Answer THREE questions.
All questions carry equal marks.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
In the following, units of $\hbar = c = \varepsilon_0 = 1$ are used, so that

- 1 metre $\equiv 5.068 \times 10^{15} \text{ GeV}^{-1}$
- 1 second $\equiv 1.519 \times 10^{24} \text{ GeV}^{-1}$
- 1 Joule $\equiv 6.242 \times 10^9 \text{ GeV}$
- 1 kilogramme $\equiv 5.610 \times 10^{26} \text{ GeV}$
1. (i) Explain what is meant by the parity operation. Explain the terms polar vector and axial vector and give an example of each from classical physics. [3 marks]

(ii) Briefly discuss the connection between the covariance of a system under the parity operation and the conservation of the parity quantum number in quantum mechanics. Show that the eigenvalues of parity are \( \pm 1 \). [3 marks]

(iii) The Dirac equation can be written as

\[
i\gamma^\mu \partial_\mu \psi - m\psi = i\gamma^0 \partial_0 \psi + i\gamma^\nu \nabla_\nu \psi - m\psi = 0,
\]

where the \( \gamma^\mu \) matrices satisfy

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^\mu\nu.
\]

Write down the Dirac equation after a parity inversion, assuming that, if \( \psi \) is a solution of the original equation, then \( \psi' = \hat{P}\psi \) is a solution of the parity-inverted equation. Hence, show that the parity operator can be written as \( \hat{P}\psi = \gamma^0 \psi \). [4 marks]

(iv) Within the Standard Model, all interactions take place through the four-vector currents

\[
J^\mu_X = \bar{\psi} \gamma^\mu \phi, \quad J^\mu_Y = \bar{\psi} \gamma^\mu \gamma^5 \phi,
\]

where \( \bar{\psi} = \psi^\dagger \gamma^0 = (\gamma^0 \psi)^\dagger \) and the matrix \( \gamma^5 \) satisfies

\[
\gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu = 0.
\]

Determine how the time and spatial components of these currents change under a parity operation and hence state whether they are polar or axial vectors. [5 marks]

(v) Which of the above currents participate in the electromagnetic, strong and weak interactions? Explain why this means that weak interactions do not conserve parity. [3 marks]

(vi) Give an example of a measurement which demonstrates parity is not conserved. [2 marks]

[Total 20 marks]
2. We consider electron-muon scattering in the high energy limit, where we can neglect the masses of the particles. Within Quantum Electrodynamics (QED), the expression for the matrix element $M$ is given by

$$iM = \bar{u}_c (ie\gamma^\mu) u_a \left( \frac{-ig_{\mu\nu}}{q^2} \right) \bar{u}_d (ie\gamma^\nu) u_b.$$ 

Here $a$ and $b$, and $c$ and $d$, are labels for the incoming electron and muon, and the outgoing electron and muon, respectively.

We will consider backward ($\theta = \pi$) scattering, where $\theta$ is the angle between the incoming and outgoing electrons in the centre-of-mass frame.

(i) Show that the $q^2$ (square of the four-momentum transferred) is $-4E^2$ for backward scattering, where $E$ is the energy of each of the incoming leptons. What is the value of $q^2$ for forward ($\theta = 0$) scattering? [4 marks]

(ii) Each of the incoming and outgoing particles can have one of two helicity states. What does the conservation of helicity in the massless limit imply for the 16 possible helicity combinations? [3 marks]

(iii) For the combination where incoming and outgoing leptons all have positive helicity, evaluate the matrix element for backward scattering. [7 marks]

(iv) Each helicity combination contributes to the scattering cross section through the following formula:

$$\frac{d\sigma_i}{d\Omega} = \frac{|M_i|^2}{64\pi^2 s},$$

where the subscript $i$ represents each helicity combination. Explain clearly how the contributions from the incoming and outgoing helicity states are combined to give the full cross section $d\sigma/d\Omega$. [4 marks]

(v) Why is it not valid to replace the muon with another electron in the above calculation to find the QED cross section for electron - electron scattering? Use Feynman diagrams to explain this if necessary. [2 marks]

Information for this question.

The gamma matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
The adjoint spinor: $\bar{u} = u^\dagger \gamma^0 = (\gamma^0 u)^\dagger$

The general solutions for the Dirac equation, for particles with helicities along an axis with angle $\theta$ defined as above:

$$u_1 = \sqrt{E + m} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \quad u_2 = \sqrt{E + m} \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$$

$$+ \frac{p}{E + m} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}$$

$$- \frac{p}{E + m} \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$$

[Total 20 marks]
3. The figure below indicates the mean energy loss ($|dE/dx|$) of positive-muons in copper. Here, $\beta$ is the muon velocity in natural units, and $\gamma$ is the relativistic parameter $\gamma = 1/\sqrt{1 - \beta^2}$. In this question, we do not make any distinction between positive and negative muons.

![Graph showing energy loss as a function of velocity](image)

(i) State and explain the appropriate units for the vertical axis, taking into consideration the value of the minimum near $\beta \gamma = 3$, and the fact that the plot is independent of the density of the copper. [3 marks]

(ii) In particle physics, we often encounter muons with momenta in the range of several MeV/c to a few GeV/c. In this range, the “Bethe-Bloch” formula is a good approximation. Name and describe the dominant mechanism for energy loss at these momenta. Given that matter consists of nuclei and electrons, comment on how the muons are affected by each. [3 marks]

(iii) The density-normalised energy loss through this mechanism, for heavy charged particles with $\beta \gamma \sim 3$ and unit charge, has only a weak dependence on the particle type and the composition of the material traversed. Explain why this is the case, comparing the examples of solid copper and gaseous helium media. [3 marks]

(iv) Name a type of long-lived particle with unit charge which does not lose energy in this way at the same value of $\beta \gamma$, and give the reason. [3 marks]
(v) In the limit of the approximation that the $|dE/dx|$ does not depend on the material composition, the vertical axis of the figure can be taken as MeV/cm, when the material being traversed is pure water.

Super-Kamiokande is a water-based particle detector with linear dimensions of the order of several tens of metres. What is the approximate kinetic energy of muons which leave a 10m long track on average inside Super-Kamiokande? Give the reason for your answer.

[5 marks]

(vi) Name a major discovery in particle physics that was made by Super-Kamiokande, using its ability to measure muon tracks of many metres contained inside its sensitive volume. Explain why this would have been difficult with a water detector with dimensions of $10m \times 10m \times 10m$.

[3 marks]

[Total 20 marks]
4. The tau lepton can decay leptonically to give one charged lepton and two neutrinos. It can also decay semi-hadronically to either: a) a tau neutrino and one or more pions or b) a tau neutrino, a kaon and zero or more pions.

(i) Draw the lowest-order Feynman diagram for each of the two possible leptonic decays indicating clearly the particle represented by each line on the diagram. The type of neutrinos produced should also be clearly indicated.

Ignoring mass effects, what are the relative rates of these decays?

(ii) For a leptonic tau decay, draw a diagram showing the orientation of the momenta of the three outgoing particles when the charged lepton has the maximum energy that it can have in such a decay. Show that this maximum energy, $E_{l\text{max}}$, is given by:

$$E_{l\text{max}} = \frac{m_\tau^2 + m_l^2}{2m_\tau}$$

where $m_\tau$ is the tau mass and $m_l$ is the mass of the charged lepton produced in the decay. You may assume that the neutrinos are massless.

What is the minimum energy of the charged lepton?

(iii) Examples of the two types of semi-hadronic decays are $\tau^- \rightarrow \nu_\tau \pi^-$ and $\tau^- \rightarrow \nu_\tau K^-$. Draw the lowest-order Feynman diagram for each of these decays. Ignoring mass effects, estimate their relative rates.

(iv) Neglecting the electron mass, the partial width for the leptonic decay to an electron plus neutrinos is given by

$$\Gamma(\tau^- \rightarrow e^- + \text{neutrinos}) = \frac{G_F^2 m_\tau^5}{24(2\pi)^3}$$

Estimate the total width of the tau, stating clearly any assumptions that you make. Hence, show that the tau lifetime is approximately $0.3 \times 10^{-12}$ seconds.

(v) Draw the lowest-order Feynman diagram for the decay of a $B^0$ meson (quark content $d\bar{b}$) into $\pi^+\pi^-$ and explain why this would be expected to be a rare decay.
(vi) The total width of the tau is proportional to $m_\tau^5$. Using this, and assuming the effect of the $d$ quark is negligible (“spectator model”), estimate the $B^0$ lifetime. The mass of the bottom quark can be taken to be 4.5 GeV. State clearly any assumptions that you make.

[3 marks]

[Total 20 marks]

Information for this question.
The Cabibbo-Kobayashi-Maskawa (CKM) unitary matrix for the flavour-dependent relative couplings for the charged-current weak interactions of quarks, where $V_{ij}$ is the factor for interactions involving quarks $i$ and $j$. The numerical values of the magnitudes of the matrix elements can be taken to be

\[
\begin{pmatrix}
|V_{ud}| & |V_{us}| & |V_{ub}| \\
|V_{cd}| & |V_{cs}| & |V_{cb}| \\
|V_{td}| & |V_{ts}| & |V_{tb}|
\end{pmatrix}
\approx
\begin{pmatrix}
0.975 & 0.223 & 0.003 \\
0.222 & 0.974 & 0.040 \\
0.009 & 0.039 & 0.999
\end{pmatrix}
\]

The Fermi constant: $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$.
The tau mass: $m_\tau = 1.777$ GeV.
The $\pi^-$ mass: $m_\pi = 0.140$ GeV.
The $K^-$ mass: $m_K = 0.494$ GeV.
The tau lifetime: $\tau_\tau = 0.29 \times 10^{-12}$ seconds.
5. (i) The figure below shows cross section measurements from various $e^+e^-$ annihilation experiments (CESR, DORIS, LEP etc.) over a range of centre of mass energies.

(a) The cross-sections shown specifically are for $e^+e^- \rightarrow$ hadrons, $e^+e^- \rightarrow \mu^+\mu^-$ and $e^+e^- \rightarrow \gamma\gamma$. Draw the underlying lowest-order Feynman diagrams for these processes; label all lines with their associated particles and indicate clearly all force carriers that may be involved.

[3 marks]

(b) Explain briefly the main features of the cross-sections in the figure with reference to the underlying interactions and their force particles.

[3 marks]

(ii) In the centre of mass, and neglecting all masses, the electromagnetic cross section, $\sigma$, for the reaction $e^+e^- \rightarrow \mu^+\mu^-$ is

$$\sigma = \frac{1}{|\beta_a - \beta_b|} e^4 (1 + \cos^2 \theta) \rho \frac{1}{4E_aE_b}$$

where $\beta_a$ and $\beta_b$ are the velocity three-vectors of the incoming particles, $E_a$ and $E_b$ are their energies and $\rho$ is the phase space of the final state. In the centre of mass, the differential phase space is

$$\frac{d\rho}{d\Omega} = \frac{1}{32\pi^2}$$

(a) Show that

$$|\beta_a - \beta_b| = \frac{P(E_a + E_b)}{E_aE_b}$$

where $P$ is the magnitude of the momentum of one of the incoming particles.

[2 marks]
(b) Obtain an expression for the differential cross section \( d\sigma/d\Omega \) and hence show that, neglecting masses, the \( e^+e^- \rightarrow \mu^+\mu^- \) cross section is

\[
\sigma = \frac{4\pi\alpha^2}{3s}
\]

where \( \alpha = e^2/4\pi = 1/137 \) is the fine structure constant and \( s \) is the square of the centre of mass energy.

(iii) Explain how \( e^+e^- \) annihilation experiments can be used to study the strong force, QCD, in particular the gluon and the strong force coupling \( \alpha_s \).

(iv) Explain how \( e^+e^- \) annihilation experiments, at centre of mass energies in the region of the \( Z \) mass (\( \sim 91 \) GeV), can be used to determine the number of light neutrino families.

[Total 20 marks]
6. We consider using the Higgs mechanism to give a mass to the photon without violating gauge invariance.

(i) If we take the free Lagrangian density for the photon, and add a mass term with non-zero mass $m$, we obtain

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu, \quad (1)$$

where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$.

By applying the gauge transformation $A^\mu \rightarrow A^\mu + \partial^\mu \Lambda$, show that Eq. 1 is not gauge invariant.

(ii) From the Klein-Gordon equation, we can write down the free Lagrangian density for a massive complex scalar field

$$\mathcal{L} = (\partial^\mu \phi)^* (\partial_\mu \phi) - m^2 \phi^* \phi. \quad (2)$$

Using the minimal substitution $\partial^\mu \rightarrow \partial^\mu + iq A^\mu$, find the gauge invariant interaction terms between the scalar and photon fields. These include the quartic interaction term

$$q^2 A^\mu A_\mu \phi^* \phi. \quad (3)$$

(iii) The Higgs mechanism arises when the potential in Eq. 2 is modified to allow the vacuum expectation value of $\phi^* \phi$ to be non-zero. Consider the replacement

$$m^2 \phi^* \phi \rightarrow -a \phi^* \phi + b (\phi^* \phi)^2,$$

in the potential term, where $a$ and $b$ are positive constants.

Show that the potential is a minimum for $|\phi| = \sqrt{a/(2b)} \equiv v$.

(iv) Expand $|\phi|$ in a small range around this minimum in the form

$$|\phi| = v + \frac{H}{\sqrt{2}}, \quad (4)$$

to obtain the mass of the real field $H$.

(v) Substitute Eq. 4 into Eq. 3 to give three terms. Use these to:

(a) Find the mass of the photon in this model.
(b) Draw the Feynman diagrams corresponding to the two terms other than that for the photon mass.
(c) Give the strengths of the couplings for these diagrams.

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End of examination paper