluid Helium 4

The simplest physical system for which a gauge theory of defects can be developed is that of superfluid \(^4\)He. It possesses the ordered ground state at zero temperature. Small disturbances introduce soft long-range excitations which are observable as sound waves. Stronger disturbances, for instance, the rotation of the container of the superfluid, lead to the formation of vortex lines which may be viewed as line-like defects. Moreover, there exists a critical temperature at which a phase transition takes place leading to the disappearance of the superfluid state.

Liquid \(^4\)He is an exceptional liquid. It does not become solid when cooled down to absolute zero temperature at atmospheric pressure. Instead, at a temperature \( T_c = 2.18K \), called the \( \lambda \)-point, it undergoes a second order phase transition below which it becomes superfluid. The temperature of 2.18\( K \) is a result that appreciates the Bose condensation that the \(^4\)He atoms undergo and the effects of their strong atomic repulsion.

In the weak coupling approximation, the phenomenon of superfluid phase transition can be studied within a second quantized boson field theory. The quantum fluid is now being described by a Bose condensate wavefunction \( \Psi \):

\[ \Psi = |\Psi| e^{i\theta}. \]

When \( \Psi \) satisfies the Schrödinger equation we get:
\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + \mu \Psi \]  \hspace{1cm} (3.2)

where \( m \) can be taken to be the mass of the \( He^4 \) atom, and \( \mu \) is the chemical potential.

The equation for superfluid flow can be obtained from the Schrödinger equation by computing the rate of change of \( |\Psi|^2 \) where this probability density is now interpreted as the density of the superfluid. Thus, one can show that the velocity \( \vec{u} \) of the superfluid and the phase \( \theta \) of \( \Psi \) are connected with a simple equation:

\[ \vec{u} = \frac{\hbar}{m} \nabla \theta \]  \hspace{1cm} (3.3)

The formal connection between superfluid helium and our field theoretic consideration of symmetry breaking, can be established when it is assumed that the free energy density can be expanded in powers of \( |\Psi|^2 \), identified now as our order parameter, and that it has the Landau-Ginzburg form:

\[ F(\vec{r}) = a |\Psi(\vec{r})|^2 + \frac{\beta}{2} |\Psi(\vec{r})|^4 + \frac{\hbar^2}{2m} |\nabla \Psi(\vec{r})|^2 \]  \hspace{1cm} (3.4)

One can rewrite the Schrödinger equation as:

\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + a\Psi + \beta |\Psi|^2 \Psi \]  \hspace{1cm} (3.5)

This relation is known as the Gross-Pitaevskii equation which differs from the equation of motion one would obtain from the globally gauge invariant Lagrangian, as in the cosmic string case above (2.30), only in that it has a first, rather than second, time derivative. As a result, the Euler-Lagrange equations in the dimensionless form become:

\[ i \dot{\eta} = -\nabla^2 \eta + (|\eta|^2 - 1)\eta \]  \hspace{1cm} (3.6)
with the following equations for the correlation length and relaxation time:

\[ \tau = \frac{\hbar}{|a|} = \frac{\tau_0}{|\epsilon|} \]  (3.7)

and

\[ \xi = \frac{\hbar}{\sqrt{2m|a|}} = \frac{\xi_0}{\sqrt{|\epsilon|}} \]  (3.8)

with \( \sigma^2 = -a/\beta \) and \( \epsilon \) the relative temperature with

\[ \epsilon = 1 - \frac{T}{T_c} \]  (3.9)

The \( \xi_0 \) and \( \tau_0 \) are given by

\[ \tau_0 = \frac{\hbar}{a^\prime} \]  (3.10)

and

\[ \xi_0 = \frac{\hbar}{\sqrt{2ma^\prime}} \]  (3.11)

There have been numerical estimations of the values for \( a^\prime \) and \( \beta [13] \) and in CGS units \( a^\prime \simeq 10^{-16} \text{erg} \) and \( \beta \simeq 4 \times 10^{-40} \text{erg cm}^3 \).

For a bulk superfluid HeII the critical slowing down is exhibited by the behavior of the speed of second sound

\[ s = \frac{\xi}{\tau} = \frac{\xi_0}{\tau_0} \sqrt{|\epsilon|} \]  (3.12)

which limits the speed with which different regions of the emerging superfluid become correlated. In the vicinity of \( T_c \), \( s \) is much smaller than \( \frac{\xi_0}{\tau_0} \), so that the “second sound horizon”, calculated as

\[ \int_0^t s dt' \]  (3.13)

can be quite a bit smaller than the size of our sample. Second sound gives the rate of propagation of perturbations of the density perturbations of the
superfluid. Hence, it also limits the rate at which the order parameter will be able to adjust. In this sense, one is able to reproduce the “causal” nature of the cosmological phase transitions in the superfluid HeII.

Thus, in the course of the temperature quench we have in mind the time scale on which the order parameter can adjust to the new thermodynamic parameters (especially to the new value of $\epsilon$) is becoming very long in the vicinity of the critical temperature. As a result, two regimes can be distinguished: (i) Sufficiently far from $T_c$ the relaxation time scale $\tau$ is much smaller than the time on which the quench is proceeding. In this adiabatic regime, the order parameter will be characterized by an equilibrium configuration with the correlation length $\xi$ determined by the instantaneous value of $\epsilon$. By contrast, very near $T_c$ the equilibrium relaxation time scale will be much larger than the time spent by the system with the corresponding value of $\epsilon$. As a consequence, we can define (ii) the impulse region, in which $\tau$ is so large that the configuration of the order parameter will be in effect immobilized on the timescale of interest.

### 3.1.1 The freeze-out time

The boundary between these two regimes will occur at the freeze-out time $\tilde{t}$. We can compute it by assuming that, in the vicinity of $T_c$, the relative temperature $\epsilon$ is approximately proportional to time:

$$\epsilon = \frac{t}{\tau_Q}$$

(3.14)

The quench timescale $\tau_Q$ can be controlled by the rate at which the temperature is lowered. The critical temperature is reached at $t = 0$.

The freeze-out time is set by the equality:

$$\tau(\tilde{t}) = \tilde{t}$$

(3.15)

which, given the relation $\tau = \tau_0/|\epsilon|$, gives
\[ \tilde{t} = \sqrt{\frac{\tau_0}{\tau Q}} \]  

(3.16)

Consequently, the transition between the adiabatic and impulse regimes occurs twice during the quench at the relative temperature

\[ \tilde{\epsilon} = \epsilon(\tilde{t}) = \sqrt{\frac{\tau_0}{\tau Q}} \]  

(3.17)

Thus, before \(-\tilde{t}\) (3.16), the relaxation time scale must be small, but increasing. For \( |t| \leq \tilde{t} \), \( \tau \) should be too large for the field to be able to follow the changes in the temperature, in the course of the quench, but when \( t \geq \tilde{t} \) the relaxation time is small enough again. The correlation length in (3.8) at \( \tilde{t} \) becomes

\[ \xi(\tilde{t}) = \frac{\xi_0}{\tau_0} \sqrt{\frac{\tau_0}{\tau Q}} \]  

(3.18)

The way the equilibrium correlation length (3.8) and the true correlation length \( \xi_r(t) \), change with time can be schematically seen in the figure below. We are going to come back to the ways one can derive this plot in the next chapter.

Figure 3.1: The equilibrium correlation length of (3.8) against the true correlation length \( \xi_r(t) \) as postulated by Zurek.
Scaling laws

The relation between the correlation length plus the relaxation time and the relative temperature, are in general

\[ \xi(\epsilon) = \frac{\xi_0}{|\epsilon|^\nu} \]  \hspace{1cm} (3.19)

and

\[ \tau(\epsilon) = \frac{\tau_0}{|\epsilon|^{\kappa}} \]  \hspace{1cm} (3.20)

In the mean field theory, \( \nu = 1/2 \), as we assumed already, and \( \kappa = 1 \). However, experiments seem to point to a better accord with the renormalization group prediction for \( \nu = 2/3 \) and \( \kappa = 4/3 \). As we will see in the context of Josephson Tunnel Junctions those numbers are system dependent.

3.1.2 TDGL and Defects

The Gross-Pitaevskii equation (3.5) considered so far did not affect the analysis of the rapid phase transition provided. Rather, the qualitative analysis was general and based on causal rather than any other conditions that could be model related. It might be that in the context of superconductors, for example, equation (3.5) is not a good starting point for a reliable theory. It was nevertheless good to write it down and use it to extract several important pieces of information with the help, though, of the Landau-Ginzburg free energy. However, the TDGL theory is a better model for superconductors and for the superfluid helium, it can provide a valid qualitative guide to the behavior of that system.

In the context of the TDGL thus[30], we assume that the qualitative dynamics are conditioned by the field’s equilibrium free energy, of the form

\[ F(T) = \int d^3x \left( |\nabla \phi|^2 + m^2(T)|\phi|^2 + \lambda|\phi|^4 \right). \]  \hspace{1cm} (3.21)
Intentionally, with the early universe in mind we have written $F(T)$ of (3.21) in the form appropriate for a relativistic quantum field. The coefficient $m^2(T)$ has the interactions with the real particles of the heat bath taken into account, and vanishes at $T = T_c$. We are considering the $\varphi$ field as an open system, in which changes in the external environment lead to changes in the parameter $m^2(T)$, which takes the value $-M^2$ when $T = 0$.

For relativistic QFT the change in temperature that leads to the change in the sign of $m^2$ is most simply understood as a consequence of the system expanding. Thus, in the early universe, once thermalization is possible, a weakly interacting relativistic plasma at temperature $T \gg M$ has an entropy density $s \propto T^3$. As long as thermal equilibrium can be maintained, constant entropy $S$ per comoving volume, $S \propto s \alpha(t)^3$, gives $T \propto \alpha(t)^{-1}$ and falling, for increasing scale factor $\alpha(t)$.

Specifically, at time $t$ the temperature $T(t)$ satisfies

$$tT(t)^2 = m_P \left( \frac{45}{16\pi^4 N^*} \right)^{1/2}$$

(3.22)

where $m_P$ is the Planck mass, and $N^*$ is the number of effective field degrees of freedom. Models that attempt to take inflation into account, however, lead to ‘preheating’ that is not Boltzmannian[14]. Nonetheless, even in such cases it is possible to isolate an effective temperature for long-wavelength modes. This is all that is necessary, but is too sophisticated for the simple scenarios that we shall present here. Even the inclusion of an FRW metric would complicate the issue at this stage, and we assume flat space-time, with decreasing temperature.

Prior to the transition, we assume a uniform temperature $T > T_c$, for which $m^2(T) > 0$. After the transition, $m^2(0) = -M^2 < 0$ enforces the $U(1)$ symmetry-breaking, with field expectation values $\langle \varphi \rangle = \pm \eta$, $\eta^2 = M^2/\lambda$. The Compton wavelength $\xi_0 = M^{-1}$ is the natural distance scale. For the sake of argument we assume mean-field behavior $m^2(T) = M^2(T/T_c - 1)$, whereby the equilibrium correlation length $\xi_{eq}(T) = |m(T)|^{-1} = \xi_0(T/T_c - 1)^{-1/2}$.

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Equally well, after rescaling, $F$ could be the Ginzburg-Landau free energy

$$ F(T) = \int d^3x \left( \frac{\hbar^2}{2m} |\nabla \varphi|^2 + a(T)|\varphi|^2 + \beta|\varphi|^4 \right)$$

(3.23)

for a non-relativistic condensed matter field, in which the chemical potential $a(T) = a'(T/T_c - 1)$ vanishes again at the critical temperature $T_c$. In this case we envisage the change in $a(T)$ as a consequence of an external cooling of the system or a change in the pressure of the system that leads to a change in $T_c$. In either case, we again assume circumstances in which, in a finite time, $T/T_c$ varies from greater than unity to less. The fundamental length scale $\xi_0$ is given from (3.11) as $\xi_0^2 = \hbar^2/(2ma')$ whereby $\xi_{eq}(T) = \xi_0(T/T_c - 1)^{-1/2}$ as before.

3.1.3 Experimental predictions

The end result of the simple causality arguments is that, both for QFT and condensed matter, when the field begins to order itself its correlation length has the form

$$ \xi = \xi_0 \left( \frac{T}{T_0} \right)^\gamma. $$

(3.24)

for appropriate $\gamma$. Whereas, for $^3He$, the critical behavior of (3.8) survives, for $^4He$, $\gamma = 1/3$. On the other hand, for QFT the relevant parameter is $\bar{\xi}T_c$, the ratio of the maximum correlation length to the thermal length $\beta_c = T_c^{-1}$. In equilibrium theory, when the temperature is high enough the theory is essentially three-dimensional, with critical index $\gamma$ different from its four-dimensional mean-field value. At our level of discussion, it is sufficient to keep (four-dimensional) mean-field values.

Correlation lengths in the early universe are not amenable to direct observation. Kibble made a second assumption, that the correlation length (3.24) also sets the scale for the typical minimum intervortex distance at the time that vortices are produced.
That is, the initial vortex density $n_{\text{def}}$ is

$$n_{\text{def}} = O\left(\frac{1}{\xi^2}\right) = \frac{1}{f^2\xi^2} \left(\frac{\tau_0}{\tau_Q}\right)^{2\gamma},$$

(3.25)

for $\gamma = 1/3$, where $f = O(1)$ estimates the fraction of defects per ‘domain’. Equivalently, the length of vortices in a box volume $v$ is $O(n_{\text{def}}v)$.

What is striking and suspect about these predictions is that they are universal. They do not use any information about the strength ($\lambda$ or $\beta$) of the interactions, and hence the magnitude of the order parameter after the transition or, in consequence, the existence of the Ginzburg regime. The latter might affect the defect density for when $n_{\text{def}}$ is eventually measured, at a much later stage, little or no string would have survived unwinding through thermal activation. Moreover, the strength of the interactions can affect the competition between the long and short range order which indeed affects the string density.

One might argue that the initial string density is valid at the time of stabilization, $\tilde{t}$, when the relevant $\xi$ can be calculated from (3.24). From then on, the string density could be subject to influences such as the width of the Ginzburg regime that could indeed affect its final magnitude at the time of measurement. In general, one could wonder up to what level the expression (3.25) reflects all known processes that can affect the string density by $\tilde{t}$. Causality is, of course, valid within and outside the Ginzburg regime and if one could assume that thermal activations can not, on average, affect the string density then (3.25) provides a good estimation.

Thermal activations can be responsible for both unwinding any given string and forming it. As usually envisaged, within the Ginzburg regime, a certain lowest energy field configuration can ‘hop’ over the potential barrier $\Delta V$ (2.32) due to the absorption of some $K_B T$, with $T_c > T > T_G$. That can have either destructive or constructive results. Thus, one can simplistically assume that, on average, the amount of string that formed due to thermal activation corresponds to the amount of string that got destroyed by it. Nevertheless, one needs to
investigate this issue further.

One way to check the validity of (3.25) is by doing experimental tests. It is still, however, not possible to compare the density (3.25) with experiment. What is more amenable to experiment, in principle, is the length distributions of string networks, and their ability to show scaling behavior. This only impinges indirectly on the correlation length of the field, but would have to be commensurate with any density calculations.

It was Zurek[8] who first suggested that, if this relationship (3.25) between defect density and correlation length were true, it could be tested directly in condensed matter systems, particularly in liquid helium.

### 3.2 Experiments in Condensed Matter

#### 3.2.1 Vortices in superfluid helium

Vortex lines in both superfluid $^4He$ and $^3He$ are analogues of global cosmic strings. A crude but effective model is to treat the system as composed of two fluids, the normal fluid and the superfluid, which has zero viscosity. In $^4He$ the bose superfluid is characterized by a complex field $\varphi$, whose squared modulus $|\varphi|^2$ is the superfluid density. The superfluid fraction is unity at absolute zero, falling to zero as the temperature rises to the lambda point at 2.17K. The Landau-Ginzburg theory for $^4He$ has, as its free energy $F(T)$ of (3.23).

The situation is more complicated, but more interesting, for $^3He$, which becomes superfluid at the much lower temperature of 2mK. The reason is that the $^3He$ is a fermion. Thus the mechanism for superfluidity is very different from that of $^4He$. Somewhat as in a BCS superconductor, these fermions form the counterpart to Cooper pairs. However, whereas the (electron) Cooper pairs in a superconductor form a $^1S$ state, the $^3He$ pairs form a $^3P$ state. The order parameter $A_{\alpha\beta}$ is a complex $3 \times 3$ matrix $A_{\alpha\beta}$. There are two distinct superfluid phases, depending on how the $SO(3) \times SO(3) \times U(1)$ symmetry is broken.
the normal fluid is cooled at low pressures, it makes a transition to the \(^3\text{He} - B\) phase, in which \(A_{ai}\) takes the form \(A_{ai} = R_{ai}(\omega)e^{i\phi}\), where \(R\) is a real rotation matrix, corresponding to a rotation through an arbitrary \(\omega\)[15] \(^1\).

The Landau-Ginzburg free energy is, necessarily, more complicated[17], but the effective potential \(V(A_{ai}, T)\) has the diagonal form \(V(A, T) = \alpha(T)|A_{ai}|^2 + O(A^4)\) for small fluctuations, and this is all that we need for the production of vortices at very early times. Beyond that it can be mimicked by (3.23) for our purposes (e.g. see [16]). Thus the Zurek analysis leads to the prediction (3.25), as before, for appropriate \(\gamma\). However, for \(^3\text{He}\) the mean-field approximation is good and the mean-field critical index \(\gamma = 1/4\) is not renormalised, whereas for \(^4\text{He}\) a better value is \(\gamma = 1/3\), as for the naive relativistic theory.

### 3.2.2 Experiments in \(^3\text{He}\).

Although \(^3\text{He}\) is more complicated to work with, the experiments to check (3.25) are cleaner in that, because the nucleus has spin 1/2, even individual vortices can be detected by magnetic resonance. Further, because vortex width is many atomic spacings the Landau-Ginzburg theory is reliable.

So far, experiments have been of two types. In the Helsinki experiment[18] superfluid \(^3\text{He} - B\) in a rotating cryostat is bombarded by slow neutrons. Each neutron entering the chamber releases 760 keV, via the reaction

\[
n + ^3\text{He} \rightarrow p + ^3\text{He} + 760\text{keV}.
\]

The energy goes into the kinetic energy of the proton and triton, and is dissipated by ionization, heating a region of the sample above its transition temperature. The heated region then cools back through the transition temperature, creating vortices. Vortices above a critical size (dependent on the angular velocity of the cryostat) grow and migrate to the center of the apparatus, where

\(^1\)At large distance scales there is a complication in that the small spin-orbit coupling becomes important, to fix \(\omega\) at \(\arccos(-1/4)\), but this will not concern us here.
they are counted by an NMR absorption measurement. Suffice to say that the quench is very fast, with $\tau_Q/\tau_0 = O(10^3)$. The defect density is measured effectively very shortly after the transition takes place and need not be limited by the “infinitely long string”. Agreement with (3.25) is very good, at the level of less than an order of magnitude. This is even though it is now argued[16] that the Helsinki experiment should not show agreement because of the geometry of the heating event.

The second type of experiment has been performed at Grenoble and Lancaster[19]. Rather than count individual vortices, the experiment detects the total energy going into vortex formation. As before, $^3$He is irradiated by neutrons. After each absorption the energy released in the form of quasiparticles is measured. When the medium is very cold, energy dissipation will slow exponentially and vortices can be stabilized by a coherent flow resulting from their motion through the superfluid. If the energy released is found to be less than the total 760 keV, then this missing energy is assumed to have been expended on vortex production. Again, agreement with Zurek’s prediction (3.25) is good.

\subsection*{3.2.3 Experiments in $^4$He.}

The experiments in $^4$He, conducted at Lancaster, follow Zurek’s original suggestion. The idea is to expand a sample of normal fluid helium, in a container with bellows, so that it becomes superfluid at essentially constant temperature. That is, we change $1 - T/T_c$ from negative to positive by reducing the pressure, thereby increasing $T_c$. As the system goes into the superfluid phase a tangle of vortices is formed, because of the random distribution of field phases. The vortices are detected by measuring the attenuation of second sound within the bellows. Second sound scatters off vortices, and its attenuation gives a good measure of vortex density. A mechanical quench is slow, with $\tau_Q$ some tens of milliseconds, and $\tau_Q/\tau_0 = O(10^{10})$. Two experiments have been performed[20, 21].
In the first fair agreement was found with the prediction (3.25), although it was not possible to vary \( \tau_Q \). However, there were potential problems with hydrodynamic effects at the bellows, and at the capillary with which the bellows were filled. A second experiment, designed to minimize these and other problems has failed to see any vortices whatsoever.

Some care is needed. Not only is the Landau-Ginzburg effective theory more suspect for \(^4\text{He}\), but its Ginzburg regime is so wide, at \( O(1K) \) that the transition takes place entirely within it. Thus, the crucial question is whether enough long string would persist at the time of measurement to yield a positive signal. It has been proposed[32] that in order to measure a positive signal in these circumstances the measurements would have to be made sooner, after a much faster quench, or with a higher sensitivity. The validity of this suggestion will be confirmed later on.

**Short summary and further motivation**

The Zurek scenario is built on a mainly causal framework where the order parameter orders on the scale of some correlation length \( \xi_r \), at a rate determined by a relaxation time \( \tau \). Close to the transition point, \( \xi_r \) can be considered as a constant simply because of the large value of \( \tau \) which makes the field change very slowly. The explicit formulation of the parameters involved can be derived from the appropriate theory for our system. The TDGL equation for the free energy dynamics, can verify Zurek’s considerations in a mean field approximation for the potential and a gaussian thermal noise as we will see in the next chapter.

Within a certain fixed potential, the field configuration can be associated with various energy distributions which then should tend to evolve to the lowest possible stable energy state. In this picture, \( \varphi \) follows its own intrinsic laws. In Zurek’s platform, it is clear that one talks about the dynamics of the phase transition which impinge on the order parameter behavior. One can postulate that actually, \( \varphi \) will have to balance the external with the internal driving forces.
of its consequent evolution.

The experiments performed in superfluid $He$ had mixed results which could not provide watertight evidence in favor or against the Zurek scenario. One suggestion was to consider experiments in annular configurations where one would simply be able to measure the phase difference along the annulus $A$, and postulate about the number of defects that could have been created within the area bounded by $A$. The main assumption was that the field inside the annulus itself, orders to some lowest energy configuration. As we will see next, there are various length scales relevant to the defect density created. These are mainly the result of the geometry of the annulus.
Chapter 4

Kibble-Zurek: A Field Theoretic Approach

In thermal equilibrium\[31\] the behavior of simple systems experiencing a continuous phase transition is generic, as manifest in the utility of Landau-Ginzburg theory. The early universe it is thought of as having proceeded through a sequence of phase transitions whose consequences are directly observable, but whose detailed dynamics is unknown. Although it is difficult to measure an order parameter as it changes, many transitions generate topological charge or topological defects which can be detected. Motivated in part by Kibble’s mechanism for the formation of cosmic strings in the early universe, Zurek suggested\[8\] that we measure the density of vortices produced during a pressure quench of liquid $^4$He into its superfluid state, as well as the variance of superflow velocity.

The scenario, as proposed by Zurek, is very simple. It is exemplified by assuming that the dynamics of the transition can be derived from an explicitly time-dependent Landau-Ginzberg free energy of the form

$$ F(t) = \int d^3x \left( \frac{-\hbar^2}{2m} |\nabla \varphi|^2 + a(t)|\varphi|^2 + \frac{1}{4}\beta|\varphi|^4 \right). \quad (4.1) $$

In (4.1) $\varphi = (\varphi_1 + i\varphi_2)/\sqrt{2}$ is the complex order-parameter field, whose magnitude determines the superfluid density. We identify $a(t)$ as an externally driven...
time-dependent chemical potential. In equilibrium at temperature $T$, in a mean field approximation $a(T)$ takes the form $a(T) = a' \epsilon(T_c)$, where $\epsilon = (T/T_c - 1)$ measures the critical temperature $T_c$ relative to $T$. In a pressure quench at approximately constant $T$, $T_c$ will vary with time $t$, and we assume that $\epsilon$ can be written as

$$\epsilon(t) = \epsilon_0 - \frac{t}{\tau_Q} \theta(t) \quad (4.2)$$

for $-\infty < t < \tau_Q (1 + \epsilon_0)$, after which $\epsilon(t) = -1$. $\epsilon_0 = (T/T_c^{in} - 1)$ measures the original critical temperature $T_c^{in}$ against the temperature $T$ at which the quench takes place, and $\tau_Q$ defines the quench rate. The quench begins at time $t = 0$ and the transition from the normal to the superfluid phase begins at time $t = \epsilon_0 \tau_Q$.

With $\xi_0^2 = \hbar^2 / 2ma'$ and $\tau_0 = \hbar / a'$ setting the fundamental distance and time scales, the equilibrium correlation length $\xi(\Delta t)$ and the relaxation time $\tau(\Delta t)$ diverge at the relative time $\Delta t = t - \epsilon_0 \tau_Q = 0$ as

$$\xi(\Delta t) = \xi_0 \left( \frac{\Delta t}{\tau_Q} \right)^{-1/2},$$
$$\tau(\Delta t) = \tau_0 \left( \frac{\Delta t}{\tau_Q} \right)^{-1}. \quad (4.3)$$

As we approach the transition, eventually the relaxation time will be so long that the system will not be able to keep up with the temperature change. We estimate the time $t_Z$ (and the relative time $\bar{t} = \Delta t_Z = t_Z - \epsilon_0 \tau_Q$) at which the change from equilibrium to non-equilibrium behavior occurs by identifying $\tau(\Delta t_Z)$ with $-\Delta t_Z$ i.e. $-\Delta t_Z = \sqrt{t_0 \tau_Q}$. After this time it is assumed that the relaxation time is so long that the field correlation length $\bar{\xi} = \xi(\bar{t}) = \xi(\Delta t_Z) = \xi_0 (\tau_Q / \tau_0)^{1/4}$ is more or less frozen in until the system is again changing slowly, at time $\Delta t \approx +\Delta t_Z$.

The correlation length of the field can only be measured indirectly. One of Zurek’s proposals, as yet unfulfilled, is to measure the variance in the superflow in an annulus after a quench. Since superflow velocity is proportional to the
gradient of the field phase $\theta$, a random walk in phase would suggest that the measurable $(\Delta \theta)^2$ along a perimeter of length $L$ has the form

$$(\Delta \theta)^2 = O\left(\frac{L}{\xi_{var}}\right), \quad (4.4)$$

where $\xi_{var}$ measures the effective phase-winding length. If, as Zurek does, we assume that $\xi_{var} \approx \bar{\xi}$, then $(\Delta \theta)^2$ is large enough to be observed.

A more accessible experiment is to measure the density of vortices at their formation. If the initial density of vortices, the defects of $^4He$, is $n_{def}$, with separation $\xi_{def}$, then

$$n_{def} = O\left(\frac{1}{\xi_{def}^2}\right). \quad (4.5)$$

Zurek makes the assumption that $\xi_{def} \approx \bar{\xi}$ whereby

$$n_{def} = O\left(\frac{1}{\xi^2}\right) = O\left(\frac{1}{\xi_0^2} \sqrt{\frac{\tau_Q}{\tau_0}}\right). \quad (4.6)$$

Since $\xi_0$ also measures cold vortex thickness, $\tau_Q \gg \tau_0$ corresponds to a measurably large number of widely separated vortices.

### 4.1 Reproducing the Zurek behavior for $\xi_r(t)$. 

The primary ingredient for all those predictions is the qualitative picture of the freezing in of the correlation length as seen in figure (3.1). Since all equations of motion have causality built into them we should be able to confirm the first predictions of Kibble and Zurek explicitly, as we shall now see.

#### 4.1.1 Condensed matter: the TDLG equation

We assume that, for the condensed matter systems of interest to us, the dynamics of the transition can be derived from the explicitly time-dependent Landau-
Ginzburg free energy

\[ F(t) = \int d^3x \left( \frac{\hbar^2}{2m} (\nabla \varphi_a)^2 + a(t) \varphi_a^2 + \frac{1}{4} \beta (\varphi_a^2)^2 \right). \] (4.7)

in which we substitute \( T(t) \) for \( T \) directly in (3.23). In (4.7) \( \varphi = (\varphi_1 + i \varphi_2)/\sqrt{2} \) \((a = 1, 2)\) is the complex order-parameter field, whose magnitude determines the superfluid density. As before, in a mean field approximation, the chemical potential \( a(T) \) takes the form \( a(T) = a' \epsilon(t) \), where \( \epsilon = (T/T_c - 1) \). In a quench in which \( T_c \) or \( T \) changes it is convenient to shift the origin in time, to write \( \epsilon \) as

\[ \epsilon(t) = \epsilon_0 - \frac{t}{\tau_Q} \theta(t) \] (4.8)

for \(-\infty < t < \tau_Q(1 + \epsilon_0)\), after which \( \epsilon(t) = -1 \). \( \epsilon_0 = -1 + T_0/T_c \) measures the original temperature \( T_0 \) and \( \tau_Q \) defines the quench rate. The quench begins at time \( t = 0 \) but the transition from the normal to the superfluid phase only begins at time \( t_0 = \epsilon_0 \tau_Q \). When it is convenient to measure time from the onset of the transition we use the notation \( \Delta t = t - t_0 \).

Motivated by Zurek’s later numerical\[10\] simulations, we adopt the time-dependent Landau-Ginzburg (TDLG) equation for \( F \),

\[ \frac{1}{\Gamma} \frac{\partial \varphi_a}{\partial t} = - \frac{\delta F}{\delta \varphi_a} + \eta_a, \] (4.9)

where \( \eta_a \) is Gaussian thermal noise, satisfying

\[ \langle \eta_a(\mathbf{x}, t) \eta_b(\mathbf{y}, t') \rangle = 2 \delta_{ab} \Gamma(t) \delta(t - t'). \] (4.10)

This is a crude approximation for \(^4\text{He}\), and a simplified form of a realistic description of \(^3\text{He}\) but it is not a useful description of QFT, as it stands.

It is relatively simple to determine the validity of Zurek’s argument since it assumes that freezing in of field fluctuations occurs just before symmetry breaking begins. At that time the effective potential \( V(\varphi, T) \) is still roughly
quadratic and we can see[31] that, for the relevant time-interval \(-\tilde{t} \leq \Delta t \leq \tilde{t}\) the self-interaction term can be neglected ($\beta = 0$). The $\Delta t = t - \epsilon_0 \tau_Q$.

In space, time and temperature units in which $\xi_0 = \tau_0 = k_B = 1$, the equation of motion for $\varphi$ then becomes

$$\dot{\varphi}_a(x, t) = -[-\nabla^2 + \epsilon(t)] \varphi_a(x, t) + \tilde{\eta}_a(x, t).$$

(4.11)

where $\tilde{\eta}$ is the renormalized noise. The solution of the ‘free’-field linear equation is straightforward, giving a Gaussian equal-time correlation function

$$\langle \varphi_a(r, t) \varphi_b(0, t) \rangle = \delta_{ab} G(r, t) = \int \delta^3 k e^{i k r} P(k, t).$$

(4.12)

in which the power spectrum $P(k, t)$ has a representation in terms of the Schwinger proper-time $\tau$ as

$$P(k, t) = \int_0^\infty d\tau \tilde{T}(t - \tau / 2) e^{-\tau k^2} e^{-\int_0^\tau ds \epsilon(t-s/2)},$$

(4.13)

where $\tilde{T}$ is the renormalized temperature. In turn, we can invoke Schwinger’s proper time method, which is based on Euler’s form

$$\frac{1}{a^2} = \frac{1}{\Gamma(z)} \int_0^\infty \frac{d\tau}{\tau} \tau^z e^{-\tau q}$$

to write[29]

$$G(r, t) = \int_0^\infty d\tau \tilde{T}(t - \tau / 2) \left( \frac{1}{4\pi\tau} \right)^{3/2} e^{-r^2 / 4\tau} e^{-\int_0^\tau ds \epsilon(t-s/2)}.$$

(4.14)

For constant $\epsilon$, (i.e $\epsilon = \epsilon_0$), as happens at early times, on rescaling in (4.14) we recover the usual Yukawa correlator

$$G(r, t) = \frac{T_0}{4\pi r} e^{-r / \xi_0},$$

(4.15)
where \( T_0 = T_c(1 + \epsilon_0) \) is the initial temperature and \( \xi_0 \) the appropriate length scale for the correlation length.

To evaluate the integral we find that we need to calculate the following quantity

\[
\int_0^{2t} d\tau \tilde{T}(t - \tau/2)(\frac{1}{4\pi \tau})^{3/2} e^{f_1(\tau)}
\]

with

\[
f_1(\tau) = -\frac{r^2}{4\tau} - \tau(\epsilon_0 - \frac{t}{\tau_Q}) - \frac{\tau^2}{4\tau_Q}
\]

We can perform a saddle point approximation of the integral keeping the first two terms in the expansion of \( f_1(\tau) \).

We find that the correlation function is dominated by its smaller-\( r \) behavior at early times. At time \( t_0 = \epsilon_0 \tau_Q \), when the transition begins, provided the quench is not too fast,

\[
G(r, t_0) \approx \frac{T_c}{4\pi r} e^{-b(r/\bar{\xi})^{4/3}}, \tag{4.16}
\]

with \( b = O(1) \). The expression for \( \bar{\xi} \) is

\[
\bar{\xi} = \xi_0(\frac{T_Q}{T_0})^{1/4}
\]

We will take that \( \bar{\xi} = \xi(\bar{t}) \) confirming Zurek’s result. Zurek’s prediction is robust, since further calculation shows that \( \xi(\bar{t}) \) does not vary strongly in the interval \(-\bar{t} \leq \Delta t \leq \bar{t}\), where \( \Delta t = t - t_0 \) and we can identify \( \bar{\xi} \) with \( \xi(\bar{t}) \).

The formula in (4.14) gives us a way to check the behavior for the correlation length over times when the self-interaction term can be neglected and the gaussian approximation holds. The plot (3.1) was produced by means of (4.14) and cannot be taken at face value when \( t > \bar{t} + \epsilon_0 \tau_Q \). However, for \( t \rightarrow (\bar{t} + \epsilon_0 \tau_Q)^- \) one can postulate that (4.14) can indicate a valid behavior for the correlation length.

As one can see, the function \( f_1(\tau) \) cannot be approximated to \(-r^2/(4\tau) - \tau/(4\tau_Q)\) as before. If the term \( \tau(\epsilon_0 - t/\tau_Q) \) is included, we find that for \( t < \tau_Q\epsilon_0 \)
the saddle point calculation gives

\[ \tau_s^3 + 3a(t)\tau_s^2 - \frac{r^2\tau_Q}{2} = 0 \]  

(4.17)

with

\[ a(t) = \frac{2}{3}(\epsilon_0\tau_Q - t) \rightarrow \frac{2}{3}\epsilon_0\tau_Q \]

Taking that \( \epsilon_0\tau_Q \) is very large, one can approximate (4.17) to \( \tau_s = \frac{r}{2\sqrt{\epsilon_0 - t/\tau_Q}} \)

which gives

\[ f_1(\tau_s) \rightarrow -r\sqrt{\epsilon_0 - t/\tau_Q} \]

Thus, the saddle point approximation will yield a term of the form

\[ \exp\left(-r/\left(\frac{\xi_0}{\sqrt{\epsilon_0 - t/\tau_Q}}\right)\right)\]

which indicates a correlation length of exactly the expected form

\[ \xi = \xi_0\frac{1}{\sqrt{\epsilon_0 - t/\tau_Q}} \]

for \( t << \epsilon_0\tau_Q \).

For times \( t >> \bar{t} + \epsilon_0\tau_Q \), as we said, one has to be very careful with the gaussian approximation for the field and the saddle point calculation for the integral. However, for times close to \( \bar{t} + \epsilon_0\tau_Q \) one can postulate that the behavior of \( \xi \) will be more or less commensurate with (4.14), due to continuity. Then, (4.17) will become

\[ \tau_s^3 + 3a(t)\tau_s^2 + 3a^2(t)\tau_s + a(t)^3 = \frac{r^2\tau_Q}{2} + 3a^2(t)\tau_s + a^3(t) \]

which can be approximated to give \( \tau_s + a(t) = (\frac{r^2\tau_Q}{2})^{1/3} \). Thus, \( \tau_s = (\frac{r^2\tau_Q}{2})^{1/3} + |a(t)| \) and

\[ f_1(\tau) \propto -c_1(\frac{\tau}{\xi_1})^{4/3} - c_2(\frac{\tau}{\xi_2})^{2/3} \]
with
\[ \bar{\xi}_1 = \xi_0 \left( \frac{\tau_0}{\tau_0} \right)^{1/4} \]
and
\[ \bar{\xi}_2 = \xi_0 \left( \frac{\tau_0}{\tau_Q} \right)^{1/2} \]
with \(|a(t)|\) increases with \(t\). Assuming that \(c_1 \sim c_2 \sim O(1)\) we find that one can define an effective correlation length of the form
\[ \bar{\xi} = \frac{\bar{\xi}_1^2}{\bar{\xi}_2} \]
which increases as time goes by.

Even though this can be rather ambiguous for \(t > \tau_Q(1 + \epsilon_0)\) one can postulate that for \(\tau_Q \epsilon_0 << t < \tau_Q(1 + \epsilon_0)\) the correlation length does experience the increase that was seen numerically in the plot (3.1) earlier.

### 4.2 Annular Geometry

Due to the ambiguity in the experimental results especially about \(^4He\), Zurek suggested\(^32\) that superflow in \(^4He\) is still measurable, provided some parameters, determining important measurable quantities, are adjusted. We are going to show that this statement is valid, in an annular geometry.

As we saw, a concrete realization of how the freezing sets in is provided by the time-dependent Landau-Ginzburg (TDLG) equation for \(F\) of (3.23)\(^10\),

\[ \frac{1}{\Gamma} \frac{\partial \varphi_a}{\partial t} = -\frac{\delta F}{\delta \varphi_a} + \eta_a, \]  
(4.18)

for \(\varphi = (\varphi_1 + i\varphi_2)/\sqrt{2}\), where \(\eta_a\) is Gaussian noise. Since the field fluctuations are approximately Gaussian because in the relevant time interval the self interaction term can be neglected, the field phases \(e^{i\theta(r)}\), where \(\varphi(r) = |\varphi(r)|e^{i\theta(r)}\), are then correlated on the same scale as the fields. We will assume that the
time here measures the time difference from the moment when the transition takes place \((t = \Delta t\) with \(\Delta t\) introduced before).

Consider a closed path in the bulk superfluid with circumference \(C \gg \xi(t)\). Naively, the number of ‘regions’ through which this path passes in which the phase is correlated is \(N = O(C/\xi(t))\). Assuming an independent choice of phase in each ‘region’, the r.m.s phase difference along the path is

\[
\Delta \theta_C \approx \sqrt{N} = O(\sqrt{C/\xi(t)}). \tag{4.19}
\]

If we now consider a quench in an annular container of similar circumference \(C\) of superfluid \(^4\)He and radius \(l \ll C\), Zurek suggested that the phase locked in is also given by (4.19), with \(\bar{\xi}\) of (3.24) with \(\gamma = 1/3\) the mean field critical index.

Since the phase gradient is directly proportional to the superflow velocity we expect a flow after the quench with r.m.s velocity

\[
\Delta v = O\left(\frac{\hbar}{m} \sqrt{\frac{1}{C \bar{\xi}}} \right). \tag{4.20}
\]

provided \(l = O(\bar{\xi})\). Although in bulk fluid this superflow will disperse, if it is constrained to a narrow annulus it should persist, and although not large is measurable.

In addition to this experiment, Zurek also suggested that the same correlation length \(\bar{\xi}\) should characterize the separation of vortices in a quench. As shown in[31] that consideration is too simple. Causality arguments are not enough, and whether vortices form on this scale is also determined by the thermal activation of the Ginzburg regime, in which all \(^4\)He experiments take place. Experimentally, this seems to be the case[21]. Our aim is to see whether thermal fluctuations interfere with the prediction (4.20), for which experiments have yet to be performed.

Again consider a circular path in the bulk fluid (in the 1-2 plane), circum-
ference $C$, the boundary of a surface $S$. For given field configurations $\varphi_a(\mathbf{x})$ the phase change $\theta_C$ along the path can be expressed as the surface integral

$$\theta_C = 2\pi \int_{\mathbf{x} \in S} d^2x \, \rho(\mathbf{x}), \quad (4.21)$$

where the topological density $\rho(\mathbf{x})$ is given by

$$\rho(\mathbf{x}) = \delta^3[\varphi(\mathbf{x})] \epsilon_{jkl} \partial_j \varphi_1(\mathbf{x}) \partial_k \varphi_2(\mathbf{x}), \quad i, j = 1, 2 \quad (4.22)$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, otherwise zero.

The ensemble average $\langle \rho(\mathbf{x}) \rangle_t$ is taken to be zero at all times $t$, guaranteed by taking $\langle \varphi_a(\mathbf{x}) \rangle_t = 0 = \langle \varphi_a(\mathbf{x}) \partial_j \varphi_b(\mathbf{x}) \rangle_t$. That is, we quench from an initial state with no rotation. For the Gaussian fluctuations that are relevant for the times of interest\cite{31, 33}, all correlations are given in terms of the diagonal \textit{equal-time} correlation function $G(r, t)$, defined by

$$\langle \varphi_a(\mathbf{x}) \varphi_b(\mathbf{0}) \rangle_t = \delta_{ab} G(r, t) \quad r = |\mathbf{x}|. \quad (4.23)$$

The correlation length $\xi(t)$ is defined by $G(r, t) = o(e^{-r/\xi(t)})$, for large $r > \xi(t)$. The TDLG does not lead to simple exponential behavior, but there is no difficulty in defining $\xi(t)$ in practice\cite{31, 33}.

The variance in the phase change around $C$, $\Delta \theta_C$ is determined from

$$\langle \Delta \theta_C \rangle^2 = 4\pi^2 \int_{\mathbf{x} \in S} d^2x \int_{y \in S} d^2y \langle \rho(\mathbf{x}) \rho(\mathbf{y}) \rangle_t. \quad (4.24)$$

The properties of densities for Gaussian fields have been studied in detail\cite{34, 35}. Define $f(r, t)$ by

$$f(r, t) = G(r, t) / G(0, t)$$
On using the conservation of charge

\[ \int d^2x \langle \rho(x)\rho(0) \rangle_t = 0 \quad (4.25) \]

it is not difficult to show, from the results of\[34, 35\], that \( \Delta \theta_C \) satisfies

\[ (\Delta \theta_C)^2 = -\int_{x \not\in S} d^2x \int_{y \in S} d^2y \ C(|x - y|, t), \quad (4.26) \]

where \( x \) and \( y \) are in the plane of \( S \), and

\[ C(r, t) = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{f^2(r, t)}{1 - f^2(r, t)} \right). \quad (4.27) \]

Since \( G(r, t) \) is short-ranged \( C(r, t) \) is short-ranged also. With \( x \) outside \( S \), and \( y \) inside \( S \), all the contribution to \( (\Delta \theta_C)^2 \) comes from the vicinity of the boundary of \( S \), rather than the whole area. That is, if we removed all fluid except for a strip from the neighborhood of the contour \( C \) we would still have the same result. This supports the assertion by Zurek that the correlation length for phase variation in bulk fluid is also appropriate for annular flow. The purpose of the annulus (more exactly, a circular capillary of circumference \( C \) with radius \( l \ll C \)) is to stop this flow dissipating into the bulk fluid.

That might seem in contrast to our earlier assertions about the nature of topological defects. Indeed, the above consideration alludes to a phase difference of \( 2\pi n \), with \( n \neq 0 \), along the annulus, to correspond to a defect. In principle, we believe, that is an overestimation. A topological defect has to have some assimilated energy which, though, a phase difference of \( 2\pi n \), with \( n \neq 0 \), along an essentially one dimensional annulus does not guarantee. However, for the purposes of identifying the proper correlation length, one can assume that \( \xi \) represents the area with the energy concentration, which, if removed, will not affect the field configuration that produced it. In general, the field configuration over \( R^n - C \) will be, essentially, unaffected by the presence or not of \( C \) itself.
Thus, the same correlation length for $R^n$ can be applicable to $R^n - C$. The $C$ here is the area of defects talked about in the first chapter and should not be confused with the circumference of the annulus.

Suppose now that the circumference satisfies the relation $C \gg \xi(t)$. Then, if we take the width $2l$ of the strip around the contour to be larger than the correlation length of $C(r, t)$, (4.26) can be written as

$$(\Delta \theta_C)^2 \approx -2C \int_0^\infty dr r^2 C(r, t). \tag{4.28}$$

The linear dependence on $C$ is purely a result of Gaussian fluctuations.

From (4.19) and (4.26), (4.28) since $(\Delta \theta_C)^2 \sim (\xi)^2 = 2C \int_0^\infty dr r \frac{\partial}{\partial r} \left( \frac{f^2(r, t)}{1 - f^2(r, t)} \right)$ we can identify a correlation length $\xi$ as

$$\frac{1}{\xi} = 2 \int_0^\infty dr r \frac{\partial}{\partial r} \left( \frac{f^2(r, t)}{1 - f^2(r, t)} \right). \tag{4.29}$$

We can separate the above by parts to arrive at the expression for $\xi$

$$\frac{1}{\xi(t)} = 2 \int_0^\infty dr \frac{f^2(r, t)}{1 - f^2(r, t)}. \tag{4.29}$$

Insofar as we can identify the bulk correlation with the annular correlation, instead of (4.20), we have

$$\Delta v = h - \frac{1}{m} \sqrt{\frac{1}{C \xi_\text{s}(t)}}. \tag{4.30}$$

The step length $\xi_\text{s}(t)$ is given by (4.29) where $\xi = \xi_\text{s}$. There are two important differences between (4.30) and (4.20). The first is in the choice of time for which $\Delta v$ of (4.30) is to be evaluated. In (4.20) the time is the time $-\bar{t}$ of freezing in of the field correlation. Since $\xi(t)$, the Zurek correlation length calculated in the first section, does not change much in the interval $-\bar{t} < t < \bar{t}$ we can as well take $t = 0$. We shall argue below that for (4.30) a more appropriate time is the spinodal time $t_\text{sp}$ at which the transition has completed itself in the sense that
the fields have begun to populate the ground states.

Secondly, a priori there is no reason to identify $\xi_s(t_{sp})$ with either $\tilde{\xi} = \xi(\tilde{t})$. In particular, because $\tilde{\xi}$ in (4.20) is defined from the large-distance behavior of $G(r, t)$, and thereby on the position of the nearest singularity of $G(k, t)$ in the $k$-plane, it does not depend on the scale at which we observe the fluid. This is not the case for $\xi_s(t)$ which, from (4.29), explores all distance scales. Because of the fractal nature of the short wavelength fluctuations, $\xi_s(t)$ will depend on how many are included, i.e. the scale at which we look. If we quench in an annular capillary of radius $l$ much smaller than its circumference, we are, essentially, coarsegraining to that scale. That is, the observed variance in the flux along the annulus is $\pi l^2 \Delta v$ for $\Delta v$ averaged on a scale $l$. We make the approximation that that is the major effect of quenching in an annulus. This cannot be wholly true, but it is plausible if the annulus is not too narrow for boundary effects to be important.

Provisionally we introduce a coarsegraining by hand, modifying $G(r, t)$ by damping short wavelengths $O(l)$ as

$$G(r; t; l) = \int \delta^3 k \, e^{i k \cdot r} G(k, t) \, e^{-k^2 l^2}. \quad (4.31)$$

with

$$G(k, t) = \int_{-\infty}^{\tau_0 + \Delta t} dt' \, \exp \left[ -2 \int_{t'}^{\tau_0 + \Delta t} dt'' [k^2 + \epsilon(t'')] \right]$$

We shall denote the value of $\xi_s$ obtained from (4.31) as $\xi_s(t; l)$. It permits an expansion in terms of the moments of $G(k, t) \, e^{-k^2 l^2}$,

$$G_n(t; l) = \int_0^\infty dk \, k^{2n} \, G(k, t) \, e^{-k^2 l^2}. \quad (4.32)$$

For small $r$ it follows that $f^2(r, t; l)/(1 - f^2(r, t; l))$

$$= \frac{G_2}{3G_1} \left[ 1 - \left( \frac{3G_3}{20G_2} - \frac{G_2}{12G_1} \right) r^2 + O(r^4) \right]. \quad (4.33)$$
Although, for large $r$, $f'(r,t;l)^2 = o(e^{-2r/\xi(t)})$, we find that the bulk of the integral (4.29) lies in the forward peak, and that a good upper bound for $\xi_s$ is given by just integrating the quadratic term, whence

$$\frac{1}{\xi_s(t;l)} \geq \frac{1}{\xi_s^{\min}(t;l)} = \frac{4G_2}{9G_1} \left( \frac{3G_3}{20G_2} - \frac{G_2}{12G_1} \right)^{-1/2}, \quad (4.34)$$

with the equality slightly overestimated.

The above expression comes from the minimization of the expression resulting from the integration

$$\frac{2G_2}{2G_1} w - \frac{2G_2}{9G_1} \left( \frac{3G_3}{20G_2} - \frac{G_2}{12G_1} \right) w^3 \quad (4.35)$$

which happens at $w_r \sim \left( \frac{3G_3}{20G_2} - \frac{G_2}{12G_1} \right)^{-1/2}$. The relation for $\xi_s^{\min}$ comes after calculating (4.35) at $w_r$.

In units of $\xi_0$ and $\tau_0$ we have, in the linear regime[31],

$$G_n(t;l) \approx \frac{I_n}{2^{n+1/2}} \xi^{(t/l)^2} \int_0^\infty dt' \frac{e^{-(t'-t)^2/\ell^2}}{[t' + \ell^2/2]^{n+1/2}} \frac{T(t')}{T_c}. \quad (4.36)$$

where $I_n = \int_0 dkk^{2n} e^{-k^2}$. The presence of the $T(t')/T_c$ term is a reminder that the strength of the noise $\eta$ is proportional to temperature. However, for the time scales $O(\ell) \ll \tau_Q$ of interest to us this ratio remains near to unity and we ignore it. For small relative times the integrand gets a large contribution from the ultraviolet cutoff dependent lower endpoint, increasing as $n$ increases.

If we return to the Landau-Ginzburg equation (4.18) we find that $\langle |\phi|^2 \rangle_t \ll \alpha_0/\beta$ in the interval $-\bar{t} \leq t \leq \bar{t}$. Although the field has frozen in, the fluctuations have amplitudes that are more or less uniform across all wavelengths. As a result, what we see depends totally on the scale at which we look. Specifically, from (4.36) $\xi_s^{\min}(0;l) = O(\ell)$, as shown in the lowest curve of figure (4.1).

If, as suggested by Zurek, we take $l = O(\xi)$ we recover (4.20) qualitatively, although a wider bore would give a correspondingly smaller flow. However, this
is not the time at which to look for superflow since, although the field correlation length $\xi(t)$ may have frozen in by $t = 0$, the symmetry breaking has not begun.

Assuming the linearized\textsuperscript{31} (4.18) for small times $t > 0$ we see that, as the unfreezing occurs, long wavelength modes with $k^2 < t/\tau_Q$ grow exponentially and soon begin to dominate the correlation functions. How long a time we have depends on the self-coupling $\beta$ which, through $G_1$, sets the shortest time scale. This is because, at the absolute latest, $G_1$ must stop its exponential growth at $t = t_{sp}$, when $(|\phi|^2)_{t_{sp}}$ satisfies $(|\phi|^2)_{t_{sp}} = \alpha_0/\beta$. We further suppose that the effect of the backreaction that stops the growth initially freezes in any structure. In figure (4.1) we also show $\xi_{s\text{ min}}^\text{min}(t; l)$ for $t = 3l$ and $t = 4l$, increasing as $t$ increases.

For $^4\text{He}$ with quenches of milliseconds the field magnitude has grown to its equilibrium value before the scale-dependence has stopped\textsuperscript{31}. For vortex formation, for which the scale is $O(\xi_0)$, the thickness of a vortex, the dependence of the density on scale makes the interpretation of observations problematic. This is not the same here. That the incoherent $\xi_s$ depends on radius $l$ is immaterial. The end result is that

$$\Delta v = \frac{\hbar}{m} \sqrt{\frac{1}{C\xi_s(t_{sp}; l)}}. \quad (4.37)$$

We saw that the expression (4.34) for $\xi_s$ assumed that $2l$ is larger than

$$\xi_{eff}(t; l) = \left(\frac{3G_3}{20G_2} - \frac{G_2}{12G_1}\right)^{-1/2}. \quad (4.38)$$

Otherwise the correlations in the bulk fluid from which we want to extract annular behavior are of longer range than the annulus thickness. Numerically, we find that $\xi_{eff}(0, l) = 2l$ very accurately at $t = 0$, but that $\xi_{eff}(t, l) \geq 2l$ for all $t > 0$. A crude way to accommodate this is to cut off the integral (4.28).

This gives an expression of the form

$$\frac{2G_2}{3G_1}((w - l) - \frac{1}{3}\frac{3G_3}{20G_2} - \frac{G_2}{12G_1})(w^3 - l^3))$$
which is extremized at $w = w_r$ as before.

With a little effort, we see that the effect of this is that $\xi_s^{\text{min}}(t_{sp}, l)$ of (4.34) is replaced by

$$
\xi_s^{\text{max}}(t_{sp}, l) = \xi_s^{\text{min}}(t_{sp}, l)[1 - (1 - 4l^2/\xi_{eff}(t_{sp}, l)^2)^{3/2}]^{-1},
$$

(4.39)
greater than $\xi_s^{\text{min}}(t_{sp}, l)$ and thereby reducing the flow velocity for narrower annuli. These are the dashed curves in figure (4.1). The effect is largest for small radii $l \leq \bar{\xi}$, for which the approximation of trying to read the behavior of annular flow from bulk behavior is most suspect. A more realistic approach for such narrow capillaries is to treat the system as one-dimensional[8]. For this reason we have only considered $l \geq \bar{\xi}$ in figure (4.1). We would expect, from (4.34), that $\xi_s(t_{sp}; l)$ has an upper bound that lies somewhere between the curves.

Once $l$ is very large, so that the power in the fluctuations is distributed strongly across all wavelengths we recover our earlier result, that $\xi_s(t_{sp}; l) = O(l)$. In figure (4.1) this corresponds to the curves becoming parallel as $l$ increases for fixed $t$. However, the change is sufficiently slow that annuli, significantly wider than $\bar{\xi}$, for which experiments are more accessible, will give almost the same flow as narrower annuli. This would seem to extend the original Zurek prediction of (4.20) to thicker annuli, despite our expectations for incoherent flow. However, we stress again that caution is necessary, since in the approximation to characterize an annulus by a coarse-grained ring without boundaries we have ignored effects in the direction perpendicular to the annulus. In particular, the circular cross-section of the tube has not been taken into account. One consequence of this is that infinite (non-selfintersecting) vortices in the bulk fluid have no counterpart in an annulus. Removing such strings will have an effect on $\Delta \theta_C$, since the typical fraction of vortices in infinite vortices is at the level of 70%. However, at the spinodal time the fluctuations in $^4He$ are relatively enhanced in the long wavelengths, and such an enhancement is
known to reduce the amount of infinite vortices, perhaps to something nearer to 20%. The details of this effect (being pursued elsewhere) are unclear but, for the sake of argument we take the predictions of the curves in figure (4.1) as a rough guide in the vicinity of their minima.

So far we have avoided the question as to which time curves we should follow. This is because \( t_{sp} \) itself depends on the scale \( l \) of the spatial volume for which the field average achieves its ground state value. In practice variation is small, with \( t_{sp} \) for \(^4\text{He}\) varying from about \( 3\bar{l} \) to \( 4\bar{l} \) as \( l \) varies from \( \xi_0 \ll \bar{\xi} \) to \( l = 10\bar{\xi} \). Since the curves for \( \xi_s(t_{sp}; l) \) lie so close to one another in figure (4.1) once \( l \geq 4\bar{\xi} \) the scale at which the coarse-grained field begins to occupy the ground states becomes largely irrelevant.

Since \( \Delta v \) only depends on \( \xi_s^{-1/2} \) it is not sensitive to choice of \( l > 2\bar{\xi} \) at the relevant \( t \). Given all these approximations our final estimate is (in the cm/sec units of Zurek[8])

\[
\Delta v \approx 0.2(\tau_Q[\mu s])^{-\nu/4}/\sqrt{C[cm]} 
\] (4.40)

for radii of \( 2\bar{\xi} - 4\bar{\xi} \), \( \tau_Q \) of the order of milliseconds and \( C \) of the order of centimeters. \( \nu = 1/2 \) is the mean-field critical exponent above. In principle \( \nu \) should be renormalised to \( \nu = 2/3 \), but the difference to \( \Delta v \) is sufficiently small that we shall not bother. Given the uncertainties in its derivation the result (4.40) is indistinguishable from Zurek’s[8] (with prefactor 0.4), but for the possibility of using somewhat larger annuli. The agreement is, ultimately, one of dimensional analysis, but the coefficient could not have been anticipated. How experiments can be performed, even with the wider annuli that (4.40) and figure (4.1) suggest, is another matter.

**Short summary and further motivation**

An annular configuration has the potential of influencing the topological flux going through an area \( \pi l^2 \), with \( l \) the radius of the annulus, because of its geometrical features. That can have useful consequences if one had secure leads...
Figure 4.1: $\xi_{\text{min}}(t, l)$ of (4.34) (solid lines) plotted against $l$, in units of $\xi$, for $t=0, 3\xi$ and $4\xi$. $\xi_{\text{max}}(t, l)$ of (4.39) (dashed lines) plotted against $l$ for $t=0, 3\xi$ and $4\xi$. In each case the higher lines correspond to higher values of time as to which length scales to look for.

The exclusion of defects happening inside the annulus, for example considering kink densities along $C$, can be an issue for further investigation. This is being studied in the Josephson Tunnel Junction situation we will be considering further.
Chapter 5

Josephson Tunnel Junctions

5.1 Introduction

As we have seen, an effective theory based on the Landau-Ginzburg free energy (3.23) and the time dependent equation for $F$ of (4.18) can provide the means to both formulate a plausible framework for defect production and predict quantities like the initial defect density which could, in principle, be possible to test experimentally. The fact that, so far, there has been no definite experimental verification of the proposed theory one could attribute more to the setting up of the experiments rather than the validity of the under investigation theory.

Kibble observed that causality imposed useful constraints on domain growth and the density of defects at the time of their formation. However, since notions of causality are not specific to the relativistic quantum field theory (QFT) appropriate to the early universe, Zurek argued that similar causal bounds were valid in condensed matter systems for which direct experiments on defect formation could be performed. Although the causal bounds are robust, the extent to which they are saturated depends on the details of the microscopic dynamics. From the microscopic level, causality, as proposed by Zurek, is not explicit, although encoded in the relevant dynamical equations.

Dimensional analysis, in the mean field approximation, provides an estimation of the causal time scale of defect formation and a causal correlation length
which is associated with defect density at the causal time of formation. The dynamics embedding such causal behavior can be relevant to various systems as we have seen. Here we will focus on a particular physical system where the theoretical framework is very similar to Landau-Ginzburg and, in accordance to early Zurek postulations, one could associate with a possible platform for further experimental tests of defect formation. That will be the system of annular Josephson Tunnel Junctions. Before going on to study the Josephson effect we would like to outline some basic properties of the superconducting state that are important to our discussion.

5.2 Basic Superconductivity

Superconductivity[39] is a macroscopic quantum phenomenon occurring at low temperatures of the order of a few degrees kelvin. The main characteristic of a superconductor is the “perfect” conductivity it exhibits. For example, an electric current in a closed mercury ring would circle around for a very long time without observable decay. Lifetimes of $10^5$ years have been seen experimentally. Another very important property of a superconductor is that it is a perfect diamagnet i.e it cannot be penetrated by a magnetic field. Such fields can enter only a very thin surface layer, decaying exponentially as $\exp\left(-\frac{x}{\lambda}\right)$ with $\lambda$ the so called penetration depth being of the order of a few hundred angstroms. The superconductors that have this characteristic are called London superconductors and $\lambda$ corresponds to the London penetration depth.

5.2.1 London theory

A phenomenological description of the superconductors was first provided by the London brothers in 1935. They assumed that a certain fraction of electrons is able to form a permanent electric current density $\vec{j}_s = n_s e \vec{v}_s$ with a number
density $n_s$ and a velocity $\overrightarrow{v_s}$ so that their energy is

$$E = \frac{m}{2} \int d^3x n_s |\overrightarrow{v_s}|^2 = \frac{m}{2} \int d^3x \frac{1}{n_s e^2} |\overrightarrow{j_s}|^2$$  \hspace{1cm} (5.1)

Such a permanent current can be coupled to a magnetic field of the form

$$\nabla \times \overrightarrow{H} = \frac{1}{c} \overrightarrow{j_s}$$  \hspace{1cm} (5.2)

From this they found the total energy to be

$$E = \frac{1}{2} \int d^3x (H^2 + n_s m v_s^2) = \frac{1}{2} \int d^3x (H^2 + \frac{mc^2}{n_s e^2} (\nabla \times H)^2)$$  \hspace{1cm} (5.3)

with

$$\lambda = (\frac{mc^2}{n_s e^2})^{1/2}$$  \hspace{1cm} (5.4)

the London penetration depth, which is temperature dependent. At low temperatures $\lambda = \lambda_0$ a characteristic constant of the particular metal. Above about 0.8 of the transition temperature, however, the penetration depth increases rapidly and approaches infinity as the temperature approaches the transition temperature. From the experimental data, it has been found that the expression

$$\lambda = \frac{\lambda_0}{(1 - (\frac{T}{T_c})^4)^{1/2}}$$  \hspace{1cm} (5.5)

is a good fit.

At the mean field level this energy can be extremized to give the field equation

$$H - \lambda^2 \nabla^2 H = 0$$  \hspace{1cm} (5.6)

since $H$ is divergence-less. This is solved by an exponential exp $(-x/\lambda)$ and explains the finite penetration depth of a magnetic field into the superconductor.

The number density $n_s$ corresponds to the density of the superelectrons, i.e the electrons which pass through the metal suffering no collision with the lattice.
The London theory assumes that in parallel to the superelectrons, normal electrons also exist which can be scattered just as in a normal metal. The ratio between the super and normal electrons goes from zero to infinity as the temperature drops from \( T = T_c \) to \( T = 0 \) so that the superconducting properties are stronger at lower reduced temperatures \( T/T_c \).

With this picture in mind, at finite reduced temperature both kinds of electrons could carry a current in a superconductor, but, in the case of direct current the superelectrons “short circuit” the normal electrons and the resistivity is zero. This is not any longer true if an alternative current is fed into the superconductor: in fact, in this case, due to their smaller inertial mass the normal electrons are accelerated by the electric field more than the superelectrons are and a small (with respect to the normal state) resistivity appears which grows as we increase the frequency of the alternating current.

### 5.2.2 Landau Ginzburg Theory

The concept of the order parameter which allowed for the thermodynamic understanding of the superconductors was originally introduced by Gorter and Casimir. The order parameter, denoted as \(|\psi|^2\), is proportional to the density of superelectrons \( \rho_s \). Close to the transition temperature \( T_c \), at which the superconductivity sets in, the free-energy density is described by

\[
f = \frac{m^2}{2} |\psi|^2 + \frac{g}{2} |\psi|^4 \tag{5.7}
\]

where \( m^2 \) is the parameter that contains the temperature dependence in the form

\[
m^2 = m_0^2 \left( \frac{T}{T_c} - 1 \right) \tag{5.8}
\]

From that we see that a second order phase transition takes place with the free energy behaving quadratically in \( (\frac{T}{T_c} - 1) \), the entropy \( s = -\frac{\partial f}{\partial T} \) linearly and the specific heat \( c = T \frac{\partial s}{\partial T} \) jumping upwards when cooling through \( T_c \).
Landau realized that it is possible to study long wavelength variations of an order parameter by introducing the ordered field \( \varphi(x) \) and adding to the free energy density a gradient \( \nabla \varphi(x) \) leading to a free energy functional

\[
F = \int d^3x f = \int d^3x \left( \frac{1}{2} |\nabla \varphi|^2 + \frac{m^2}{2} |\varphi|^2 + \frac{g}{4} |\varphi|^4 \right) \quad (5.9)
\]

Moreover, the London equations can be derived from the Landau-Ginzburg model if one couples the vector potential of magnetism with the energy in a gauge invariant way to arrive at:

\[
f = \frac{1}{2} |(\nabla - uq A)| \varphi|^2 + \frac{m^2}{2} |\varphi|^2 + \frac{g}{4} |\varphi|^4 \quad (5.10)
\]

The microscopic interpretation of the superconducting state came after the theoretical breakthrough of Bardeen-Cooper-Schieffer. They found that due to the electric polarization caused by an electron in a crystal, pairs of electrons are attracted to each other. For the active electrons close to the Fermi surface the attraction could overcome the coulomb repulsion since this is rather small for such electrons: typically \( O(e^2p_F/h) \). There exists a temperature below which the electrons form bound states called cooper pairs. All (and only) the electrons belonging to a small energy band centered around the Fermi energy are involved in the formation of cooper pairs so leaving a small gap of energy levels forbidden to normal electrons. Since the cooper pairs form a gas of bosons the permanent current postulated by the Londons turned out to be the superflow of this Bose gas.

The order parameter field \( \varphi(\vec{x}) \) can now be written as \( \varphi(\vec{x}) = \rho e^{i\theta} \) with \( \rho^2 \) the density of the cooper pairs. It has a natural unit length of variation known as the coherence length. The phase \( \theta \) is left undetermined and represents a redundant variable for an isolated superconductor. In JTJ language, this arbitrariness is being removed when we consider a system composed of two weakly coupled superconductors, as we shall see.
5.3 About Josephson Tunnel Junctions (JTJs)

Consider two planar conductors, our electrodes, that are separated by a thin dielectric barrier. The latter prevents the free exchange of electrons between the electrodes. However, when the barrier is thin enough, around $10 - 20 \, \text{Å}$, by applying some external voltage difference between the electrodes, we can force a current to appear through the dielectric. The relationship between the current and the voltage is linear, for small values of the applied voltage.

In 1962, Brian Josephson[37] studied how that particular phenomenon could translate to a system of two superconductors. Following the Feynman way of explaining the Josephson phenomenon, consider a system of two London superconductors separated by a thin layer of an insulating oxide (the dielectric barrier). Call $\psi_1$ to be quantum mechanic wave function representing the physical state the one superconductor is at, and $\psi_2$ the corresponding wave function for the other superconductor. They should satisfy the time dependent Schrödinger equations:

\[
\begin{align*}
    i\hbar \frac{\partial}{\partial t} \psi_1 &= E_1 \psi_1 + K \psi_2 \\
    i\hbar \frac{\partial}{\partial t} \psi_2 &= E_2 \psi_2 + K \psi_1
\end{align*}
\] (5.11)

with $E_1$, $E_2$ the energy states of each superconductor and $K$ a parameter that depends on the barrier and signifies the coupling between the two electrodes, determining, therefore, how much each wave function can extend to the opposite superconductor. For simplicity, one considers that $K$ is only sensitive on the barrier and thus, is common for both superconductors. That should not be the case in general, but we are here going to assume that the electrodes are basically mirror images of one another and therefore have all intrinsic properties the same (see figure (5.1)).
Figure 5.1: The two superconductors separated by the oxide (dashed area). The wave functions in each superconductor extend to the other by a small degree. The light dotted contour will be used further head.

When we apply a certain voltage difference $V$ on the barrier, we have that

$$E_1 - E_2 = 2E = 2eV$$  \hspace{1cm} (5.12)

which means that the two superconductors are in different energy states because of the applied voltage. Suppose that $\psi_1 = \sqrt{p}e^{i\theta_1}$ and $\psi_2 = \sqrt{p}e^{i\theta_2}$ thus, both wave functions have the same magnitude something that stems from the symmetric nature of our superconductors. Substituting those expressions for $\psi_1$ and $\psi_2$ to the Schrödinger equations above, we find that

\[
\begin{align*}
    i\hbar \frac{1}{2} (\frac{\partial}{\partial t} \rho) - \hbar \rho (\frac{\partial}{\partial t} \theta_1) & = E_1 \rho + K \rho e^{i(\theta_2 - \theta_1)} \\
    i\hbar \frac{1}{2} (\frac{\partial}{\partial t} \rho) - \hbar \rho (\frac{\partial}{\partial t} \theta_2) & = E_2 \rho + K \rho e^{-i(\theta_2 - \theta_1)}
\end{align*}
\]  \hspace{1cm} (5.13)

Call

$$\varphi = \theta_2 - \theta_1$$  \hspace{1cm} (5.14)

and expand the exponent into a real and an imaginary part. Thus, the equations become:
\[
\begin{align*}
\frac{i\hbar}{2} \left( \frac{\partial}{\partial t} \rho \right) - \hbar \rho \left( \frac{\partial}{\partial t} \theta_1 \right) &= E_1 \rho + K \rho \cos \varphi + iK \rho \sin \varphi \\
\frac{i\hbar}{2} \left( \frac{\partial}{\partial t} \rho \right) - \hbar \rho \left( \frac{\partial}{\partial t} \theta_2 \right) &= E_2 \rho + K \rho \cos \varphi - iK \rho \sin \varphi
\end{align*}
\]

which, if we separate the real from the imaginary factors we find:

\[
\begin{align*}
\frac{i\hbar}{2} \left( \frac{\partial}{\partial t} \rho \right) - K \rho \sin \varphi &= E_1 \rho + K \rho \cos \varphi + \hbar \rho \left( \frac{\partial}{\partial t} \theta_1 \right) \\
\frac{i\hbar}{2} \left( \frac{\partial}{\partial t} \rho \right) + K \rho \sin \varphi &= E_2 \rho + K \rho \cos \varphi + \hbar \rho \left( \frac{\partial}{\partial t} \theta_2 \right)
\end{align*}
\]

(5.16)

which says that

\[
\frac{\hbar}{2} \left( \frac{\partial}{\partial t} \rho \right) \pm K \rho \sin \varphi = 0
\]

(5.17)

and that

\[
\begin{align*}
E_1 \rho + K \rho \cos \varphi + \hbar \rho \left( \frac{\partial}{\partial t} \theta_1 \right) &= 0 \\
E_2 \rho + K \rho \cos \varphi + \hbar \rho \left( \frac{\partial}{\partial t} \theta_2 \right) &= 0
\end{align*}
\]

(5.18) (5.19)

From (5.17), one takes that

\[
\left( \frac{\partial}{\partial t} \rho \right) = \frac{2K \rho}{\hbar} \sin \varphi
\]

(5.20)

assuming that the minus (or the plus) sign is due to the direction of the current that passes the barrier and not any other reason that has to do with the nature of the system. Further, if one subtracts the last two equations, (5.18)-(5.19) and takes into account both (5.12) and (5.14), one finds that \(2eV - \hbar \frac{\partial}{\partial t} \varphi = 0\). Thus, by means of the phenomenological Feynman model we have derived the following equations\(^1\)

\(^1\)Their original derivation was based on the BCS theory, focusing on the microscopics of
\[
\frac{\partial}{\partial t} \varphi = \frac{2eV}{\hbar} \tag{5.21}
\]

and

\[
\frac{\partial}{\partial t} \rho = \frac{2K\rho}{\hbar} \sin \varphi \tag{5.22}
\]

known as the Josephson equations.

The \(\psi_i \psi_i^*\) with \(i = 1, 2\) determines the density of the Cooper pairs in either superconductor and therefore, \(\frac{\partial}{\partial t} \rho\) defines the Cooper pair current through the oxide. The \(\frac{2K\rho}{\hbar}\) corresponds to the maximum current that could pass from the one electrode to the other, at any given moment\(^2\), and is called critical Josephson current. For

\[
J = \frac{\partial}{\partial t} \rho \tag{5.23}
\]

and

\[
J_c = \frac{2K\rho}{\hbar} \tag{5.24}
\]

we have

\[
J = J_c \sin \varphi \tag{5.25}
\]

with \(K\) having energy dimensions. The Josephson current \(J\) corresponds to Cooper pair tunnelling through the barrier and is not, in general, equal to the total current that passes through the oxide. Other transport phenomena are relevant, such as the ohmic passage of single electrons but in our discussion here, those types of contributions to the total current will be neglected. We will come back to this matter further ahead.

The current density \(J_c\) (5.24) is a critical quantity above which the Josephson current cannot grow. It depends on the local properties of the barrier and on its thickness. The thicker the barrier the smaller is the critical current density.

The total Josephson current \(I\) is given by the surface integral of \(J\) over the superconductors. The Feynman model is just a phenomenological approach that can be helpful to understand the basic -macroscopic- principles behind the Josephson equations.

\(^2\)One, usually, considers that \(\rho\) is so large that even if \(\frac{\partial}{\partial t} \rho \neq 0\), it induces so little change in the actual value of \(\rho\) that one can consider \(J_c\) to be, essentially, a constant.
barrier area:

\[ I(t) = J_c \int_0^L \int_0^W dydx \sin \varphi(x, y, t) \]

Assuming a uniform barrier, \( J_c(x, y) \) a constant with \( L \) the length of the junction and \( W \) its width. The most striking feature of JTJs derives from the \( \varphi(t) \) a constant solution to (5.21) when \( V = 0 \). In general, we find that

\[ \varphi = \varphi_0 + \frac{2e}{\hbar} \int V dt \tag{5.26} \]

with \( \varphi_0 \) the phase difference when \( V = 0 \). Physically meaning that a D.C current due to the tunnelling of cooper pairs may flow through the barrier with no voltage drop across it: this phenomenon is called the D.C Josephson effect. Its essence is that the presence of the interaction between two superconductors removes the arbitrariness in the choice of phase of the wave functions: the relative phase of the coherent state is fixed by the current flowing \( \varphi = \arcsin(J/J_c) \). The existence of a zero voltage current (also called the supercurrent or Josephson current) suggests that the Josephson effect can be regarded at least qualitatively as an extension of the superconductive properties over the barrier. However, typical values for the Josephson current density \((10^6 A/m^2)\) are much lower than the supercurrent density in homogeneous superconductors \((10^{11} A/m^2 \text{ or more})\).

It is for this and other similarities that often the Josephson effect is called “weak superconductivity”.

### 5.3.1 JTJ in the presence of some external magnetic field

Suppose we apply now an external magnetic field \( \vec{B} \) parallel to the plane of the barrier (being, thus in the \( y \) direction (see figure (5.1)). \( \vec{B} \) will be associated with a vector field \( \vec{A} \) such that \( \nabla \times \vec{A} = \vec{B} \). The field \( \vec{A} \) will penetrate each superconductor up to the relevant London penetration depth, common to both electrodes. The Cooper pair electric current density that goes through each
superconductor is given by the equation:

\[
\overrightarrow{J} = \frac{\rho e}{m}(\hbar \nabla \theta - 2e \overrightarrow{A}) \tag{5.27}
\]

with \(m\) and \(e\) the mass and electric charge of a single electron. Thus, the spatial gradient of the phase of the wave function \(\psi\) associated with the energy state of the superconductor and the vector field \(\overrightarrow{A}\) due to the applied magnetic field can contribute to the current density. From that equation, we find the spatial gradient of the phase, relevant to the energy state of each superconductor. If one chooses an appropriate path on the plane perpendicular to that of the barrier (assume the \((x, z)\) plane, see figure (5.1)), one can reduce the integrals:

\[
-\theta_1(x + dx) + \theta_1(x) = \frac{2e}{\hbar} \int_{C_1} (\overrightarrow{A} + \frac{m}{2e^2 \rho} \overrightarrow{J})d\overrightarrow{l}
\]

\[
\theta_2(x + dx) - \theta_2(x) = \frac{2e}{\hbar} \int_{C_2} (\overrightarrow{A} + \frac{m}{2e^2 \rho} \overrightarrow{J})d\overrightarrow{l} \tag{5.28}
\]

to

\[
-\theta_1(x + dx) + \theta_1(x) = \frac{2e}{\hbar} \int_{C_1} (\overrightarrow{A})d\overrightarrow{l}
\]

\[
\theta_2(x + dx) - \theta_2(x) = \frac{2e}{\hbar} \int_{C_2} (\overrightarrow{A})d\overrightarrow{l} \tag{5.29}
\]

assuming that the current \(\overrightarrow{J}\), whenever non zero, is actually vertical to \(d\overrightarrow{l}\) and, thus, does not contribute to the integration. The variation of the phase difference (5.14) is

\[
\delta \varphi = \delta \theta_2 - \delta \theta_1
\]

\[
= (\theta_2(x + dx) - \theta_2(x)) - (\theta_1(x + dx) - \theta_1(x))
\]

\[
= \frac{2e}{\hbar} \int_{C_2} (\overrightarrow{A})d\overrightarrow{l} + \frac{2e}{\hbar} \int_{C_1} (\overrightarrow{A})d\overrightarrow{l} \tag{5.30}
\]

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which, neglecting the thickness of the oxide, will be equal to

$$\delta \varphi = \frac{2e}{\hbar} \int_C (\vec{A}) d\vec{l}$$  \hspace{1cm} (5.31)$$

with $C$ the closed path we consider. From Stokes theorem we know that

$$\int_C (\vec{A}) d\vec{l} = \int_S (\nabla \times \vec{A}) d\vec{s}$$

which, along a rectangular contour in the chosen plane, gives

$$\begin{align*}
\frac{\partial \varphi}{\partial x} &= \frac{2ed}{\hbar} B_y \\
\frac{\partial \varphi}{\partial y} &= -\frac{2ed}{\hbar} B_x
\end{align*}$$  \hspace{1cm} (5.32)$$

with $d$ a quantity with the dimension of length. In fact, assuming the thickness of the superconducting films to be much larger than their London penetration depths a magnetic field penetrates not only the dielectric barrier but also into both the films for a distance of about one London length.

With the aid of the Ampere-Maxwell law \( \nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon \mu_0 \frac{\partial}{\partial t} \vec{E} \) which in our case becomes

$$\frac{\hbar}{2ed} (\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}) = \mu_0 J_c \sin \varphi + \frac{\mu_0 \epsilon}{t_{ox}} \frac{dV}{dt}$$

and the consideration that

$$|\vec{E}| = \frac{V}{t_{ox}}$$

with $t_{ox}$ the width of the barrier and $V$ the applied voltage, one can find[36] that

$$\varphi_{tt} - c_0^2 \nabla^2 \varphi + \Omega_J^2 \sin \varphi = 0$$  \hspace{1cm} (5.33)$$

with

$$c_0 = \sqrt{\frac{t_{ox}}{\epsilon \mu_0 d}}$$  \hspace{1cm} (5.34)$$

and

$$\Omega_J = \sqrt{\frac{2e J_c t_{ox}}{\hbar \epsilon}}$$  \hspace{1cm} (5.35)$$
with $\lambda_i$ with $i = 1, 2$ the London penetration depth for each superconductor.

One defines the quantity known as the Josephson length as

$$\lambda_J = \frac{c_0}{\Omega_J}$$

(5.36)

which basically represents the length scale over which the spatial changes of $\varphi$ are significant. For junctions smaller than $\lambda_J$ we can safely declare $\varphi$ as uniform throughout the junction and thus, take $J = J_c \sin \varphi$ to be spatially uniform as well. The $c_0$ corresponds to some characteristic velocity whereas the $\Omega_J$ to the inverse of some characteristic time scale.

The above definitions are valid when the thickness of each superconductor is infinite in which case $c_0$ is the speed of light in the barrier and $\lambda_J = c_0/\Omega_J$. However, for finite electrodes those equations have to be appropriately modified as we shall see. The parameter $d$ here is equal to $d = t_{ox} + \lambda_1 + \lambda_2$ the magnetic thickness of the JTJ so long as $l_1 \to \infty$, $l_2 \to \infty$ with $l_r, r = 1, 2$, the thicknesses of the electrodes.

### 5.4 Short and Long Josephson Tunnel Junctions

Depending on the relationship between the Josephson length $\lambda_J$ (5.36) and the length $L$ of the JTJ one can separate between short and long junctions. In principle, for $L >> \lambda_J$ the JTJ is considered to be long, otherwise called a Josephson Transmission Line (JTL). For $L = O(\lambda_J)$ one can ignore spatial variations in $\varphi$ and thus arrive at the equation

$$\frac{\partial^2 \varphi}{\partial t^2} + \Omega_J^2 \sin \varphi = 0$$

which describes the short JTJ.
5.5 Swihart velocity for a JTL

Considering $L >> \lambda_J$ one can study the propagation of an electromagnetic wave moving along the $x$ direction (see figure 5.1) by adopting the formula:

$$F = F_c(z) \exp i(\omega t - ax)$$

(5.37)

along the superconducting electrodes and the oxide. We require $a > 0$ so that the wave travels and is attenuated in the plus $x$ direction. We must have the same propagating constant $a$ in all the media in order to ensure the continuity of the tangential fields $E_x$ and $H_y$ at the boundaries.

Assume that the thickness of each superconductor is $l_r$ with $r = 1, 2$ and that of oxide $t_{ox}$. Consider $\lambda_r$ with $r = 1, 2$ to be the London penetration depths for each superconductor and $\epsilon$ the dielectric constant in the oxide barrier. The combination of the London and Maxwell equations assuming $H_x = H_z = 0 = E_y = j_y$ in all media easily brings us to the following equations:

$$E_x = -\frac{i}{a} \frac{dE_z}{dz}$$  \hspace{1cm} (5.38)

$$H_y = \frac{1}{a \mu_0 \omega} (a^2 - k^2) E_z$$  \hspace{1cm} (5.39)

$$0 = \frac{d^2 E_z}{dz^2} - k^2 E_z$$  \hspace{1cm} (5.40)

For the dielectric we have

$$k_{ox}^2 = a^2 - \omega^2 \epsilon \mu_0$$

whereas for the superconductors we find

$$k_r^2 = \lambda_r^{-2} + a^2 - \omega^2 \mu_0 \epsilon_0$$

with $r = 1, 2$. Requiring continuity of the electric and magnetic fields at the boundaries $z = -\frac{t_{ox}}{2}$, $z = \frac{t_{ox}}{2}$, $z = -l_1 - \frac{t_{ox}}{2}$ and $z = l_2 + \frac{t_{ox}}{2}$ we get an equation
for the speed of the electromagnetic wave inside the dielectric as

\[ c_0 = \frac{\omega}{a} = \frac{\tilde{c}_0}{\sqrt{1 + \frac{\lambda_1}{t_{ox}} \coth\left(\frac{t_{ox}}{\lambda_1}\right) + \frac{\lambda_2}{t_{ox}} \coth\left(\frac{t_{ox}}{\lambda_2}\right)}} \]  

(5.41)

with \( \tilde{c}_0 = \frac{1}{\sqrt{\mu_0}} \) the speed of light in the dielectric. For \( t_{ox} = O(\lambda) = O(l) \), for \( l_r = l \) and \( \lambda_r = \lambda \) with \( r = 1, 2 \), we recover approximately the equation \( c_0 = O(\tilde{c}_0) \). Expression (5.41) was originally calculated assuming the one superconductor to have an infinite thickness. However, it is of no particular difficulty to derive the (5.41) using the original principles. From equations (5.34) and (5.41) we find that

\[ d = t_{ox} + \lambda_1 \coth\left(\frac{t_{ox}}{\lambda_1}\right) + \lambda_2 \coth\left(\frac{t_{ox}}{\lambda_2}\right). \]

Thus, \( c_0 = \tilde{c}_0 \sqrt{\frac{t_{ox}}{d}} \)

which means that the speed of propagation of an electromagnetic wave of some frequency \( \omega \) will be the speed of light in the barrier reduced by \( \sqrt{\frac{t_{ox}}{d}} \). We can call \( \tilde{c} \) the effective dielectric constant which equals to \( \epsilon \frac{d}{t_{ox}} = \tilde{c} \) which the electromagnetic wave propagating in the barrier feels. We see that \( d \) diverges at \( T = T_c \) and thus \( \tilde{c} \) becomes so large that no external wave can penetrate the junction.

The expression in (5.41) holds so long as

\[ \lambda_r^{-2} \gg (1 + \frac{\lambda_1}{t_{ox}} \coth\left(\frac{t_{ox}}{\lambda_1}\right) + \frac{\lambda_2}{t_{ox}} \coth\left(\frac{t_{ox}}{\lambda_2}\right))\omega^2\epsilon\mu_0 \]  

(5.42)

This is a very crucial constraint and will be discussed further when we make use of the expression (5.41) shortly. In principle, the speed of the electromagnetic wave in the superconductor will be further affected by losses due to the surface resistance of the superconductor and a conductance per unit length to represent the ohmic losses in the barrier. Also radiation losses should be considered, but they cannot be cast in a dissipation per unit length being connected to the discontinuities of the line. Such contributions will be be ignored for the moment not introducing any significant change in the essence of the JTL behavior, but their total effect will be considered later.

As we see the Swihart velocity depends on the London length. When \( T \to T_c \)
then the London length diverges and for \( l_r \) finite we see that the relation (5.42) may not hold for some wave frequency \( \omega \). However, (5.41) is quite accurate for a wide range of conditions. We shall consider as an example a transmission line in which \( \lambda_0 = l = d = 500 \ Å \) and the dielectric constant \( \epsilon = 4\epsilon_0 \), with \( \epsilon_0 \) the dielectric constant of the vacuum. At a temperature below \( 0.99T_c \) we find that \( \lambda \leq 2500 \ Å \). We can calculate from (5.42) that \( \omega << 10^{14} \text{sec}^{-1} \). Thus, for appropriate frequencies the Swihart velocity as calculated without taking into account losses, can be accurate. To be sure, the approximation is better the lower the frequency, the temperature or the conductivity.

In our calculation, we have assumed that no matter how close to the transition temperature we are, the expression (5.41) is accurately describing the speed of the electromagnetic wave. To be on the safe side, this can be valid in a lossless junction and that is what we will assume. A more rigorous approach is required and we will talk about ways forward further ahead.

5.6 JTJs and the Kibble-Zurek Mechanism

The equation for the phase difference \( \varphi \) in a lossless system can be written now in terms of the Josephson length and \( c_0 \) as:

\[
\frac{1}{c_0^2} \varphi_u - \nabla^2 \varphi + \frac{1}{\lambda_J^2} \sin \varphi = 0 \tag{5.43}
\]

an equation which stems from a Lagrangian \( L \) with terms like

\[
L = \frac{1}{c_0^2} \partial_i \varphi \partial^i \varphi - \partial_i \varphi \partial^i \varphi + \frac{1}{\lambda_J^2} \cos \varphi \tag{5.44}
\]

where the \( \partial_i \) derivative indicates spatial differentiation. For small values of \( \varphi \) the potential term of \( \frac{1}{\lambda_J^2} \cos \varphi \) can be expanded around \( \varphi = 0 \) to become

\[
\frac{1}{\lambda_J^2}(1 - \frac{1}{2} \varphi^2 + \frac{1}{24} \varphi^4)
\]

which is equivalent to a symmetry breaking potential. The equation (5.43) is known as the Sine-Gordon (SG) equation. A small field
approximation, $|\varphi(x,t)| << 1$, has been shown to be true only when the energy of the wave is much less than the fluxon rest energy which can be computed by integrating the Hamiltonian

$$H = \frac{1}{2}\left((\partial_t \varphi)^2 + (\partial_x \varphi)^2 - 2\cos \varphi\right)$$  \hspace{1cm} (5.45)

which comes from the original sine-Gordon equation. However, we shall not restrict our order parameter in such a way.

The system we are going to study has an *annular* structure of radius $r_a$. The width $\Delta r_a$ of the annulus should be much smaller than $r_a$ and the relevant $\lambda_f(T)$. The first condition ensures that our system can be considered as one dimensional whereas the second maintains the probability of a non zero vortex flux along the annulus. Focusing on the variation of $\varphi$ in order to gain information about the existence of topological defects can have its caveats. The reason is that the experiments will not measure $\varphi$ itself but rather the Josephson current $J$ as it flows through the oxide. Since $J = \frac{\partial \rho}{\partial t}$ (5.23) one can wonder whether actually any spatial variation in $\rho$ could be responsible for any observable effects, rather than $\varphi$ itself. The answer is that one can have $\rho$ a constant and maintain $J \neq 0$. In the simple Feynman model, that can be possible by assuming that the loss of $\frac{\partial \rho}{\partial t}$ Cooper pairs at a time unit from the one superconductor to the other, is immediately balanced by new Cooper pairs. Thus, the $\frac{\partial \rho}{\partial t}$ represents a local effect, homogeneous along the barrier, rather than a global one.

Another point of importance is that whereas the Zurek scenario talks about the freezing-in of the correlation length **before** the transition, in a JTJ we cannot talk about a superconducting system above $T_c$. The reason is that the Josephson effect disappears for higher than $T_c$ temperatures and our framework changes. However for temperatures above but close to $T_c$ one can still observe a very small Cooper pair current, which though cannot be used long enough for the Zurek scenario to be applicable as originally proposed.
To start, we observe that for an infinitely long JTL equation (5.43) invariant under Lorentz transformations:

\[ x \rightarrow x' = \gamma(x - ut) \quad (5.46) \]
\[ t \rightarrow t' = \gamma(t - \frac{x}{u}) \quad (5.47) \]

where \( \gamma = \frac{1}{\sqrt{1 - \left(\frac{c_0}{c}\right)^2}} \) is the so-called Lorentz factor and \( c_0 \) is the Swihart velocity which represents the limiting value for the relativistic speed \( u \) of the wave solution. Because of the Lorentz invariance, simple travelling wave solutions of the kind \( \varphi(x, t) = \varphi(\xi) \) with \( \xi = x - ut \) can be found. They should satisfy the equation

\[ \frac{d^2\varphi}{d\xi^2} = \left(\frac{\gamma}{\lambda_J}\right)^2 \sin \varphi \quad (5.48) \]

This equation is equivalent to an one-dimensional time independent sine-Gordon model. The simplest solution obtained for \( T < T_c \) is the JTJ classical topological defect (‘fluxon’): the ‘kinks’ of the Sine-Gordon theory

\[ \varphi_{\pm}(x, t) = 4 \arctan \left[ \pm \frac{\frac{x - ut}{\lambda_J \sqrt{1 - (u/c_0)^2}}} \right] + n\pi \quad (5.49) \]

with \( u \) the speed of that configuration at it moves inside the barrier, along the annulus. The \( x \) variable measures the distance along the annulus, with periodic boundary conditions. It has also been assumed that along the barrier only a single isolated fluxon propagates.

We can re-scale the length \( \xi \) to \( \tilde{\xi} = \gamma \frac{\xi}{\lambda_J} \) which gives the width of the dispersionless travelling supercurrent vortex. It indicates that the fluxon undergoes a Lorentz contraction as the speed \( u \) approaches the limit speed \( c_0 \). As in the case of a static vortex it can be shown that the total Josephson current carried by such a \( \varphi \) profile is null and the linked magnetic flux corresponds to one magnetic flux quantum \( \Phi_0 \). In other words, in an infinite JTL, neglecting dissipation mechanisms, relativistic voltage pulse propagation is obtained through
a travelling magnetic flux quantum or fluxon.

### 5.6.1 Causality and JTJs

As with other established models of defect formation, the classical equations are only valid once the transition is complete. We do not use the SG (5.43) equation to study the evolution of defects. In fact, we are not going to be concerned about that for the moment. Instead, the SG (5.43) and the fluxon solution (5.49) will help us identify the necessary quantities such as the equilibrium correlation length and the relaxation time scale. The type of phase transition we will be thinking about is second order. However, the limits of those postulations will be discussed further ahead.

Since notions of causality are not specific to the relativistic quantum field theory (QFT) appropriate to the early universe, Zurek [8, 10] argued that similar causal bounds were valid in condensed matter systems for which direct experiments on defect formation could be performed. Identifying the causal quantities is a crucial matter and at a primary level will be done by re-scaling the length and time scales in the Lagrangian following Zurek.

As an ordering transition begins to be implemented, from time \( t = 0 \), say, there is a maximum speed \( c(t) \) at which the system can become ordered. This leads to causal horizons, outside of which there are no correlations. Zurek proposed that the earliest time (the ‘causal time’) at which defects can form is when a local causal horizon is large enough to accommodate a single defect at that time. Since the defect size \( \xi(t) = \xi(T(t)) \) depends on temperature \( T \), which depends on time, the causal time \( \tilde{t} \) is determined from

\[
\xi(\tilde{t}) \approx 2 \int_0^\tilde{t} dt \ c(t).
\]  

(5.50)

There are other ways to formulate causality[10], most simply by requiring that \( \xi(t) \) cannot grow faster than \( c(t) \). This can be imposed either before the
transition, or after. In simple systems all bounds agree, up to numerical factors approximately unity[10]. To the level at which we are working (better than an order of magnitude, but to a factor of a few), they are indistinguishable.

The strength of Zurek’s predictions lies in the assumption that, in general, these can be read off from the time and space derivatives alone in the equations of motion in the adiabatic regime. To see how well the bounds are saturated requires further knowledge, but we shall assume that they will be well saturated here, as exemplified by the $^3$He experiments. Further, in this scenario the adiabatic approximation is pushed to its limits, by which we take it to be approximately valid from the causal time onwards.

Whatever the case, although the causal bounds are robust, the extent to which they are saturated depends on the details of the microscopic dynamics. From the microscopic level, causality along the lines above is not explicit, although encoded in the relevant dynamical equations. The picture is rather one of order being established through the growth of the amplitudes of long-wavelength instabilities. The earliest time at which we can identify defects from this viewpoint is when the order parameters have achieved their equilibrium magnitudes. Qualitatively, for simple models this time is also in agreement with the causal time above, showing that the causal bounds are approximately saturated. There is no real surprise in this. The causal time and distance scales $\bar{t}$ and $\bar{\xi} = \xi(\bar{t})$ are just as we would expect from dimensional analysis (in the mean-field approximation) and unstable modes grow exponentially, whereby the dependence of the causal time (and corresponding defect density) on the microscopic parameters is only (square-root) logarithmic. However, before the causal time we now have a picture in which there is a fractal thermal fuzz of potential defects, whose density depends on the scale at which we look. By the causal time some of these have matured into the (scale-independent) defects that we see subsequently.

We will assume that the temperature $T_c$ below which we will perform our
calculations, is well within the range of temperatures where the sine-Gordon model has a sensible continuum limit and a true ground state. That is, when we have gone through the JTJ phase transition. Originally, Zurek proposed that the causal constraints are valid above and below the transition. Here, the causal picture for the behavior of the correlation length, for example, will be applied only if we go backwards in time. However, even if qualitatively this is plausible, quantitatively one should expect to be able to reproduce the freezing in of $\xi$, for instance, using the appropriate equations as we did earlier using the time dependent Landau-Ginzburg (4.18) equation.

At the moment, our aim is to show that the theory of Josephson Junctions incorporates quantities that are remarkably similar to those one uses in the standard framework for defect creation. However, the extent of the saturation of the causal bounds we will assume, will depend on the microscopics of our system. The latter will be discussed further ahead.

### 5.7 Applying Kibble-Zurek to Josephson Junctions

In the steps of the Zurek analysis, we write the Sine-Gordon equation (5.43) in a dimensionless form. Thus:

$$\varphi_{TT} - \varphi_{XX} + \sin \varphi = 0 (5.51)$$

taking that $T = \frac{t}{\tau}$ and $X = \frac{x}{\lambda_J}$ with

$$\tau = \frac{\lambda_J}{c_0} (5.52)$$

Following the methodology outlined in (2.19), (2.20), (2.21) and (2.22), we will identify the relevant relaxation time with $\tau$ and the appropriate correlation length with $\lambda_J$. 

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The critical slowing down of (3.1) will be now found by

\[ \xi(t) \sim 2 \int_0^t dsc(s) \]  \hspace{1cm} (5.33)

with \( c(s) \) the relevant causal horizon velocity. For the times of interest, we will assume that \( \lambda_J = \xi(t) \). That is, for \( t \geq \bar{t} \), \( \lambda_J = \xi(t) \). The earliest time for the stabilization of defects will be found by the equality

\[ \frac{d\lambda_J(t)}{dt}_{t=\bar{t}} = -c_0(\bar{t}) \]  \hspace{1cm} (5.34)

We will assume that the relevant speed is the Swihart velocity. Since we will be looking at temperatures \( T \) close to \( T_c \), we have to make the additional assumption that \( c_0(t) \) is a valid speed to consider. Our approximations will be tested further.

### 5.7.1 Symmetric JTJs

The Josephson length changes with time as \( \lambda_J(t) = \lambda_J(T(t)) \) is given by

\[ \lambda_J(t) = \sqrt{\frac{\hbar}{2e\mu_0d_e(t)J_c(t)}} \]  \hspace{1cm} (5.35)

in which \( J_c \) is the critical Josephson current density. The \( d_e(t) \) is the magnetic thickness. Specifically, if \( \lambda(t) \) is the London penetration depth of the two (identical) superconducting sheets, then

\[ d_e(t) = t_{ox} + 2\lambda(t) \tanh \frac{l}{\lambda(t)} \]  \hspace{1cm} (5.36)

where \( l \) is the thickness of the identical superconductors

\[ \lambda(t) = \lambda(0)/\sqrt{1 - (T(t)/T_c)^4} \]

which for \( \frac{T}{T_c} = 1 - \frac{t}{\tau_Q} \) goes like \( \lambda(t) \sim \frac{\lambda(0)}{2} \sqrt{\frac{\tau_Q}{t}} \). Neglecting the barrier thickness
\( t_{ox} \ll l, \) \( \lambda \) gives \( d_e = 2l \) close to \( T_c \), i.e., the magnetic thickness equals the films’ thickness and can be set constant.

The equation for the magnetic thickness comes from the consideration that when \( \lambda \) goes to infinity at \( T_c \), the magnetic thickness has to correspond to the physical thickness of the whole junction alone.

All the \( t \)-dependence of \( \lambda_J \) resides in \( J_c \) which, for the symmetric JTJ has the form[36]

\[
J_c(t) = \frac{\pi \Delta(t)}{2 e \rho_N} \tanh \frac{\Delta(t)}{2 k_B T(t)}. \tag{5.57}
\]

In (5.57) \( \Delta(t) \) is the superconducting gap energy and varies steeply near \( T_c \) as

\[
\Delta(t) \simeq 1.8 \Delta(0) \left( 1 - \frac{T(t)}{T_c} \right)^{1/2} = 1.8 \Delta(0) \sqrt{\frac{t}{\tau_Q}},
\]

and \( \rho_N \) is JTJ normal resistance per unit area. Introducing the dimensionless quantity \( \alpha = 1.6 \Delta(0)/k_B T_C \) whose typical value\(^3\) is between 3 and 5, enables us to write \( J_c(t) \) as

\[
J_c(t) \simeq \alpha J_c(0) \left( 1 - \frac{T(t)}{T_c} \right) = \alpha J_c(0) \frac{t}{\tau_Q}. \tag{5.58}
\]

Thus, in the vicinity of the transition,

\[
\lambda_J(t) = \xi_0 \left( \frac{\tau_Q}{t} \right)^{1/2}, \tag{5.59}
\]

where

\[
\xi_0 = \frac{1}{2} \sqrt{\frac{h}{e \mu_0 \alpha J_c(0)}}. \tag{5.60}
\]

On the other hand, for a finite electrode thickness tunnel junction, the Swihart velocity takes the form[36]

\[
c_0(t) = c \sqrt{t_{ox}/\epsilon_r d_4(t)},
\]

\(^3\Delta(0)\) and \( J_c(0) \) denote the respective values at \( T = 0 \).
where
\[ d_i(t) = t_{ox} + 2\lambda(t) \coth \frac{l}{\lambda(t)} \approx \frac{2\lambda^2(0)}{l} \tau_Q, \tag{5.61} \]
near the transition, \( \epsilon_r \) the relative dielectric constant of the barrier \((\frac{\epsilon_r}{\epsilon_0})\) and \( c \) the speed of light in the vacuum. Thus \( c_0(t) \) shows critical slowing down at the transition, as
\[ c_0(t) = \bar{c}_0 \left( \frac{t}{\tau_Q} \right)^{1/2}, \]
where \( \bar{c}_0 = c\sqrt{H_{ox}/2\epsilon_r\lambda^2(0)} \).

The relaxation time from (5.52), thus, will be
\[ \tau = \frac{\xi_0 \tau_Q}{\bar{c}_0 t}. \]

The freeze-out time will be calculated from (5.54) or, one could derive an order of magnitude for \( \bar{t} \) by simply equating \( \tau \) with \( t \). The end result is that
\[ \bar{t} = \sqrt{\xi_0 \tau_Q} = \sqrt{\frac{1}{2}\xi_0 \tau_Q} \]
with \( \tau_0 = \frac{\xi_0}{\bar{c}_0} \). The correlation length at \( \bar{t} \) will be
\[ \bar{\xi} = \lambda_j(\bar{t}) = \xi_0 \left( \frac{2\tau_Q}{\tau_0} \right)^{1/4}, \tag{5.62} \]

For reasonable values for \( \tau_0, \tau_Q \) and \( \xi_0 \) we get (in SI): \( \tau_0 = 1\text{ps}, \tau_Q = 1\text{s}, \xi_0 = 10\mu m \Rightarrow \bar{\xi} = 11.8\text{mm} \) and \( \bar{t} = 0.7\mu s \). For \( \tau_0 = 0.65\text{ps}, \tau_Q = 1\text{s}, \xi_0 = 6.5\mu m \Rightarrow \bar{\xi} = 8.6\text{mm} \) and \( \bar{t} = 0.6\mu s \). We see that those values for \( \bar{\xi} \) are too large indeed and for the quench rate we considered, are quite forbidding.

### 5.7.2 Non Symmetric JTJ

Fortunately, the manufacture of JTJs typically yields non-symmetric devices with more acceptable properties. Suppose the two superconductors, 1 and 2, now have different critical temperatures \( T_{c2} > T_{c1} \). Fluxons only appear at
temperatures $T < T_{c1}$, from which we measure our time $t$. At this time

$$
\Delta_2(T_{c1}) \simeq 1.8 \Delta_2(0) \left(1 - \frac{T_{c1}}{T_{c2}}\right)^{1/2},
$$

and $\Delta_1(t) \simeq 1.8 \Delta_1(0) \sqrt{t/\tau_Q}$. The critical Josephson current density $J'_c(t)$ for a non-symmetric JTJ, being proportional to $\Delta_1(t)\Delta_2(t)$ [36], behaves just after the transition as

$$
J'_c(t) \approx \left(1 - \frac{T_{c1}}{T_{c2}}\right)^{1/2} \alpha' J'_c(0) \left(\frac{t}{\tau_Q}\right)^{1/2},
$$

(5.63)

where $J'_c(0) = \pi \Delta_1(0) \Delta_2(0)/[\Delta_1(0) + \Delta_2(0)]e\rho_N$, and $\alpha' = [\Delta_1(0) + \Delta_2(0)]/k_B T_{c1}$, provided $\Delta_2(T_{c1}) \ll 2\pi k_B T_{c1}$. This is the case here.

The crucial difference between (5.63) and (5.58) is in the critical index. Near $t = 0$, from equation (5.55) we now find

$$
\lambda_J(t) = \xi_0 \left(1 - \frac{T_{c1}}{T_{c2}}\right)^{-1/4} \left(\frac{T_Q}{t}\right)^{1/4},
$$

(5.64)

where $\xi_0$ is as in (5.59), since $J'_c(0)$ is indistinguishable from $J_c(0)$ and $\alpha'$ is comparable to $\alpha$. We have assumed that the thickness of the superconductors is the same. For the critical behavior (5.64) to be valid, rather than (5.59) we need $1 - T_{c1}/T_{c2} \gg O(\bar{t}/\tau_Q) = O(10^{-6})$, which is always the case. A typical value for $(1 - T_{c1}/T_{c2})$ is 0.02.

The Swihart velocity from (5.41) is found to vary with time as

$$
c_0(t) = \overline{c}_0 \left(\frac{t}{\tau_Q}\right)^{1/2},
$$

where $\overline{c}_0 = c\sqrt{\mu_0/e_r(\lambda_1^2(0) + \lambda_2^2(0))}$ with $\lambda_r$ with $r = 1, 2$ the London lengths for each superconductor. We will assume that the term $\lambda_1^2(0) + \lambda_2^2(0)$ is not changing the value of $\overline{c}_0$. This is commensurate with the properties of the metals that were used in the experiments.

The critical time $\bar{t}$ is now determined by
\[ \bar{t} = \left( \frac{\tau_0}{4} \right)^{4/7} \tau_Q^{3/7} \left( 1 - \frac{T_{c1}}{T_{c2}} \right)^{-1/7}, \]

In turn,

\[ \bar{\xi} = \lambda_f(\bar{t}) \simeq \xi_0 \left( 1 - \frac{T_{c1}}{T_{c2}} \right)^{-1/4} \left( \frac{4\tau_Q}{\tau_0} \right)^{1/7} \tag{5.65} \]

For \( \tau_0 = 1ps, \tau_Q = 1s, \xi_0 = 10\mu m \), we find \( \bar{\xi} = \lambda_f(\bar{t}) \simeq 1.7mm \) which is an order of magnitude smaller than \( \bar{\lambda}_J \) of (5.62) and \( \bar{t} \simeq 0.1\mu sec \). For \( \tau_0 = 0.65ps, \tau_Q = 1s, \xi_0 = 6.5\mu m \Rightarrow \bar{\xi} = 1.1mm \) and \( \bar{t} = 0.1\mu s \).

### 5.8 Discussion and further motivation

Experiments\(^4\) were devised using annular JTJs in which the creation of fluxons was the main aim. The intention was to produce fluxons for further experiments and the density and frequency with which they were produced was secondary. The experiments that have been performed, and which we have in mind, are simple in principle. As we cool an annular JTJ uniformly\(^4\) from above its critical temperature \( T_c \), fluxons \( \varphi \) will form. The mean flux is zero. The relevant observational quantity here is the variance in the flux or, equivalently, the variance \( \Delta N \) in the number of fluxons minus antifluxons.

If the circumference of the annulus is \( C = 2\pi \bar{r} \), and the mean \( \varphi \) correlation length at the time \( \bar{t} \) of the fluxon formation is \( \bar{\lambda}_J = \lambda_f(\bar{t}) \) then, in parallel with our earlier observations, we would expect, for \( C \gg \bar{\lambda}_J \),

\[ \Delta N = \frac{\Delta \varphi}{2\pi} = O \left( \frac{1}{2\pi} \sqrt{\frac{C}{\bar{\lambda}_J}} \right), \tag{5.66} \]

as a result of a random walk in phase around a ring. It is the prediction (5.66) that we would like to test for a variety of junctions and temperature quenches, with different \( \bar{\lambda}_J \). Unfortunately, for the experiments to date it has not been

\(^4\) Even though, from what we already said, the drop in \( E_T \) should be uniform around the \( t_1 = \frac{\tau_0 \tau_Q}{\tau_1} \) time. Much before and much after that, one might wonder whether uniformity is really a necessary condition.

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possible to have $C \gg \bar{\lambda}_J$.

The time scale $\bar{t}$ tells us when the mean flux of defects can start deviating from the zero. At that time we can be sure that the order parameter has acquired a formula that corresponds to the external $E_T(\bar{t})$ and any changes it might endure afterwards, will have to balance the gradient with the potential terms in its associated energy. The latter, thus, will not have to follow the changes in $E_T$ and therefore let $\varphi$ evolve by itself balancing or re-adjusting the gradient with the potential contributions to the total energy.

For the experiments of non-symmetric annular Nb/Al-AlO$_x$/Nb JTJs were fabricated with a whole-wafer process in which the junctions are formed in the window of the SiO insulating layer between the base and the counter electrodes. The ring-shaped junctions had a mean radius $\bar{r} \approx 80 \mu m$, width $\Delta r = 4 \mu m$ and a geometry in which both the base and top electrode have a hole concentric to the ring. The junction temperature was raised above its critical value ($T_c \simeq 9.2 K$) by means of a heating resistor placed close to the (unbiased) sample; then the temperature was lowered by letting the sample cool down by exchanging heat with the liquid helium bath through some helium gas. By changing the helium gas pressure it was possible to vary the thermal constant and hence the quench time $\tau_Q$ in the range $10^{-2}$ s-1s. The samples were measured in a very well electrically and magnetically shielded environment. The number of fluxons trapped during the transition was determined simply by feeding a proper current to the junction and measuring its voltage.

Two different samples were produced, both with $1 - T_{c1}/T_{c2} \simeq 0.02$. For sample B, we estimated $\xi_0 = 6.5 \mu m$, $\tilde{c}_0 = 10^7 m/s$ and $\tau_0 = 0.65 ps$.

With $C \sim \bar{\lambda}_J$ we cannot use (5.66) as it stands. However, we would expect to see a fluxon several percent of the time. In practice (invariably single) fluxons formed once every 10-20 times.

In Sample A, having a similar circumference and similar superconductors coupled differently, whereby $\bar{\lambda}_J$ was 2.5 times larger, there was much less fluxon
trapping, with no reliable fluxons seen in several quenches (enough to have seen something in the former case). This demonstrates that the superconductors cannot be treated individually for fluxon production, as our earlier discussion confirms. We are imprecise in the quantitative details since the aim was not to count the frequency with which defects occurred, but just to have defects at all. Nonetheless, it is apparent that, in order to have $\tilde{\lambda}_J \ll C$ for annular JTJs of a size comparable to the one above, we must find ways to reduce $\tilde{\lambda}_J$ further, as well as possibly increase $C$.

The quench time $\tau_Q$ is difficult to establish accurately. Fortunately, the small power of $\tau_Q/\tau_0$ in (5.65) makes the uncertainty largely irrelevant. However, it has the consequence that a huge reduction in the quench time is necessary to have any observable effect on $\tilde{\lambda}_J$. Similarly, we can gain little from increasing the asymmetry of the JTJ, even if that is easily possible experimentally. In consequence, the only realistic way to reduce $\tilde{\lambda}_J$ is by reducing $\xi_0$ of (5.60). Primarily, we need to increase $J_c(0)$ to the largest possible value which does not degrade the barrier quality, $J_c \simeq 10^4 A/cm^2$. New experiments with such values are under active consideration at this time.

Although the absence of fluxons in our Sample A, with larger $\tilde{\lambda}_J/C$, suggests that the observed fluxons are not artefacts of the heating and cooling environment, we might be concerned at the effects of temperature inhomogeneities.

The critical slowing down of $c_0(t)$ weakens the effect of any large scale inhomogeneities in temperature, even though it is modified slightly for idealized non-symmetric JTJs. It is difficult to put quantitative limits on the permissible inhomogeneities but, with empirically comparable $\xi_0$, $\tau_0$, and $\tau_Q$, the situation is no better of worse for JTJs than for any other superconducting system undergoing a mechanical quench. A possible check of these ideas could be performed by cooling with a temperature gradient across the ring circumference, although this would cause its own experimental problems. We postpone such considerations to the far future.
5.8.1 How good is the approximation for the Swihart velocity?

From the relation (5.42) and the values for the $\tilde{t}$, $\tilde{\xi}$ and $\tau_Q$ above we can check the approximation for the Swihart velocity. We find that the temperature $\tilde{T}$ at $t = \tilde{t}$ should be $(1 - 10^{-6})$ times smaller than $T = T_c$ which can safely be taken to be $\tilde{T} = T_c$. Actually, whereas the non symmetric case the correlation length is much more encouraging, the toll comes in the form of $\tilde{t}$ which becomes smaller too. The Swihart approximation of low losses brakes down for temperatures so close to the transition temperature $T_c$, unless the value of $\omega$ is taken to be close to zero. Our calculation has to be redone with a revised formula for the velocity.

Moreover, one should be quite careful here, since the expression for the magnetic thickness should not, in principle, be the same with that of the London penetration depth (for $\lambda >> t_{\omega c}$). What determines the magnetic penetration depth is the relationship between $\Xi_0$ and $\lambda(T)$ with $\Xi_0$ the ‘size’ of the Cooper pair, otherwise called the coherence length, and $\lambda(T)$ the generalized magnetic penetration depth. For $T \rightarrow T_c$ the relation

$$\lambda(T) > \Xi_0$$  \hspace{1cm} (5.67)

holds for all superconductors and so the actual magnetic penetration depth is given by the London formula we have adopted. However, for lower temperatures there could be deviations, for the relation (5.67) is not necessarily true. In fact, pure superconductors generally have $\Xi_0 > \lambda(T)$ but the niobium type that has been used in the experiments is an exception. Therefore, we are quite confident that the London length describes the penetration depth for the temperatures and the materials we have in mind. However, one should be careful in applying our results in different types of situations.

In order to derive the Josephson equations we made sure that, no matter how close to the transition we are, we can always draw a closed contour that
stays inside the superconductors but is outside the magnetic thickness. In other
words, our contour should have a width \( w \) which should satisfy the relation:

\[
t_{ox} + l_1 + l_2 > w > t_{ox} + \lambda_1 + \lambda_2
\]

with \( \lambda_r \) the penetration depths. With \( l_r \to \infty \) we can always be sure that this
equation holds. Alternatively, we have to introduce the reduced penetration
depth \( d_e = t_{ox} + \lambda_1 \tanh \frac{l_1}{\lambda_1} + \lambda_2 \tanh \frac{l_2}{\lambda_2} \) as in (5.56). Thus, the relevant relation
for finite electrodes becomes:

\[
t_{ox} + l_1 + l_2 > w > t_{ox} + \lambda_1 \tanh \frac{l_1}{\lambda_1} + \lambda_2 \tanh \frac{l_2}{\lambda_2}
\]

In case \( \lambda_r^{-1} \to 0 \) then the equation for the propagation velocity becomes:

\[
c_0 = \frac{\tilde{c}_0 \sqrt{2}}{\sqrt{f_1 + \sqrt{f_1^2 + f_2^2}}} \tag{5.68}
\]

with \( f_1, f_2 \) being functions of the oxide thickness, the conductivity of the superconductors and their thickness. The calculation here will be done assuming identical electrodes.

The expressions for \( f_1 \) and \( f_2 \) are[2]:

\[
f_1(\delta, t_{ox}, l) = 1 + \frac{\delta}{\sqrt{2t_{ox}}} \left( \frac{\sinh \left( 2^{\frac{1}{2}} \left( \frac{1}{3} \right) \right) - \sin \left( 2^{\frac{1}{2}} \left( \frac{1}{3} \right) \right)}{\cosh \left( 2^{\frac{1}{2}} \left( \frac{1}{3} \right) \right) - \cos \left( 2^{\frac{1}{2}} \left( \frac{1}{3} \right) \right)} \right) \tag{5.69}
\]

and

\[
f_2(\delta, t_{ox}, l) = \frac{\delta}{\sqrt{2t_{ox}}} \left( \frac{\sin \left( 2^{\frac{1}{2}} \left( \frac{1}{3} \right) \right) + \sin \left( 2^{\frac{1}{2}} \left( \frac{1}{3} \right) \right)}{\cosh \left( 2^{\frac{1}{2}} \left( \frac{1}{3} \right) \right) - \cos \left( 2^{\frac{1}{2}} \left( \frac{1}{3} \right) \right)} \right) \tag{5.70}
\]

with \( \delta = \frac{2}{\sqrt{\omega \mu_0 g}} \) where \( \sigma \) is the conductance of the normal electrons in the superconductor and \( \omega \) the frequency of the electromagnetic wave. We will take that \( \delta = \frac{2}{\sqrt{\omega \mu_0 \sigma_n}} \left( \frac{T}{T_c} \right)^2 \) with \( \sigma_n \) the conductance of the normal electrons in the normal metal which will be assumed a constant. That is true for most metals but for a short temperature range \( \delta T \) with \( T \epsilon U(T_c, \delta T) \). We will be using the
estimations $\sigma_n \simeq 10^9 (\Omega m)^{-1}$, $T_c \simeq 10 K$ and $\omega \simeq 10^{10} Hz$ a time independent parameter. The equation for $\sigma$ is just a result of the Gorter and Casimir two fluid model for a superconductor according to which

$$\sigma = \left(\frac{T}{T_c}\right)4\sigma_n$$

for temperatures in the above temperature range. This is an empirical law and even though it is not strictly accurate, it does give a qualitative description of the facts.

We will call $\delta_0 = \frac{2}{\sqrt{\mu_0 \sigma_n}}$ which is in the range of $\mu$ meters. The time dependence of $\delta = \delta_0 \left(\frac{T}{T_c}\right)^2$ comes from the expression

$$\frac{T}{T_c} = 1 - \frac{t}{\tau_Q}$$

which gives for $t \to 0$ an approximated expression as

$$\delta \approx \delta_0 + \delta_1 t$$

(5.71)

with $\delta_1 = \frac{2\delta_0}{\tau_Q}$. As temperature increases then $\delta$ must decrease. Equivalently, at time decreases (we go backwards in time) $\delta$ decreases. We are going to calculate the Swihart velocity at $t = 0$ when $\delta = \delta_0$. We see that it is finite of the order of $0.04\tilde{c}_0$ assuming $t_{ox} = O(10^{-9}) m$ and $l = O(10^{-6}) m$ meters. This can indicate that the order parameter in the Kibble-Zurek picture can have long-range interactions something that alludes to a non random selection of phase in space and consequently a different picture. We will assume that since for the study of losses on has to base the theory on a modified Lagrangian, we can be sure that our results hold so long as the junction can be described by a non dissipative field equation.

In general, the equation for the Swihart velocity is a function of $h_\pm = \sqrt{\lambda^{-4} + 16\delta^{-4}} \pm \lambda^{-2}[2]$. We see that both $\lambda$ and $\delta$ are functions of temperature
and at $T \to T_c$, $\lambda$ goes to infinity whereas $\delta$ says finite. The converse ($\lambda$ being finite and $\delta$ going to infinity) happens when $T \to 0$. The equation for the Swihart velocity (5.41) we used in our original calculation assumed that $h_+ \simeq 2\lambda^{-2}$ whereas $h_- \simeq 0$. We see that those expressions for $h_\pm$ can be recovered not only when $T \to 0$ but also when

$$b^4 \lambda^{-4}(\bar{T}) = 16\delta^{-4}(\bar{T})$$

(5.72)

for some temperature $\bar{T}$ and a parameter $b < 1$. We can be confident that our calculation holds when the equation (5.72) can be satisfied for some temperature $\bar{T} \to T_c$. For example, when the frequency $\omega$ is very small, $\delta$ goes to infinity and $b$ is zero. Thus, we can recover the form (5.41) we used. However, when $\lambda$ goes to infinity, we have to consider the (5.68) formula for the velocity whereby $h_+ \simeq h_- \simeq 4\delta^{-2}$. The critical slowing down in (5.68) will be exhibited when $\delta \to \infty$.

To see this consider the lim $f_1$ and lim $f_2$ when $\delta \to \infty$. It is easy to see that the velocity will go as

$$c_0 = 2\bar{c}_0 \sqrt{\frac{l u_{\alpha \tau}}{\delta^2}}$$

which becomes zero when $\delta \to \infty$. The latter will be the limit of low frequency or low conductivity so that any temperature dependence to be due to the varying $\lambda$. From (5.72) we find that

$$\bar{T} = T_c (1 - \frac{1}{1 + (\frac{b\delta}{2\lambda_0})^2})^{1/4}$$

which can satisfy the condition that $\bar{T} \to T_c$ when $\delta_0$ is very large and $\lambda_0$ is very small. In that situation, $(\frac{b\delta_0}{2\lambda_0})^2 >> 1$ for some appropriate $b$ and $\bar{T} \to T_c$. The $\lambda_0 = \lambda(0)$. 

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Losses, the Swihart velocity and beyond

So far we assumed that our fluxon propagates without any dissipation. However, this is an idealized approach and should be improved to accommodate more realistic features of the junction. In practical devices the effect of ohmic losses in the dielectric barrier and of the superconductors’ surface impedance is a viscous damping which slows down the fluxons and eventually stops them, if no forcing mechanism is present on the line. Such forcing mechanism is usually provided by an externally applied magnetic field or bias current which can accelerate the fluxons by means of the Lorenz force acting at right angles to the directions of the bias current and of the magnetic flux. The Lorenz force is between the superelectrons carrying the transport current and the vortices generating the magnetic flux.

Being the damping proportional to the fluxon speed, for a given value of the forcing term, a steady velocity is reached in which the power losses are balanced by the externally supplied power: the larger the force the higher the speed. However, because the current carried by a Josephson transmission line is limited by its critical current density $J_c$, the fluxon speed $u$ is prevented from reaching its limiting value $c_0$. Consequently, the kinetic energy of a fluxon moving on a Josephson transmission line in the presence of bias and losses is limited.

To study the losses we have to introduce a more general formula for the current density. This is given by\[40\] the expression

$$J = J_c(V)\sin\varphi + \left(\frac{1}{R_1(V)}\cos\varphi + \frac{1}{R_0(V)}\right)V$$

which incorporates the interactions of the cooper pairs (second term) and the ohmic current contributions (last term). Usually, the second term is much smaller than the third and is being neglected. In that situation, the resistivity $R_0$ and the critical current density $J_c$ are not functions of the voltage $V$. 

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The sine-Gordon equation that derives from equation (5.73) by means of the Maxwell equations, as shown earlier, will have the form

\[
\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{\lambda_J^2} \sin \varphi = \mu_0 d \left( \frac{1}{R_0} \right) \frac{\partial \varphi}{\partial t} - \epsilon R_s \frac{\partial^3 \varphi}{\partial t \partial x^2} \tag{5.74}
\]

We can introduce the dimensionless quantities \( \alpha = \frac{k_{\text{B}}}{\lambda_J R_0} \) and \( \beta = \epsilon R_s \Omega_J \) and measuring space and time in terms of the Josephson length and the inverse frequency \( \Omega_J \) given by equation (5.35), we can write the sine-Gordon in the form

\[
\varphi_{XX} - \varphi_{TT} - \sin \varphi = \alpha \varphi_T - \beta \varphi_{XT} - \gamma \tag{5.75}
\]

where \( X = x/\lambda_J , T = t \Omega_J \) and \( \gamma = \frac{J_{\text{ex}}}{4} \) a parameter that represents the external bias due to the applied current \( J_{\text{ex}} \). The resistivity \( R_s \) corresponds to the surrounding currents generated inside the London area of each superconductor.

We see that in the absence, for example, of surrounding currents, we can call our relevant Zurek length as \( \lambda_J/\alpha \) and the relaxation time as \( \Omega_J \alpha \) with \( \alpha \) the earlier introduced quantity that depends on \( R_0 \). With changing temperature the ohmic losses may reduce due to the lower energy of the moving electrons. Thus, one could derive a very different picture about the feasibility of the Kibble-Zurek mechanism in junctions where losses are present and affect the field behavior. More investigation on this has to take place.

### 5.10 Field theoretic approach to the sine-Gordon model

First of all, the approach we presented earlier had the major assumption in it that the relevant time for the transition from the ordered to the disordered phase \( (t \text{ decreases}) \) to take place is the temperature when the metals become superconductors. However, this is not necessarily true and one should be able to calculate the temperature when the sine-Gordon model exhibits a phase
transition. In principle, one should be anticipating that the general expression for the time dependence of the temperature to be

\[ 1 - \frac{T}{T_{c,\text{JTJ}}} = \frac{t}{\tau_Q} - \epsilon_0 \]

with \( T_{c,\text{JTJ}} \) the temperature of the \( JTJ \) phase transition. At \( T = T_c = T_{c,\text{sc}} \) the temperature of the superconducting phase transition our quench starts. Fluxons will be created only after a temperature \( T_{c,\text{JTJ}} \) which must be smaller than \( T_c \).

Suppose that \( 1 + \epsilon_0 = O(1) \) then, our calculations earlier will be valid having in mind now that \( T_c \) should be substituted with \( T_{c,\text{JTJ}} \). The time \( t \) we found must now be inserted in the new expression for \( T(t) \). Thus, the relevant temperature for the earliest stabilization of fluxons will be \(((1 + \epsilon_0) - \frac{t}{\tau_Q})T_{c,\text{JTJ}}\) which should be less than \( T_c \) by a higher factor than the one calculated earlier. Therefore, our approximation of the Swihart velocity might still hold.

However, we have little leads to calculate the \( T_{c,\text{JTJ}} \) and the only rescue can come from a field theoretic approach. However, the uncertainty in the accurateness of a given Lagrangian for various junctions and different conditions makes the situation more complicated. Here we will briefly talk about a method that has been implemented and which depicts how a phase transition can take place in the sine-Gordon model.

The original Hamiltonian density of (5.45) can be written as \( H_0 + H_{\text{int}} \) with

\[ H_0 = \frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}(\partial_x \varphi)^2 + \frac{m^2}{2} \varphi^2 \]

and

\[ H_1 = -\cos \varphi - \frac{m^2}{2} \varphi^2 \]

where \( \tilde{\pi} \propto \partial_x \varphi \) is the canonical momentum of the \( \varphi \) field, which satisfies the quantum mechanical commutation rules

\[ [\tilde{\pi}, \varphi] = -i\delta_{x,x'} \]

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Coleman attempted to find a good ground state by expanding $\varphi$ into free particles of momentum $k$ with an unknown trial mass $m$, as above.

Thus,

$$\varphi = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi\sqrt{2k_0}} [e^{ik_1x}a(k_1, m) + e^{-ik_1x}a^+(k_1, m)]$$

where $k_1$ are the spatial momenta and $k_0 = \sqrt{k_1^2 + m^2}$. The commutation relation for $a(k_1, m)$ and $a^+(k_1, m)$ is

$$[a(k_1, m), a^+(k', m)] = 2\pi \delta^4(k_1 - k'_1)$$

Kleinert[39] has shown that for an infinitesimally fine spatial lattice of spacing $a \to 0$ where the mass of the free particles is very small compared to $1/a^2$, the ground state defined by the equation

$$a(k_1, m)\mid_0 \geq 0$$

can correspond to a finite energy only when

$$\beta = \frac{1}{T} < \frac{2}{\pi}$$

If $\beta$ exceeds this value, the sine-Gordon model has no ground state and thus no sensible continuum limit[39].

**Short summary and further motivation**

Lossless Josephson Tunnel Junctions can easily incorporate the Kibble-Zurek framework for defect creation and evolution. One can identify the necessary quantities like the correlation length and the relaxation time for the order parameter and can appropriately adapt the qualitative causal considerations. However, losses can play a very important role in the field evolution in the junction and must be incorporated. Moreover, the existence of a phase transition in our circumstances must be checked and one has to check whether the
existence of fluxons should be attributed to the Kibble-Zurek mechanism.

Recently, an experiment was performed[38] which showed promising results. There was clear evidence that i) the trapping of a fluxon occurs on a purely statistical basis for identical conditions of each thermal cycle and ii) the probability to trap one fluxon is larger when the transition is performed at a faster speed (smaller quenching time) in accordance with the causality principle. However, the Kibble-Zurek mechanism could be responsible for those results but the converse is rather more dubious. Therefore, one is not certain that the results themselves can unambiguously justify the assumption that Kibble-Zurek is relevant here.
Chapter 6

About some homotopic future

6.1 Past and Present

This thesis was about various theoretical tests to the current picture of defect formation. We considered the standard Kibble-Zurek approach where topological defects are the result of the competition between long and short range order. Ideas like the freezing in of the correlation length and the relaxation time for the order parameter allude to a situation where the order parameter follows its own evolutionary procedure which is, though, subject to various environmental constraints. The domain picture itself allows for the postulation that not all areas in our medium are equivalent if we are to impose continuity to our order parameter.

We saw that the Kibble-Zurek picture can be reproduced by the dynamics of a gaussian field in the Landau-Ginzburg theory for second order phase transitions. We also proved that there are various length scales one could associate with defect density calculations depending on the strength of short and long wavelength fluctuations. Assuming that they can affect the end result, we found that the larger the scale at which we look the smaller the defect density we will observe.

Given the apparent failure of the so far conducted experiments to provide any watertight arguments in favor or against the current picture, we proposed
a new system which looks remarkably similar to the ones we investigated in the beginning. However, the issue of whether the dynamics of the order parameter evolution should determine if the system really is like the ones we investigated earlier remains unanswered. Nonetheless, Josephson tunnel Junctions look quite promising.

6.2 Future Work

It seems to us that the current picture affords a lot of investigation. First of all, the picture of domains is rather unclear. One does not know whether they are a result of the phase transition, in which case they are imposed on our system, or whether they were its generic characteristic in the first place. The former is what one usually assumes but there is then the problem of defining domains in a continuous order parameter space. It is common place to define the domains after the order parameter has acquired a certain configuration over them\(^1\). If the domains can be identified only after the field has acquired some configuration over our physical space, then one can see that they cannot be relevant to the occurrence of the relevant \(\varphi\) since they come after it has taken place, for any reasons of its own. Then, the domains describe the field or can help formulate a certain evolution for it rather than explain why the order parameter took the formula it took.

We believe that the domains have to come after the acquisition of a certain order parameter unless they are a generic characteristic of the physical space we are considering. In non “Big Bang” models where the Universe, for example, could be the union of different expanding domains, one could create defect creation scenarios that seem more compatible to the original Kibble postulations.

Alternatively, the occurrence of some order parameter must be dictated by, say, the energy of the system. Thus, \(\varphi\) is there because there exists enough

\(^1\)One might think of numerical simulations where this seems unapplicable. We will assume that in numerical simulations the a priori existence of domains has no generic value to the framework considered
energy to accommodate it, regardless of the existence or not of domains in its configuration. The latter will be the consequence of the energetic conditions.

Usually the domain evolution is thought of as a result of the fluctuations in the order parameter. However, if there is some generic topological reason for their existence then it could be that once the domains have been identified, they might follow a certain prescribed evolutionary procedure, regardless of $\varphi$. The Kibble-Zurek scenario is about systems that can incorporate defects, something that can be checked topologically, and talks about the way one can direct the evolution of a certain order parameter to incorporate defects at the end of the process.

So far, we found that there could be systems where the space of the minima is non trivial but, the topology of the physical space does not allow the occurrence of defects. We showed that the characteristics of the physical space and the sensitivity of the order parameter on them can play a major role in defect generation or annihilation. Higher dimensions than three to the physical manifold $A$ can be considered and, for an appropriate manifold of minima $S$, one can be sure of the feasibility of defects even if the dimensions of $A$ change but $S$ remains the same.

It is very interesting to explore order parameter configurations where the energetic behavior is trapped inside involutes of circles, generalized conchoids, curtate, prolate or common cycloids etc. Most probably (taking that the Euler-Lagrange equations, for instance, should prescribe the physical validity of any order parameter), in Euclidean space-time such behavior can be energetically forbidding. However, one could see whether premises exist that can allow their occurrence in more structured manifolds.

As far as the Kibble-Zurek Mechanism is concerned, we found that it can be applied in various systems and that it can lead to interesting results for the observed density of the relevant defect configurations. Our future goal is to formulate the qualitative framework we have in mind. In particular, consider an
order parameter $\varphi$, just below the transition temperature $T_c$ (for a temperature quench). We, also, do not ask the question of why the order parameter occurred. We say that the absorption or release of energy can modify the acquired order parameter. Moreover, we will start with an order parameter that has some non-trivial topological attributes which we would like to maintain.

Our evolutionary process will be based on the interplay between the assumed intrinsic tendency of $\varphi$ to acquire the lowest possible energy configuration, at the given temperature, and the external interference that changes the energy of the minima and consequently can increase the energy cost of any changes in $\varphi$. If one takes that the order parameter cannot absorb more energy than say some $O(K_B T)$, for any $T$, then making the cost in energy during the evolution increase while the energy the order parameter can use to continue the evolution drops, one can explain why that process stops and can, in principle, make $\varphi$ what she wishes.

As we mentioned, a defect can happen inside a certain spatial area or on a single closed contour over a certain ‘time’ interval in order to trivialize the homotopy class the corresponding loop in the order parameter space this contour belongs to. This can be most simply utilized to explain the Ginzburg postulation. The latter talks about the transition of a certain order parameter from the one minimal solution to another. We will see, that if this transition takes place on the same spatial area we will eliminate the possibility of a ‘time’-independent defect (which is anyway a transition between two minimal solutions) somewhere.

The simplest case where defects could be relevant is that of the one dimensional kink solution.

The relevant potential has the form:

$$V = \frac{1}{2} \lambda (\varphi^2 - \frac{1}{2} \eta^2)^2$$

with $\varphi$, $\eta$ real functions. To make the matter even simpler we will consider $\eta = \sqrt{2} a$ and $\varphi = \phi a$. Thus,
\[
\frac{V}{\lambda a^4} = \frac{1}{2}(\phi^2 - 1)^2
\]

We will consider three domains of equal size $2\xi$, on the real line. The domain $l$ will correspond to $-3\xi \leq x \leq -\xi$, the $m$ domain to $-\xi \leq x \leq \xi$ and the $r$ domain to $\xi \leq x \leq 3\xi$. The order parameter will take values on the $l$ and $r$ domains first, as if the $m$ domain does not correspond to real space yet. At $l$ and $r$ our function takes the values $-1$ and $1$ respectively since we want to create a non trivial situation and moreover, we only have two minimal values for our order parameter. We, first, are going to calculate how much energy is required for the domain $r$ to change its order parameter to the $-1$ value so as to make the occurrence of a discontinuity within the $m$ domain impossible. We will assume that the $2\xi$ magnitude is the largest distance at the commencement of the correlation between the $l$ and $r$ domains.

Assume that the homotopy $h_t = (1 - 2t)$ takes the order parameter at the $r$ domain from the value of $1$ to that of $-1$, with $t$ taking values from $0$ to $1$. Surely, we could consider many other types of transformations but, being faithful to our earlier attitude, we will keep matters simple. During that homotopy the potential becomes

\[
V(t) = \frac{1}{2} \lambda a^4 (\phi^2 - 1)^2 = 8\lambda a^4 t^2 (t - 1)^2
\]

and thus

\[
\frac{V(t)}{\lambda a^4} = 8t^2(t - 1)^2
\]

To keep the dimensions right we will assume that the $a\sqrt{\lambda}$ should be dimensionless in case $x$ is also dimensionless. Therefore only $a^2$ can carry energy dimensions.

As we see, the $\frac{V(t)}{\lambda a^4}$ function increases steadily from zero at $t = t_i = 0$ to $\frac{1}{2}$ at $t = t_m = 0.5$ and then drops to zero again at $t = t_f = 1$, when the
transition to the $\phi = -1$ state has been achieved. We will postulate that the increase in $\frac{V(t)}{\lambda a^4}$ signifies that indeed energy is required for that transformation. The required amount should be $\frac{1}{2}\lambda a^4(2\xi)$ which actually is the energy at the maximum of the potential multiplied by the magnitude of the domain. Since, no matter what kind of homotopy we will choose, $\frac{V(t)}{\lambda a^4}$ will always take the $\frac{1}{2}$ as its maximum value, the required amount we calculated above is universal. However, the associated value of $t_m$, the parameter of our homotopy, when this happens can vary.

The Ginzburg postulation basically argues that if we take the $\frac{1}{2}\lambda a^4(2\xi)$ to be comparable to thermal energy $K_BT_G$ then the defect is not well defined. Clearly, that is true here, even though we did not specify which the possible source for energy might be. Thus, in case of thermal activation, the condition that should be satisfied, is $K_BT_G < 2\xi V(\phi = 0) = \lambda a^4\xi$ if we are to identify any stable defect securely.

Such simple considerations can be employed to produce a self-consistent framework for the maintenance of defects in an order parameter. We hope to be able to explore those possibilities in greater detail in the future.


