Answer THREE questions. All questions are worth equal marks.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
1. A system consists of two spin-half particles. The Hilbert space that describes the spin of the particles is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where each $\mathcal{H}_i$ has basis $\{|\uparrow\rangle_i, |\downarrow\rangle_i\}$, $i = 1, 2$ and $|\uparrow\rangle (|\downarrow\rangle)$ is the eigenstate of spin in the $z$-direction with positive (negative) eigenvalue in the usual notation.

The system is in a state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2).$$

(i) Explain why the spin of particle 1 in any direction and the spin of particle 2 in any direction can be simultaneously measured. If the spin of particle 1 alone is measured in the $z$-direction what are the possible outcomes and their probabilities? If the total spin of the system is measured in the $z$-direction what are the possible outcomes and probabilities? [6 marks]

(ii) Explain why the experimental results that can be obtained for the above state, $|\psi\rangle$, and the Copenhagen Interpretation imply that there is “spooky action at a distance” in Quantum Mechanics. [4 marks]

(iii) For the state $|\psi\rangle$ above, show that the probability that the spin of particle 1 is measured to be up in the direction of unit vector $n$ and the spin of particle 2 is measured to be up in the direction of unit vector $m$ is $\frac{1}{2} \sin^2(\theta/2)$ where $\theta = \cos^{-1}(n \cdot m)$. Hint: Use the fact that $|\psi\rangle$ is invariant under rotations to choose the best $x$, $y$ and $z$ axes for the problem.

You may use the Pauli sigma matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[10 marks]

[Total 20 marks]
2. (i) Write down the Schrödinger equation in Dirac notation in terms of the Hamiltonian operator, $\hat{H}$.

Explain the difference between the Schrödinger and Heisenberg Pictures of the dynamics of a quantum system. Write down the relationships between the two Pictures. Show that expectation values of observables are the same when calculated in either Picture.

Write down the Heisenberg equation of motion for an operator $\hat{A}$ corresponding to an observable. [7 marks]

(ii) A quantum system has a 3-dimensional Hilbert space with basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ and a Hamiltonian

$$\hat{H} = 3|e_1\rangle\langle e_1| + 2|e_1\rangle\langle e_2| + 2|e_2\rangle\langle e_1| + 3|e_2\rangle\langle e_2| + 2|e_3\rangle\langle e_3|.$$ 

Find the energy eigenstates and eigenvalues.

At time $t = 0$ the state of the system is $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle + |e_3\rangle)$.

Calculate the Schrödinger Picture state of the system, as a linear combination of the basis vectors, at time $t = T$ later.

Calculate the expectation value of the energy at time $t = T$. [10 marks]

(iii) In the system of part (ii) at time $t = 0$, what is the probability for measuring the value 1 for the observable corresponding to operator $\hat{A} = |e_1\rangle\langle e_1|$? At time $t = 0$ what is the probability for measuring the value 1 for the observable corresponding to operator $\hat{B} = |e_3\rangle\langle e_3|$?

Suppose no measurement is actually made at $t = 0$. What is the earliest time, $t$ greater than zero, at which the probabilities of the outcomes of measurements of the observables corresponding to $\hat{A}$ and $\hat{B}$ are equal to what they were at $t = 0$? [3 marks]

[Total 20 marks]
3. (i) A particle of mass $m$ moving in one dimension has position operator $\hat{x}$, momentum operator $\hat{p}$ and is in state $|\psi\rangle$. Prove that

$$\langle x|\hat{p}|\psi\rangle = -i\hbar \frac{d}{dx} \langle x|\psi\rangle,$$

where $|x\rangle$ is the eigenvector of $|x\rangle$ with eigenvalue $x$.

Prove also that

$$\langle x|\hat{p}^2|\psi\rangle = - \left(\hbar \frac{d}{dx}\right)^2 \langle x|\psi\rangle.$$

You may use the formula

$$\langle x|\hat{p}|y\rangle = -i\hbar \frac{\partial}{\partial x} \delta(x-y),$$

where $\delta(x)$ is the Dirac delta function. [4 marks]

(ii) The position operator for the particle is $\hat{x}$. If $f$ is a real function of a real variable with a Taylor expansion, the function $f(\hat{x})$ of $\hat{x}$ is defined by the Taylor series

$$f(\hat{x}) = f(0) + f'(0)\hat{x} + \frac{1}{2!}f''(0)\hat{x}^2 + \ldots,$$

where $1$ is the identity operator. Show that

$$\langle x|f(\hat{x})|\psi\rangle = f(x)\langle x|\psi\rangle.$$

The Hamiltonian operator is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x}).$$

Starting with the Schrödinger equation in Dirac notation, derive the position representation for the Schrödinger equation (in other words the Schrödinger equation for the wavefunction). [6 marks]

(iii) Find the quantum mechanical energy levels of a particle of mass $m$ constrained to move on a circular ring of radius $R$. [5 marks]

(iv) Find the energy levels when the particle in part (iii) has mass $m$ and charge $q$ and there is a uniform magnetic field $B$ perpendicular to the plane of the circle. You may assume that the uniform magnetic field has a magnetic vector potential equal to $|B| R/2$ directed around the circle. [5 marks]

[Total 20 marks]
4. (i) What is a superposition of two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$? What is a statistical mixture of $|\psi_1\rangle$ and $|\psi_2\rangle$? [2 marks]

(ii) An observer, Alya, prepares quantum mechanical spin-$\frac{1}{2}$ systems, choosing their states randomly to be $|\psi_1\rangle$, $|\psi_2\rangle$ or $|\psi_3\rangle$ with probabilities $p_1$, $p_2$ and $p_3$ respectively. She passes the systems to another observer, Bai.

Given $p_1 = \frac{1}{8}$, $p_2 = \frac{1}{8}$, $p_3 = \frac{3}{4}$ and

$$|\psi_1\rangle = |\uparrow\rangle,$$

$$|\psi_2\rangle = |\downarrow\rangle,$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle),$$

where $\{|\uparrow\rangle, |\downarrow\rangle\}$ are the z-spin up and down basis states in the usual notation.

Calculate the density operator, $\rho$, that Bai has to use to describe the systems. Give $\rho$ both in Dirac notation and in its matrix representation in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. [4 marks]

(iii) If Bai measures the observable corresponding to operator $\hat{X}_1 = |\rangle\langle\phi|$, where $|\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - i|\downarrow\rangle)$, what are the possible outcomes and their probabilities?

If Bai measures the observable $\hat{X}_2 = |\uparrow\rangle\langle\uparrow| + 5|\downarrow\rangle\langle\downarrow|$ for one of the particles he receives, what are the possible outcomes of the measurement and what are their probabilities? [7 marks]

(iv) Find states $|\phi_1\rangle$ and $|\phi_2\rangle$ and probabilities $q_1$ and $q_2$ such that if Alya were to produce particles randomly in states $|\phi_1\rangle$ and $|\phi_2\rangle$ with probabilities $q_1$ and $q_2$, Bai would use the same density operator, $\rho$, as in part (ii) to describe the systems. [7 marks]

[Total 20 marks]
5. (i) Define a Hermitian (self-adjoint) operator.
   Prove that the eigenvalues of a Hermitian operator are real.
   Prove that two eigenvectors of a Hermitian operator with different eigenvalues
   are orthogonal. \[5 \text{ marks}\]

(ii) Let \(|e_1, e_2, \ldots, e_n\rangle\rangle\) be an orthonormal basis for a Hilbert space, \(\mathcal{H}\). The
   vectors of the basis are eigenstates of Hermitian operator, \(\hat{A}\):

\[\hat{A}|e_i\rangle = \lambda_i|e_i\rangle, \quad i = 1, 2, \ldots, n.\]

Prove that

\[\hat{A} = \sum_{i=1}^{n} \lambda_i|e_i\rangle\langle e_i| \]. \[3 \text{ marks}\]

(iii) State the Born Rule and the Collapse Postulate.
   Consider the system of part (ii). Suppose that all the eigenvalues \(\lambda_i\) of \(\hat{A}\) are
   distinct: \(\lambda_i \neq \lambda_j\) for \(i \neq j\). Prove that if the system is in state \(|\psi\rangle\)
   and a measurement of the observable corresponding to \(\hat{A}\) is made, the probability,
   \(q_i\), of getting the result \(\lambda_i\) is

\[q_i = \langle\psi|P_i|\psi\rangle,\]

where \(P_i\) is defined as \(P_i = |e_i\rangle\langle e_i|\).

Prove also that after the measurement, if result \(\lambda_i\) is obtained, the state collapses

\[|\psi\rangle \rightarrow |\psi'\rangle = \frac{P_i|\psi\rangle}{\langle\psi|P_i|\psi\rangle^{\frac{1}{2}}}.\] \[6 \text{ marks}\]

(iv) Consider the system of parts (ii) and (iii). Let the state at time \(t = 0\) be \(|\psi\rangle\)
   and let the Hamiltonian of the system be \(\hat{H}\). Measurements of the observable
   corresponding to \(\hat{A}\) are going to be made at times \(t_1\) and \(t_2\) and where \(0 < t_1 < t_2\).
   Prove that the probability, \(q(i, j)\), that the results of the measurements are
   \(\lambda_i\) at \(t_1\) and \(\lambda_j\) at \(t_2\) is

\[q(i, j) = \|P_j e^{-i\hat{H}(t_2-t_1)/\hbar} P_i e^{-i\hat{H}t_1/\hbar} |\psi\rangle\|^2.\] \[6 \text{ marks}\]

\[\text{Total 20 marks}\]
6. (i) Write down the commutation relations between the angular momentum operators \( J_x, J_y \) and \( J_z \).

Show that

\[
[J^2, J_i] = 0 ,
\]

for \( i = x, y \) and \( z \), where the operator \( J^2 \) is defined as \( J^2 = J_x^2 + J_y^2 + J_z^2 \).

[4 marks]

(ii) Choosing \( \{ J^2, J_z \} \) as a complete set of commuting observables, let the normalised simultaneous eigenvectors be \( |\lambda \mu \rangle \) where

\[
J^2 |\lambda \mu \rangle = \lambda |\lambda \mu \rangle ,
\]

\[
J_z |\lambda \mu \rangle = \mu |\lambda \mu \rangle .
\]

(a) Let \( J_{\pm} = J_x \pm iJ_y \). Show that \( J_{\pm} |\lambda \mu \rangle \) is an eigenvector of \( J^2 \) with eigenvalue \( \lambda \) and an eigenvector of \( J_z \) with eigenvalue \( \mu \pm \hbar \).

(b) Prove that \( \langle \lambda \mu | J_+ J_- | \lambda \mu \rangle \geq 0 \) and that \( \langle \lambda \mu | J_- J_+ | \lambda \mu \rangle \geq 0 \).

(c) Using the results of (a) and (b), prove that the possible values of \( \lambda \) are \( \lambda = \hbar^2 j(j+1) \) where \( j \) is any non-negative half-integer: \( j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \).

You may use the formulae

\[
J_+ J_- = J^2 - J_z^2 + \hbar J_z ,
\]

\[
J_- J_+ = J^2 - J_z^2 - \hbar J_z .
\]

What are the possible values of \( \mu \) for each \( j \)?

[10 marks]

(iii) Consider two particles, each with spin \( \frac{1}{2} \). What are the possible values of the total spin of the combined system of both particles? Show in detail how the Hilbert space of the combined system can be decomposed as a direct sum of subspaces corresponding to the different total spin values.

You may use the formulae

\[
J_+ |j m \rangle = \hbar \sqrt{(j-m)(j+m+1)} |j m + 1 \rangle ,
\]

\[
J_- |j m \rangle = \hbar \sqrt{(j+m)(j-m+1)} |j m - 1 \rangle .
\]

[6 marks]

[Total 20 marks]