B.Sc. and M.Sc. EXAMINATIONS 2008

SECOND YEAR STUDENTS OF PHYSICS
MATHEMATICS - M.PHYS 2

Date  Wednesday 4th June 2008   10.00 - 12.00 pm

Do not attempt more than FOUR questions.

A mathematical formulae sheet is provided

[Before starting, please make sure that the paper is complete; there should be 4 pages, with a total of SIX questions. Ask the invigilator for a replacement if your copy is faulty.]
1. (i) Show that a differentiable complex function satisfies the Cauchy-Riemann conditions.

(ii) Determine which of the following complex functions are differentiable:

(a) \( f(z) = \cos x + i \sin y \),

(b) \( f(z) = \frac{1}{z}, \ z \neq 0 \),

(c) \( f(z) = \cosh(z) \).

(iii) Determine an analytic function whose real part is

\[ u(x, y) = e^x \cos y. \]

(iv) Determine the Laurent series of the function

\[ f(z) = \frac{1}{z^2 (1 - z)} \text{ about } z = 0, \]
valid for \( 0 < |z| < 1 \).

2. (i) Consider the real function given by

\[ f(x) = \begin{cases} e^{-ax} & , \ x > 0 , \\ 0 & , \ x \leq 0 , \end{cases} \]
where \( a \) is a positive constant.

Calculate the Fourier transform \( \hat{f}(k) \) of \( f(x) \).

(ii) Using the residue theorem, determine the inverse transform \( \mathcal{F}^{-1} \left( \hat{f}(k) \right)(x) \) for \( x < 0 \) and \( x > 0 \).

(iii) Determine the value of \( \mathcal{F}^{-1} \left( \hat{f}(k) \right)(x) \) at \( x = 0 \).

(iv) Assume the function \( f(x) \) is given by

\[ f(x) = \int_{-\infty}^{\infty} g(x - y) h(y) \, dy \]
where \( g(x) \) and \( h(x) \) are two functions.

Show that

\[ \hat{f}(k) = \hat{g}(k) \hat{h}(k). \]
3. (i) Determine the integral $\oint_C z^n \, dz$ for all integer values of $n$. The contour $C$ is a positive oriented circle of radius $R$ and centre at the origin.

(ii) Let 

$$f(z) = \frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are analytic functions and $p(z_0) \neq 0$, $q(z_0) = q'(z_0) = 0$ and $q''(z_0) \neq 0$.

Determine the residue of $f(z)$ at $z_0$.

(iii) Consider 

$$f(z) = \frac{1}{\cosh z}.$$ 

Determine the position and order of all poles.

Determine the residues at the poles.

4. (i) State what is meant by a sequence $f_n$ in a normed vector space $S$ being

(a) strongly convergent,

(b) a Cauchy sequence.

(ii) Let $f_n(x) = e^{-nx}$ with $x \geq 0$. Show, using the norm $||g|| = \int_0^1 |g(x)| \, dx$, that $f_n(x)$ is strongly convergent towards the function $f(x) \equiv 0$.

(iii) Let 

$$x_n = \sin \left( \frac{A}{n} \right).$$

Show, using the norm $||x|| = |x|$, that $x_n$ is a Cauchy sequence in $\mathbb{R}$ for any $A \in \mathbb{R}$.

(iv) State and prove Schwarz’s inequality.
5. Consider a right-handed coordinate system \( S \) with orthogonal unit vectors \( e_1, e_2 \) and \( e_3 \) as basis.

Let \( S' \) denote a rotated frame with unit orthogonal basis \( e'_1, e'_2 \) and \( e'_3 \).

Assume

\[
e'_i = a_{ik} e_k.
\]

(i) Express the transformation matrix \( a_{ij} \) in terms of \( e'_i \) and \( e'_j \).

(ii) Let \( v \) be a 3-dimensional vector. Derive the relation between the coordinates \( v_i \) in \( S \) and \( v'_i \) in \( S' \).

(iii) Let \( \phi(x) \) be a scalar field.

Show that \( \nabla \phi \) is a tensor of rank 1.

(iv) Show that for any two vectors \( d \) and \( b \),

\[
e = d \times b
\]

transforms as a pseudo-vector.

You may use without proof that

\[
\varepsilon_{mjk} a_{ms} a_{jp} a_{kq} = \det A \varepsilon_{spq}.
\]

6. (i) Derive the Euler-Lagrange equation for

\[
J = \int_{x_0}^{x_1} f(y', y'', x) \, dx
\]

subject to fixed boundary conditions \( y(x_0) = c_1, \, y(x_1) = c_2, \, y'(x_0) = c_3 \) and \( y'(x_1) = c_4 \), where \( c_1, c_2, c_3 \) and \( c_4 \) are constants.

(ii) Determine the curve joining the points \( (0, 0) \) and \( (1, 0) \) for which the integral

\[
\int_0^1 \left( \frac{d^2 y}{dx^2} \right)^2 \, dx
\]

is a minimum subject to the boundary conditions \( y'(0) = 0 \) and \( y'(1) = 1 \).