Answer THREE questions.
All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
1. A simple model for a white dwarf is a constant-density sphere where the pressure due to degenerate electrons prevents its gravitational collapse.

(i) The equation of hydrostatic equilibrium is given by

\[ \frac{dP}{dr} = -\frac{GM_r \rho}{r^2}, \]

where \( M_r \) is the mass enclosed within radius \( r \). Show that the central pressure of a gas ball of mass \( M \), radius \( R \) and constant mass density \( \rho \) is given by

\[ P_c = \frac{3GM^2}{8\pi R^4}. \]

[4 marks]

(ii) The pressure counteracting the gravitational force is due to non-relativistic electrons and is given by

\[ P_e = \frac{\hbar^2 (3\pi^2)^{2/3} n_e^{5/3}}{5 m_e}, \]

where \( n_e \) is the electron number density and \( m_e \) is the electron mass.

Dropping numerical factors, derive a relation between the radius and the mass of a white dwarf. [6 marks]

(iii) Astronomers have observed an object that they believe to be a white dwarf with a luminosity of only \( 10^{-3} L_\odot \) (\( L_\odot = 3.8 \times 10^{26} \) W).

(a) The spectra suggest that the object has an effective temperature of about 10 000 K. Calculate the radius of the object and estimate its mass from the relationship derived in part (ii). [3 marks]

(b) The spectra of the object also show hugely broadened Balmer absorption lines. Their widths are in agreement with a (surface) gravitational acceleration of \( g = 2 \times 10^6 \) m s\(^{-2}\), i.e., about 4 orders of magnitude larger than the solar value. Use this information to obtain another estimate of the mass and compare it to that obtained previously. Comment on whether the derived masses and temperatures are typical for a white dwarf. [4 marks]

(iv) The white dwarf appears to have a strong magnetic field of the order of 5 Tesla. Given that the Sun has a mean magnetic field strength of about \( 10^{-3} \) T, check whether this field strength is compatible with the assumption that magnetic flux is conserved during the evolution from Sun-like star to white dwarf. [3 marks]

[Total 20 marks]

[You may wish to note that the solar mass and radius are \( M_\odot = 2.0 \times 10^{30} \) kg and \( R_\odot = 7.0 \times 10^8 \) m, respectively.]
2. Consider the following model for a type II (core-collapse) supernova. Assume that the progenitor has a total mass of 20 $M_\odot$ and that the iron core has mass 1.4 $M_\odot$ and a radius of about 7000 km, as would be typical for a white dwarf. The supernova gains its energy from the collapse of the core down to a typical neutron-star radius of 14 km.

(i) Provide a rough sketch of the supernova progenitor, labelling at least four of the burning zones. Comment on the timescales involved for progressive core burning stages as the star evolves towards the SN stage. [4 marks]

(ii) (a) Estimate the maximum energy, $E_{\text{max}}$, that can be liberated through core collapse.

(b) Calculate also an upper limit for the amount of energy used to eject the stellar envelope. Express your result in terms of the available collapse energy.

(Hint: You may treat all the material outside the iron core of the progenitor as the stellar envelope) [5 marks]

(iii) It turns out that most of the available energy is taken up by processes other than the ejection of the envelope. Name the TWO most important endothermic processes and describe both concisely, using less than 50 words in total. Comment also on their relative importance. [5 marks]

(iv) Assuming that the processes from (iii) above absorb an additional 94% of the available collapse energy, estimate the average luminosity for the first year after the supernova explosion.
Express this luminosity in terms of solar luminosities, $L_\odot = 3.8 \times 10^{26}$ W, and compare it to a typical luminosity of a galaxy such as the Milky Way. Sketch and briefly compare typical lightcurves for type Ia and type II supernovae. [6 marks]

[Total 20 marks]
3. Strömgren spheres are regions of ionised hydrogen gas around stars that are hot enough so as to emit large numbers of UV photons.

Assume that we are dealing with a hydrogen Strömgren sphere in equilibrium that gains energy through photoionisation and loses energy through recombination only. Furthermore, assume that each photoionisation event injects an energy \( Q_G = \frac{3}{2} k_B T_* \), while each recombination event loses energy \( Q_L = \frac{3}{2} k_B T_e \). Here \( T_* \) and \( T_e \) are, respectively, the effective temperature of the ionising central star and the electron temperature of the nebula.

(i) In about 100 words, describe qualitatively the THREE processes involved in the formation of hydrogen and helium lines in Strömgren spheres. [4 marks]

(ii) Bearing in mind that the nebula is in equilibrium, write down and justify the relationships between \( L, G, N, \) and \( R \). Here \( L \) and \( G \) are the total energy loss and gain (per unit time and volume) and \( N \) and \( R \) are the number of photoionisations and recombinations per unit time and volume.

Use these relationships to derive an expression for the expected electron temperature \( T_e \) of the nebula. Comment on how this expression compares to observed nebular temperatures. [6 marks]

(iii) Next consider the case where the Strömgren sphere also contains heavier trace elements, such as, e.g., oxygen. Explain how forbidden line emission can act to cool the Strömgren sphere and explain why the energy loss (per unit time and volume) through this process is given by

\[
L^c = n_e n_i C_{ij} E_{ji}.
\]

\( C_{ij} \) is the collisional excitation coefficient, \( E_{ji} = E_j - E_i \) is the difference in excitation energies between levels \( j \) and \( i \), \( n_e \) is the electron density and \( n_i \) is the number density of atoms (or ions) in level \( i \). [4 marks]

(iv) The collisional excitation coefficient is proportional to

\[
C_{ij} \propto \frac{\Omega_{ij}}{g_i} T_e^{-1/2} \mathcal{P}_{ji} / (k_B T_*) e^{-E_{ji} / (k_B T_*)},
\]

while the radiative recombination coefficient is proportional to \( T_e^{-3/4} \). Both have units of volume per time.

In the following, you may assume that the nebula is cooled through forbidden line emission of a single species emitting at one frequency only. Considering the energy loss and gain, find the relationship between the stellar temperature \( T_* \) and the electron temperature \( T_e \) and compare this to the result obtained in part (ii). Comment carefully on any expected dependencies on electron and ion number densities.

(Note: You may drop constant numerical and physical factors.) [6 marks]

[Total 20 marks]
4. For a constant source function $S_0$, the formal solution to the equation of radiative transfer simplifies to

$$I_\nu = I_{\nu,0} e^{-\tau_\nu} + S_0(1 - e^{-\tau_\nu}),$$

where $I_{\nu,0}$ is the incident intensity at frequency $\nu$ and $\tau_\nu$ is the optical depth. The optical depth is related to the geometrical depth $s$ and the absorption coefficient $\alpha_\nu$ via $d\tau_\nu = \alpha_\nu ds$.

(i) Explain the terms ‘optically thick’ and ‘optically thin’. Give an approximation for the radiative transfer equation in the optically thin case and constant incident radiation, $I_{\nu,0} = I_0$. Under what conditions would you expect to see emission and absorption lines? [5 marks]

(ii) The absorption coefficient is related to the number density of absorbers $n$ and the absorption cross section $\sigma_\nu$ through $\alpha_\nu = \sigma_\nu n$. Show that it is possible to define a column density $\mathcal{N}$, so that $\tau_\nu = \mathcal{N} \sigma_\nu$. (Give the mathematical definition of $\mathcal{N}$, as well as a description in words.) [2 marks]

(iii) Explain what is meant by the equivalent width of an absorption line and write down an expression for it in terms of $I_0$ and $I_\nu$. [3 marks]

(iv) Relate the equivalent width to the column density in the case of pure absorption (i.e., $S_0 = 0$) when the line is optically thin. Sketch the equivalent width as a function of column density and explain qualitatively what behaviour you expect as the line becomes optically thick. [5 marks]

(v) The atomic binding energy scales with the reduced mass of the nucleus and the electron, or $m_e m_{\text{nuc}}/(m_e + m_{\text{nuc}})$. Consider hydrogen and deuterium atoms, and calculate the ratio between the wavelengths of the deuterium and hydrogen lines of equivalent transitions. Comment briefly on the observability of the H and D lines and on the astrophysical relevance of deuterium EW observations. [5 marks]

[Total 20 marks]
5. (i) Describe briefly (using fewer than 150 words) the most important characteristics of spiral and elliptical galaxies respectively. [5 marks]

(ii) The surface brightness distribution (flux per unit area) of spiral galaxies is typically given by \( \Sigma(r) = \Sigma_0 e^{-r/r_0} \), where \( r \) is the distance from the galactic centre and \( r_0 \) is a characteristic length.

Calculate the fraction of the total flux that emerges within a distance of 3\( r_0 \) from the galactic centre. [3 marks]

(iii) In the following, you may assume that 100% of the flux emerges from within 3\( r_0 \) and that the orbits of stars and gas clouds around the galactic centre are circular.

Assuming further that the matter distribution follows that of the visible matter, derive the expected behaviour of the rotational velocity at a distance \( r > 3r_0 \) from the galactic centre. Sketch the expected rotation curve and compare it to a typical observed rotation curve, indicating typical velocities and distances for the observed curve.

Briefly discuss which conclusions you can draw from the comparison of the two curves. [6 marks]

(iv) A popular profile for the spherical matter density distribution is the so-called isothermal profile with

\[
\rho(r) = \frac{\rho_0}{1 + (r/a)^2},
\]

where \( a \) is the core radius. This parameter \( a \) is unrelated to \( r_0 \) from above and can be assumed to be of the order of 1 kpc here.

Show that the mass enclosed within a radius \( r \) is given by

\[
M_r = 4\pi\rho_0 a^2 (r - a \arctan r/a).
\]

Derive the expected rotation curve for \( r \gg a \) and compare it to the two curves sketched in part (iii) above.

(Hint: You may want to note the integral \( \int \frac{dx}{x^2+a^2} \) given on the formula sheet.) [6 marks]

[Total 20 marks]
6. Consider a galaxy cluster that shows strong X-ray emission within a radius of 1.9 Mpc (1 pc = 3.1 × 10^{16} m). From X-ray spectra, the gas temperature has been estimated to be 1.2 × 10^8 K.

(i) State the virial theorem and explain when it can be applied to galaxy clusters.

Give a rough estimate of the isotropic 1-d velocity dispersion of the cluster if it is virialised. [5 marks]

(ii) Consider a model where the cluster is seen as a sphere of fully ionised hydrogen gas in hydrostatic equilibrium, so that

\[ \frac{dP}{dr} = -\frac{GM_r\rho}{r^2}. \]

\( P \) is the gas pressure, \( M_r \) is the cluster mass within radius \( r \) and \( \rho \) is the gas density.

Use the ideal gas law \( P = \rho k_B T / (\mu m_H) \), where \( k_B \) is Boltzmann’s constant and \( \mu \) is the mean molecular weight) to show that the cluster mass can be obtained from the density and temperature gradient as follows

\[ M = -\frac{r k_B T}{G \mu m_H} \left( \frac{d \ln T}{d \ln r} + \frac{d \ln \rho}{d \ln r} \right). \]

[3 marks]

(iii) The observations indicate a density profile that can be approximated as \( \rho = \rho_0 r^{-2} \). Estimate the cluster mass, assuming that the gas consists of hydrogen only and is isothermal. Give your answer in terms of solar masses \( (M_\odot = 2.0 \times 10^{30}) \).

Using 50 words or less, describe briefly a method that does not rely on the virial theorem and that would allow you to check this result independently. [6 marks]

(iv) The X-ray emission is produced by Bremsstrahlung. The volume luminosity (luminosity per unit volume) emitted by a gas with temperature \( T \) and electron density \( n_e \) is given by

\[ \mathcal{L} = 1.4 \times 10^{-40} n_e^2 \sqrt{T} \text{ W m}^{-3} \text{ K}^{-1/2}. \]

The X-ray cluster emits a total luminosity or 7.4 × 10^{37} W within a radius \( r_X = 1.9 \text{ Mpc} \). Calculate the electron number density of the cluster gas and derive the total mass of the X-ray gas in solar masses.

Compare your result to that obtained in the previous section, and discuss the matter content of the galaxy cluster. [6 marks]

[Total 20 marks]