Imperial College London
BSc/MSci EXAMINATION June 2008

This paper is also taken for the relevant Examination for the Associateship

QUANTUM OPTICS

For 4th-Year Physics Students
Tuesday, 3rd June 2008: 14:00 to 16:00

Answer THREE of the SIX questions
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
1. The vector potential is commonly expanded in terms of monochromatic modes (plane waves etc.) as
\[ \hat{A}(r) = \sum_{\lambda} \left[ A_{\lambda}(r) \hat{a}_{\lambda} + A^*_{\lambda}(r) \hat{a}^\dagger_{\lambda} \right] \]
and the Hamiltonian operator of the quantized free electromagnetic field takes the form
\[ \hat{H} = \frac{1}{2} \sum_{\lambda} \hbar \omega_{\lambda} \left( \hat{a}_{\lambda}^\dagger \hat{a}_{\lambda} + \hat{a}_{\lambda} \hat{a}_{\lambda}^\dagger \right). \]

(i) Use the commutation rules for the ladder operators \( \hat{a}_{\lambda} \) and \( \hat{a}_{\lambda}^\dagger \) to bring \( \hat{H} \) into normal operator order.

(ii) On what does the expression of the ground-state energy depend? Discuss how it is possible to show experimentally the existence of this ground-state energy.

(iii) Write down the Heisenberg equations of motion for the ladder operators and subsequently the expression for the quantized electric field.

(iv) The rate of spontaneous decay of a two-level atom with transition frequency \( \omega_A \) is determined by the vacuum fluctuations of the electromagnetic field strength at the atom’s position \( r_A \). Show that the electromagnetic field fluctuations are given by
\[ \langle 0 | \hat{E}^2 (r_A) | 0 \rangle = \sum_{\lambda} \omega_{\lambda}^2 |A_{\lambda} (r_A)|^2. \]
2. A lossless beam splitter relates the amplitude operators $\hat{a}_1$ and $\hat{a}_2$ of the incoming photons to those of the outgoing photons by the matrix relation

$$
\begin{pmatrix}
\hat{b}_1 \\
\hat{b}_2
\end{pmatrix} =
\begin{pmatrix}
T & R \\
-R^* & T^*
\end{pmatrix}
\begin{pmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{pmatrix} .
$$

(i) Show that the beam splitter transformation matrix is unitary. What does unitarity mean physically in this context? [3 marks]

(ii) Suppose now that two identical photons impinge onto a beam splitter, i.e. the initial state before the transformation is $|\psi_{in}\rangle = |1_1, 1_2\rangle = \hat{a}^\dagger_1 \hat{a}^\dagger_2 |0_1, 0_2\rangle$. What is the quantum state after the light has passed through the beam splitter? [5 marks]

(iii) For a symmetric beam splitter, the degrees of transmission and reflection are equal, $|T|^2 = |R|^2$. What is the probability of finding one photon in each output beam? Give a physical explanation of the result. [3 marks]

(iv) Let us assume that the light impinging onto a Mach–Zehnder interferometer is prepared in a state $|\psi_{in}\rangle = |0_1, 1_2\rangle = \hat{a}^\dagger_2 |0_1, 0_2\rangle$, i.e. only a single photon is present at the second (lower) input port. The beam splitters have equal transmission and reflection coefficients $T = R = \frac{1}{\sqrt{2}}$.

(a) What is the quantum state after the first beam splitter? [3 marks]

(b) Let us now assume that the phase shifter in the clockwise path imprints a phase $\Theta$ onto the quantum state. The action of the phase shifter is a unitary operation of the form $\hat{U}_\Theta = e^{i\Theta \hat{n}}$ where $\hat{n}$ is the number operator of the respective light mode. What is the quantum state after the phase shifter? [2 marks]

(c) What is the quantum state after the second beam splitter? What is the probability that, after the Mach–Zehnder interferometer, the quantum state $|0_1, 1_2\rangle$ is detected? And what is the probability that $|1_1, 0_2\rangle$ is detected? Sketch both probabilities as a function of $\Theta = [0, 2\pi]$. [4 marks]

[Total 20 marks]
3. The second-order quantum correlation function describes the probability to detect the arrival of photons at a photodetector with a time delay $\tau$,

$$g^{(2)}(\tau) = \frac{\langle \hat{E}_i^-(t) \hat{E}_j^-(t+\tau) \hat{E}_j^+(t+\tau) \hat{E}_i^+(t) \rangle}{\langle \hat{E}_i^-(t) \hat{E}_i^+(t) \rangle \langle \hat{E}_j^-(t+\tau) \hat{E}_j^+(t+\tau) \rangle}.$$  

We assume that we have a single-mode field with an electric-field strength $[A$ is the mode function; we omit its spatial dependence$]$

$$\hat{E}(t) = \hat{E}_i^+(t) + \hat{E}_i^-(t) = i\omega A \hat{a}(t) - i\omega A^* \hat{a}^\dagger(t).$$

(i) Without any calculation, explain why for such a single-mode (monochromatic) field the second-order correlation function cannot depend on time. [3 marks]

(ii) Suppose the light had been prepared in a single-photon Fock state $|\psi\rangle = |1\rangle$. What is the value of $g^{(2)}$ for this single-photon state? Explain the reason for this result. [5 marks]

(iii) Calculate the second-order coherence function for an $n$-photon Fock state, $|\psi\rangle = |n\rangle$. [5 marks]

(iv) What is the range of allowed values of $g^{(2)}(0)$ for classical light? Hint: Use the fact that the average $\langle x \rangle$ is computed from a sequence of measurement results $x_i$. For just two measurements $x_1$ and $x_2$, compare $\langle x^2 \rangle = (x_1^2 + x_2^2)/2$ and $\langle x \rangle^2 = [(x_1 + x_2)/2]^2$. [7 marks]

[Total 20 marks]
4. The evolution of a two-level atom driven by a classical laser field is governed by the optical Bloch equations

\[
\begin{pmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{pmatrix} =
\begin{pmatrix}
0 & -\delta & \Omega \\
\delta & 0 & 0 \\
-\Omega & 0 & 0
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
- \frac{\Gamma}{2}
\begin{pmatrix}
u \\
v \\
2(1+w)
\end{pmatrix}
\]

where \( u = \langle \hat{\sigma} + \hat{\sigma}^\dagger \rangle \), \( v = \langle \hat{\sigma} - \hat{\sigma}^\dagger \rangle / i \), and \( w = \langle \hat{\sigma}_z \rangle \). In these equations, \( \delta \) is the detuning of the laser light from the atomic transition and \( \Gamma \) the rate of spontaneous decay.

(i) Let us assume that the laser light is resonant with the atomic transition and that we can neglect spontaneous emission. What is the solution to the Bloch equations, given that the atom was initially prepared in its ground state? Describe the evolution of the Bloch vector in words. [6 marks]

(ii) (a) What time \( T_{\pi/2} \) does it take for the Bloch vector to reach the equator of the Bloch sphere (\( \pi/2 \)-pulse)? [2 marks]

(b) Suppose the Bloch vector is now in the equatorial plane, and let us assume that there was an additional interaction that would rotate the position of the Bloch vector by an angle \( \pi \) around the \( w \)-axis. If we then applied another laser pulse for a time \( T_{\pi/2} \), where would the Bloch vector now point and why? [3 marks]

(iii) The spontaneous decay rate \( \Gamma \) is proportional to the strength of the vacuum fluctuations of the electromagnetic field. Describe in words a way in which one can make spontaneous decay anisotropic, i.e. dependent on the spatial direction. How is it possible to suppress (inhibit) spontaneous emission? [3 marks]

(iv) Including spontaneous emission and detuning back into the Bloch equations, what is their stationary state? Use the result to explain why it is impossible to build a laser with a two-level pump medium. [6 marks]

[Total 20 marks]
The Jaynes-Cummings Hamiltonian
\[ \hat{H} = \hbar \omega \hat{a}^\dagger \hat{a} + \frac{1}{2} \hbar \omega_A \hat{\sigma}_z + \hbar g (\hat{a} \hat{\sigma}^\dagger + \hat{a}^\dagger \hat{\sigma}) \]
describes the interaction of a single light mode with frequency \( \omega \) and a two-level atom with transition frequency \( \omega_A \) inside a high-quality resonator.

(i) In the case of zero detuning, i.e. when \( \omega = \omega_A \), solve the eigenvalue problem for \( \hat{H} \) in the basis of the product states \( \{|e, n\}, |g, n + 1\} \). Sketch the energy level diagram for the lowest photon numbers \( n = 0, 1, 2 \) with and without the interaction \( g \).

(ii) Suppose that an atom prepared in its excited state enters at \( t = 0 \) an empty, resonant cavity. Write the state vector as \( |\psi(t)\rangle = C_i(t) |e, 0\rangle + C_f(t) |g, 1\rangle \) and use the interaction Hamiltonian \( \hat{H}_{\text{int}} = \hbar g (\hat{a} \hat{\sigma}^\dagger + \hat{a}^\dagger \hat{\sigma}) \) to find the time evolution of the state vector \( |\psi(t)\rangle \) (the free Hamiltonian only leads to an unimportant phase factor). How is it possible to deposit exactly one photon into a cavity?

(iii) After a time \( t \), what is the probability of finding the atom in its ground state? Sketch this probability. Explain in words what happens to the atom and the cavity field during the time interval \( t = [0, \pi/g] \). Compare the atom’s time evolution in the cavity with the situation in which there was no cavity present.

[Total 20 marks]
6. In the dispersive limit of cavity QED, the cavity resonance frequency is so far detuned from the atomic transition frequency that no transitions between the atomic states can occur. The effective unitary evolution operator in the interaction picture is then

\[ \hat{U}(t) = \exp \left(-i \frac{g^2}{\delta} (\hat{n} + 1)t\right) |e\rangle \langle e| + \exp \left(i \frac{g^2}{\delta} \hat{n}t\right) |g\rangle \langle g| \]

where \( \delta \) is the detuning and \( g \) the atom-cavity coupling constant.

(i) Suppose that an atom enters the cavity prepared in a superposition \((|g\rangle + |e\rangle)/\sqrt{2}\). Explain what needs to be done in order to prepare the atom in such a quantum state, given that the atom is initially in its ground state. 

(ii) Let the cavity field be initially prepared in a coherent state \(|\alpha\rangle\). Sketch the Wigner function of a coherent state. Explain how the Wigner function can be used to compute expectation values of observables (in particular, comment on the necessary operator ordering).

(iii) Calculate the mean photon number in a coherent state, \( \bar{n} = \langle \alpha | \hat{n} | \alpha \rangle \), using the Wigner function \( W(\beta) = \frac{2}{\pi} e^{-2|\beta - \alpha|^2} \). Hint: A useful integral is

\[ \int d^2 \beta |\beta|^2 e^{-2|\beta - \alpha|^2} = \frac{\pi}{4} \left(1 + 2|\alpha|^2\right). \]

(iv) The atom in the state \((|g\rangle + |e\rangle)/\sqrt{2}\) interacts with the cavity field in the coherent state \(|\alpha\rangle\). Use the expansion of \(|\alpha\rangle\) into number states,

\[ |\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{1}{2}|\alpha|^2} |n\rangle, \]

to show that

\[ e^{i\Phi} |\alpha\rangle = |\alpha e^{i\Phi}\rangle. \]

(v) After an interaction time \( T = \frac{\pi \delta}{4g^2} \), what is the quantum state of the atom-cavity system? Sketch the cavity-field part of the Wigner function. Estimate how large \( \alpha \) must be to make the two components of the wave function orthogonal.

[Total 20 marks]