Answer ALL parts of Section A and TWO questions from Section B
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
1. (i) A train of length 160 m has stopped because of a signal. The signal is 10 m ahead of the front of the train. When the signal changes to green the train accelerates from rest with a steady acceleration of 0.9 m s$^{-2}$. How long does it take for the train to pass the signal? 

(ii) Objects moving at high speeds through fluids experience a drag force of magnitude $D = k_d v^2$, where $k_d$ is a positive constant. Show that the terminal velocity of a falling object of mass $m$ is given by $v_t = \sqrt{mg/k_d}$.

For a sphere of radius $r$ in air $k_d = 0.56 r^2$ kg m$^{-1}$. Calculate the terminal velocity of a steel sphere of radius 5 mm in air.

[Density of steel $= 7.8 \times 10^3$ kg m$^{-3}$] 

(iii) A man of mass 80 kg stands at one end of a plank of length 3 m. The mass of the plank is 50 kg. It is at rest on a frozen pond, and the friction between the plank and the ice is negligible. The man walks to the other end of the plank. How far does he move relative to the pond? 

(iv) A flywheel used for energy storage has a radius of 0.5 m. Calculate the speed of a point on the rim when it is rotating at 2000 revolutions per minute.

The flywheel delivers 250 kW of power when the rotation rate is reduced from 2000 revolutions per minute to 500 revolutions per minute in 5 s. Find the moment of inertia of the flywheel.

(v) Show that a planet following a circular orbit of radius $R$ around a star of mass $M$ has an orbital period given by

$$T = \frac{2\pi R^{3/2}}{(GM)^{1/2}}$$

where $G$ is the gravitational constant.

Calculate the orbital period of Venus, assuming that it follows a circular orbit of radius $1.08 \times 10^{11}$ m around the Sun (mass $1.99 \times 10^{30}$ kg).

[Total 27 marks]
2. A particle of mass $m$ is moving along the positive $x$ direction with momentum $p$ and energy $E = \sqrt{p^2c^2 + m^2c^4}$. It collides with a particle of the same mass at rest to form a new particle of mass $M$.

(i) Write down the equations for conservation of energy and momentum in the collision. [2 marks]

(ii) Show that

$$M^2 = \frac{2m}{c^2}(E + mc^2)$$

[4 marks]

(iii) (a) Show that $M = 2m$ in the limit $mc^2 \gg pc$. [1 mark]

(b) Explain why this is what you would expect from Newtonian mechanics. [2 marks]

[Total 9 marks]
SECTION B

3. (i) A particle of mass $m_A$, initially moving with a velocity $u_A$ in the $x$ direction, has a 1-D elastic collision with a particle of mass $m_B$, initially moving with a velocity $u_B$ in the $x$ direction. Show that after the collision the particle velocities (still in the $x$ direction) are given by:

$$v_A = \frac{(m_A - m_B)u_A + 2m_B u_B}{m_A + m_B}$$
$$v_B = \frac{(m_B - m_A)u_B + 2m_A u_A}{m_A + m_B}$$

[9 marks]

(ii) Using the equations obtained in part (i), and setting $u_A = u^*$ and $u_B = -u$, show that if $m_A \gg m_B$, then (to a very good approximation) $v_A = u^*$ and $v_B = u + 2u^*$. This result appears to suggest that momentum is not conserved in this type of collision. Briefly comment on this conclusion. [6 marks]

(iii) A ball of mass $m$ is released from rest and falls under gravity. After it has fallen a distance $d$ it has a 1-D (in the vertical direction) elastic collision with a ball of mass $M$ ($\gg m$) which is moving upwards with a speed $u^*$. Ignoring air resistance, show that the maximum height to which the ball of mass $m$ rises above the point of impact is given by

$$h = d + \frac{2u^*}{g} \left( \sqrt{2gd + u^*} \right).$$

[6 marks]

(iv) Suppose that the ball of mass $M$, referred to in part (iii), had itself fallen from rest from the same point as the ball of mass $m$, and had then undergone a 1-D (vertical) elastic collision with the ground. It was as a result of the collision with the ground that it was moving upwards with speed $u$ at the time of the collision between the two balls. Using the expression obtained in part (iii), or otherwise, show that in this case the maximum height to which the ball of mass $m$ rises above the point of impact is $h = 9d$.

[5 marks]
(v) Three balls are released from rest, one above another, from a height of 2 m. They are not joined but the separation between adjacent balls is very small and they fall together. After the lowest one (number 3) has hit the ground it starts to move upwards. It then collides with number 2, which also starts to move upwards, which then collides with number 1, which also starts to move upwards. Calculate the maximum height reached by number 1, making the following simplifying assumptions:

- all the collisions (including that of number 3 with the ground) are 1-D (vertical) and elastic,
- air resistance is negligible,
- \( m_3 \gg m_2 \gg m_1 \),
- the separation of 1 and 3 at the time of the last collision is very much smaller than the distance fallen (i.e., you can assume that all the balls have fallen a distance 2 m when they collide).

[6 marks]

[Total 32 marks]
4. (i) Rocket propulsion for spacecraft works by expelling some of the mass (in the form of burned fuel) backwards at a speed $v_e$ (the ‘exhaust velocity’) with respect to the spacecraft. Show that if the mass and velocity of the spacecraft plus its unburned fuel change by $dm$ and $dv$ respectively (where $dm < 0$ and $dv > 0$) then the total momentum change is given by

$$dp = mdv + v_e dm$$

where $m$ is the mass of the spacecraft plus its unburned fuel. [6 marks]

(ii) Manoeuvres in space are achieved by changing the spacecraft’s velocity, and are characterized by $\Delta v = v_{final} - v_{initial}$. Assuming that the spacecraft is in a region of outer space where there are no external forces, and that $v_e$ is constant, show that

$$\Delta v = v_e \ln \left( \frac{m_{init}}{m_{final}} \right)$$

where $m_{init}$ and $m_{final}$ are the masses of spacecraft plus unburned fuel before and after the manoeuvre respectively. [6 marks]

(iii) A spacecraft in outer space has a total mass of $4.0 \times 10^4$ kg. This total mass includes $3.5 \times 10^4$ kg of fuel which is expelled at a rate of $100$ kg s$^{-1}$ and an exhaust velocity of $2.0 \times 10^3$ m s$^{-1}$. The spacecraft has an initial velocity of $0.5$ km s$^{-1}$. The engine is then fired for 1 minute. Calculate the final speed. [4 marks]

(iv) When a rocket is launched from the surface of a planet, gravity must be overcome. Show that the acceleration upwards of the rocket on launch is given by

$$\frac{dv}{dt} = -\frac{v_e}{m} \frac{dm}{dt} - g$$

where $g$ is the acceleration due to gravity at the planet’s surface and $\frac{dm}{dt}$ is the rate at which fuel is burnt. [Note: $\frac{dm}{dt} < 0$.]

Hence calculate the maximum mass that could be launched from the Earth with a rocket for which $v_e = 2.0 \times 10^3$ m s$^{-1}$ and $|dm/dt| = 1.5 \times 10^4$ kg s$^{-1}$. [6 marks]

(v) Integrate the equation found in part (iv) to show that at time $t$ after the start of the launch the upward speed of the rocket is given by

$$v(t) = v_e \ln \left( \frac{m_{init}}{m(t)} \right) - gt$$

where $m_{init}$ is the total mass of the rocket and fuel before launch, $m(t)$ is the mass of rocket plus unburnt fuel at time $t$, and $g$ is assumed to be constant over the time of interest. [5 marks]
(vi) A rocket has \( v_e = 2.0 \times 10^3 \text{ m s}^{-1} \), \( m_{\text{init}} = 10^6 \text{ kg} \), and \( |dm/dt| = 1.5 \times 10^4 \text{ kg s}^{-1} \). Calculate its speed 20 s after the engines have been fired if it is

(a) launched from the Earth,
(b) launched from Mars.

[Mass of Mars = 6.42 \times 10^{23} \text{ kg}, radius of Mars = 3.40 \times 10^6 \text{ m}.]

[5 marks]

[Total 32 marks]
5. (i) By considering a rigid body of mass $M$ to be composed of a large number of small particles, show that the total torque (about the origin) exerted by gravity on the rigid body is given by

$$\tau_{\text{tot}}^{\text{grav}} = \mathbf{r}_{CM} \times (Mg)$$

where $\mathbf{r}_{CM}$ is the position vector of the centre of mass of the rigid body, and $g$ is the acceleration due to gravity.

Under what circumstance would this expression be inaccurate? [6 marks]

(ii) Find an expression for the moment of inertia of a uniform rod about an axis perpendicular to the length of the rod through its centre of mass, in terms of $M$, its mass, and $l$, its length.

By using the parallel axis theorem, or otherwise, show that the moment of inertia of the rod about one end is given by

$$I_{\text{end}} = \frac{Ml^2}{3}.$$ [8 marks]

(iii) A uniform rod of length $l$, pivotted freely at one end, is initially held in a vertical alignment with the pivot at the bottom. It is given a small displacement and allowed to fall under gravity. By considering the torque exerted by gravity on the rod, and assuming that friction and air resistance are negligible, show that the magnitude of the angular acceleration of the rod when it is at angle $\theta$ to the vertical is given by

$$\alpha = \frac{3g \sin \theta}{2l}.$$ [6 marks]

(iv) Using conservation of energy, show that the magnitude of the angular velocity of the falling rod when it is at an angle $\theta$ to the vertical is given by

$$\omega = \sqrt{\frac{3gl}{l}(1 - \cos \theta)}.$$ [6 marks]

(v) An initially vertical uniform rod of length 1.5 m, pivotted at the bottom, is given a small displacement and allowed to fall. Calculate the acceleration and speed of the upper end of the rod just before it hits the floor. [6 marks]

[Total 32 marks]
6. The Lorentz transformations for space and time coordinates can be written as

\[ ct' = \gamma (ct - \beta x), \quad x' = \gamma (x - \beta ct), \quad y' = y, \quad z' = z \]

with \( \gamma = 1/\sqrt{1 - \beta^2} \) and \( \beta = u/c \).

(i) Briefly describe under what conditions this is correct. [5 marks]

(ii) Show that under these assumptions the speed \( v' \) in reference frame \( O' \) of a body moving at speed \( v \) in reference frame \( O \) is

\[ v' = \frac{v - u}{1 - \frac{uv}{c^2}} \]

[12 marks]

(iii) A police spaceship \( P \) is chasing another spaceship \( A \). Both ships have velocities \( \beta_P = \beta_A = 3c/5 \) as measured along the \( x \) axis in the Solar System reference frame \( O \). The police ship is a distance \( L = 1 \) light-second (i.e. the distance travelled by light in one second) behind ship \( A \). What is this distance in the frame of the police ship \( O' \)? [5 marks]

(iv) The police ship fires a missile at speed \( \beta'_M = 4c/5 \) at time \( t' = 0 \) and position \( x' = 0 \) as measured in the police spaceship frame \( O' \). When and where does the missile reach ship \( A \), as measured in frame \( O \)? [10 marks]

[Total 32 marks]