Answer FIVE questions.  
All questions carry equal marks.  
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the FIVE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in FIVE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
1. (i) Show that the moment of inertia, \( I \), of a rigid rod of length \( L \) and mass \( M \) about its centre of mass is \( ML^2/12 \) \[2 \text{ marks}\]

(ii) A ladder, which can be approximated by the rod in the figure, stands vertically against a frictionless wall on a frictionless floor. It is disturbed and the bottom of the ladder slides across the floor while the top end slides down the wall. The ladder can be considered to rotate with angular velocity \( \omega \) about its centre of mass which also moves with velocity \( \mathbf{v} = (v_x, v_y) \)

(a) By considering changes in potential and kinetic energy, show that when the ladder is at an angle \( \theta \) to the horizontal (see figure) that

\[L^2 \omega^2 + 12v^2 = 12gL(1 - \sin \theta)\]

(b) As long as the ladder remains in contact with the wall, what condition must be satisfied by one of the velocity components of the top end of the ladder?

(c) A similar condition is satisfied by the bottom end of the ladder. What is it?

(d) Write down equations describing the vertical and horizontal velocity components of each end of the ladder in terms of \( \omega, \theta \) and the components of \( \mathbf{v} \).

[8 marks]

(iii) Hence show that the angular velocity obeys the equation

\[\omega^2 = \frac{3g}{L} (1 - \sin \theta)\]

[4 marks]

(iv) In fact after sliding for a while, the ladder does not remain in contact with the wall.

(a) Show that the horizontal component of the velocity of the centre of mass obeys the equation

\[\frac{dv_x}{dt} = \omega \frac{dv_x}{d\theta}\]

(b) When the ladder has no contact with the wall what is the magnitude of \( dv_x/dt \)?

(c) Hence show that the ladder loses contact with the wall when it reaches an angle such that \( \sin \theta = 2/3 \).

[6 marks]

[Total 20 marks]
2. (i) The number of decays of a radioactive nucleus in a time interval $dt$ is given by

$$dN = -\lambda N dt$$

where $N$ is the number of nuclei at time $t$ and $\lambda$ is the decay constant.

Show that the $\frac{1}{2}$-life of this nucleus is

$$t_{\frac{1}{2}} = \frac{\ln 2}{\lambda}.$$  [3 marks]

(ii) Neutrons produced in cosmic ray collisions with the atmosphere react with atmospheric nitrogen, $^{14}_7$N producing the nucleus $^{14}_6$C(C14). The C14 is radioactive. Do you expect this nucleus to decay via (a) $\alpha$-emission, (b) electron emission or (c) positron emission? Explain your answer and calculate the kinetic energy (in keV) available to the decay products, using data given below.  [3 marks]

(iii) The $\frac{1}{2}$-life of C14 is 5730 years.

(a) Estimate the number of disintegrations per second (the activity) from one gram of C14.

(b) If the fraction by weight of C14 in the carbon found in a living plant or animal is $10^{-12}$, estimate the mass of carbon required to provide an activity of 1 s$^{-1}$. (Assume all activity is due to C14 decay.)  [4 marks]

(iv) Briefly explain the principle of the Carbon Dating technique, stating the assumptions made.  [2 marks]

(v) A radiation detector is used to measure the activity from a sample of wood found in Tutankhamun’s tomb. You are limited to a sample containing 10 g of carbon and it is available for one month.

(a) Estimate the number of counts from a 10 g sample of new wood during one month, assuming the detector has a 33% efficiency.

(b) The number of counts registered by the detector in one month from the tomb sample is $10^6$. Estimate the age of Tutankhamun’s tomb.

(c) Calculate the RMS statistical error on the age.  [6 marks]

(vi) Why is this statistical error, as calculated in part v(c), likely to underestimate the true error in the age measurement? How might the burning of fossil fuels during the industrial era bias the estimate?  [2 marks]

[Total 20 marks]

Mass of $^{14}_7$N = 14003.425 MeV/c$^2$

Mass of $^{14}_6$C = 14004.091 MeV/c$^2$

Mass of electron = 0.511 MeV/c$^2$
3. This question investigates the feasibility of using a rocket engine to accelerate a space ship to relativistic speeds. As with any rocket engine, fuel is ejected at high speed and the space ship accelerates to conserve momentum. The only unusual feature is that the exhaust speed $v_{ex}$ is close to the speed of light.

(i) Express the total energy and momentum of an object of rest mass $m$ and velocity $v$ in terms of $m$, $v$, $c$ and $\gamma = 1/\sqrt{1-v^2/c^2}$. [2 marks]

(ii) Consider the inertial frame of reference in which the space ship is instantaneously at rest at time $t$. During the interval from $t$ to $t + dt$, an amount of fuel of (rest) mass $dm_f$ is ejected in the $-x$ direction at the exhaust speed $v_{ex}$ and the space ship accelerates from rest to velocity $dv$. The mass of the space ship reduces from $m$ to $m + dm$, where $dm$ is negative. Since the space ship starts at rest, its final speed $dv$ is not relativistic in this frame.

![Diagram of a rocket engine ejecting fuel]

(a) Bearing in mind that the exhaust speed is relativistic, use the principle of conservation of energy to show that $dm = -\gamma_{v_{ex}} dm_f$. Why is $|dm|$ greater than $|dm_f|$? [4 marks]

(b) Use the principle of conservation of momentum to show that $dv = -v_{ex} dm/m$, exactly as in the non-relativistic case. [4 marks]

(iii) The relativistic addition of velocities formula says that an object with velocity $v$ in frame $O$ has velocity

$$v' = \frac{v + u}{1 + uv/c^2}$$

when observed from frame $O'$, which has velocity $-u$ with respect to $O$. As measured from the Earth, which is moving at velocity $-u$ with respect to the space ship, the ship in the figure above accelerates from an initial velocity $v'$ to a final velocity $v' + dv'$. Use the transformation of velocities formula to confirm that $v' = u$ and to find $v' + dv'$. Hence show that

$$dv' = -\left(1 - \frac{(v')^2}{c^2}\right) \frac{v_{ex}}{m} dm.$$ [4 marks]

(iv) Assuming that the space ship starts at rest in the Earth’s frame, solve this differential equation to find an expression for the final velocity $v'_f$ in terms of the exhaust velocity, the speed of light, and the ratio $m_i/m_f$ of the initial and final masses. You may find the following integral useful:

$$\int \frac{dq}{1 - q^2} = \text{arctanh}(q) + \text{const.}$$ [4 marks]
(v) Making the optimistic assumption that $v_{ex} = c$, calculate the fraction of the initial mass of the space ship that would have to be used up for the ship to reach $0.9c$?

[2 marks]

[Total 20 marks]
4. The heat outflux through the surface of the Earth is \( Q = 0.06 \text{ Wm}^{-2} \).

(i) Assuming the Earth is a sphere with radius \( R = 6,360 \text{ km} \), what is the total power \( P \) of the heat passing out through its surface? [2 marks]

In order to estimate a lower bound for the proton’s mean lifetime, we assume that the total power \( P \) passing out through the surface of the Earth is produced only by proton decay in which all of the proton’s rest mass \( m_p = 938 \text{ MeVc}^{-2} \) is converted into heat within the Earth.

(ii) Show that in order to produce the total power \( P \), approximately \( 2.0 \times 10^{23} \) protons are decaying per second. [2 marks]

The mass of the Earth \( M = 6.0 \times 10^{24} \text{ kg} \).

(iii) (a) Estimate the total number of protons \( N_p \) in the Earth. [2 marks]

(b) Estimate from your results a lower bound for the proton’s mean lifetime \( \tau_p \). [2 marks]

(c) Why is this a gross under-estimate of the proton’s mean lifetime? [2 marks]

The current estimate of the lower bound for the proton’s mean lifetime is \( \tau_p = 10^{35} \) years.

(iv) In the 1980s, a tank was filled with 2,140 tons of water in order to measure the proton decay. Assuming the detectors are 100% efficient and the proton’s mean lifetime \( \tau_p = 10^{35} \) years, how long would you have to wait, on average, to observe one proton decay? [3 marks]

One of the possible decay modes of a proton is

\[
p \to \pi^+ + \nu,
\]

where the mass of the meson \( m_{\pi^+} = 140 \text{ MeVc}^{-2} \). You may assume that the neutrino \( \nu \) is massless and that the proton is decaying at rest.

(v) What is the speed \( v_{\pi^+} \) and energy \( E_{\pi^+} \) of the meson and the neutrino energy \( E_\nu \)? [7 marks]

[Total 20 marks]
5. The properties of magnetically polarizable systems in a magnetic field $B$ at constant volume can be described in terms of the following fundamental equation

$$dU = TdS - MdB,$$

where $U$ is the internal energy, $T$ is the temperature, $S$ is the entropy, and $M$ is the magnetization.

(i) Obtain an expression for the Helmholtz free energy $F = U - TS$ of this system. By taking appropriate derivatives, obtain the entropy $S(T, B)$ and the magnetization $M(T, B)$. [4 marks]

(ii) As a simple model of such a magnetic system, consider $N$ magnetic moments $\mu$ that can point either up ($\uparrow$) or down ($\downarrow$) independently of one another. This system is then placed in a magnetic field $B$. Each moment has two energy states corresponding to whether it is parallel to the magnetic field or antiparallel to the magnetic field:

$$\epsilon_\uparrow = -\mu B, \quad \epsilon_\downarrow = \mu B.$$

Determine the partition function for this system. [6 marks]

(iii) Calculate the entropy and the magnetization of the system in (ii) as functions of $T$ and $B$. [6 marks]

(iv) Suppose the system in (b) is prepared at temperature $T_i$ and in a magnetic field $B_i$. The field is then reduced adiabatically to a final value $B_f$. Use the fact that $S$ is a function only of $B/T$ to determine if the final temperature of the system is greater than or less than its initial temperature. [4 marks]

[Total 20 marks]
6. The Time Independent Schrödinger Equation for the relative motion of the electron and proton in an Hydrogen atom can be written as follows:

\[-\frac{\hbar^2}{2m}\nabla^2\psi(r) - \frac{q_e^2}{4\pi\varepsilon_0 r}\psi(r) = E\psi(r),\]  \hspace{1cm} (1)

where \( m \) is the reduced mass of the system and \( q_e \) is the electron charge.

(i) Briefly state the assumptions on the electron and proton system upon which Eq.(1) is based. \hspace{1cm} [2 marks]

(ii) Briefly explain how the solutions \( \psi(r) \) of Eq.(1) can be used to find the time evolution of any general wavefunction \( \Psi(r,t) \) of the system, given its form at time \( t = 0 \). \hspace{1cm} [3 marks]

Following standard notation, the solutions of Eq.(1) are

\[\psi_{nlm}(r) = R_{nl}(r) \times Y_{lm}(\theta, \phi),\]  \hspace{1cm} (2)

where \( n = 1, 2, ..., \) labels the energy levels, with energy eigenvalues \( E_n = -R_H\hbar c/n^2 \), where \( R_H \) is the Rydberg constant for Hydrogen.

For the 1s, 2s, and 2p states of the system, the radial functions take the following forms:

\[R_{10}(r) = 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}\]  \hspace{1cm} (3)

\[R_{20}(r) = 2\left(\frac{1}{2a_0}\right)^{3/2}\left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0}\]  \hspace{1cm} (4)

\[R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{1}{2a_0}\right)^{3/2}\left(\frac{r}{2a_0}\right) e^{-r/2a_0}\]  \hspace{1cm} (5)

where \( a_0 \) is the Bohr radius.

In a real Hydrogen atom, the assumptions behind Eq.(1) do not hold exactly. Certain deviations from these assumptions can be modelled by an additional contribution to the potential for the system at \( r = 0 \).

Here we consider this additional effect as a perturbation to the original model, \(-\lambda\delta(r)/4\pi r^2\), parametrised by a positive constant \( \lambda \), where \( \delta(r) \) is the Dirac delta function in \( r \).

(iii) Show that the first order changes \( \Delta E_{nl} \) due to this perturbation can be written

\[\Delta E_{nl} = -\frac{\lambda}{4\pi} |R_{nl}(0)|^2\]  \hspace{1cm} (6)

[4 marks]

(iv) Hence obtain expressions for \( \Delta E_{nl} \) in terms of \( \lambda \) and \( a_0 \) for the 1s, 2s, and 2p states. \hspace{1cm} [4 marks]

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In general, the first order correction \( u_n^{(1)} \) to the wavefunction \( u_n \) under a perturbation \( H' \) to the Hamiltonian is given by

\[
u_n^{(1)} = \sum_{m \neq n} \frac{\int u_m^* H' u_n d^3r}{E_n - E_m} u_m
\]

in standard notation.

(v) Show that the first-order expression for the wavefunction of the 1s state with the perturbation, in terms of the unperturbed wavefunctions \( \psi_{nlm}(r) \) is as follows:

\[
\psi_{10}^{(1)}(r) = \psi_{10}(r) + \frac{\sqrt{2} \lambda}{3\pi a_0^3 R_H hc} \psi_{20}(r).
\]

(8) [4 marks]

(vi) By considering only the contributions from the terms in Eqs. 3 to 5, and ignoring terms in \( \lambda^2 \), obtain an expression for the relative likelihood for the electron in the perturbed 1s state to be at \( r = 0 \), compared to the unperturbed case. [3 marks]

[Total 20 marks]
7. Electrons in a ferromagnet whose spins are oriented in the direction of, or opposite to, the internal magnetisation carry independent currents $I_+$ and $I_-$. This leads to the material behaving as though it has different conductivity $\sigma_+$ and $\sigma_-$ for each of the two current components. These currents may be thought of as flowing through parallel resistances. Two ferromagnetic layers with opposite magnetisation are placed next to each other as shown in the figure.

Each layer has a thickness $t$ and area $A$. When electrons pass from one layer to the other their spin-direction remains unchanged. A voltage $U$ is placed across the layers in series with an external resistor $R$.

(i) Show that the total resistance of the circuit is

$$R_0 = \frac{t}{2A} \left( \frac{1}{\sigma_+} + \frac{1}{\sigma_-} \right) + R.$$  

[5 marks]

(ii) If an external magnetic field above a certain strength is applied to the system, the two ferromagnetic layers will be magnetised in the same direction. Show that the total resistance is now

$$R_H = \frac{2t}{A(\sigma_+ + \sigma_-)} + R.$$  

[4 marks]

(iii) Now assume that $\sigma_+ \gg \sigma_-$. 

(a) Show that the change in current, $\Delta I = I_H - I_0$, between the two situations can be written as

$$\Delta I = U \left( \frac{1}{\frac{2t}{A\sigma_+} + R} - \frac{1}{\frac{2t}{2A\sigma_-} + R} \right).$$  

[4 marks]

(b) Show that we get the maximum change in current when adjusting the dimensions of the device such that

$$\frac{t}{A} = R\sqrt{\sigma_+ \sigma_-}.$$  

[5 marks]
(iv) A hard disk read head can be manufactured by using a layered device as described above and then measuring the current passing through it. Comment on how modifying the distance of the drive head to the magnetised disk surface and the dimensions of the read head will affect the maximum information density of the hard disk.

[2 marks]

[Total 20 marks]
8. A plane monochromatic electromagnetic wave is scattered by a free electron initially at rest.

(i) Assuming that it is polarized in the \( \hat{x} \) direction and propagates in the \( \hat{z} \) direction with complex amplitude \( E_0 \), write down the equation for the complex incident electric field. [2 marks]

(ii) Write down the equation of motion for the electron assuming its motion is non-relativistic. [2 marks]

(iii) The energy density of the electromagnetic wave in part (i) is \( \varepsilon_0 E_0^2 / 2 \). What is the energy flux carried by the wave? [2 marks]

(iv) The power per unit solid angle radiated by an accelerated, non-relativistic electron is

\[
\frac{dP}{d\Omega} = \frac{e^2}{16\pi^2\varepsilon_0 c^3} \langle |a|^2 \rangle \sin^2 \alpha.
\]

where \( a \) is the acceleration and \( \alpha \) is the angle between \( a \) and the outgoing radiation (direction \( \hat{n} \)). Angle-brackets denote time averages. Use this to calculate the differential scattering cross section

\[
\frac{d\sigma}{d\Omega}(\alpha) = \text{energy radiated into } \hat{n} / \text{unit time/unit solid angle} / \text{incident energy flux in energy/unit area/unit time}.
\]

Show that it is proportional to \( \sin^2 \alpha \) and find the constant of proportionality. [4 marks]

(v) Now, assume unpolarized incident radiation and show that the differential cross section, as a function of \( \theta \), the angle between the incident and outgoing radiation, is

\[
\frac{d\sigma}{d\Omega}(\theta) = \left( \frac{e^2}{4\pi\varepsilon_0 m_e c^2} \right)^2 \frac{1 + \cos^2 \theta}{2}.
\]

Figure 1 Coordinate system for subquestion (v).
(Hint: set up a coordinate system, as in the Figure, in which one axis is the polarization direction, one is the incoming direction of propagation of the wave, and the third is orthogonal to the other two. You can then write the outgoing direction in terms of this coordinate system, and hence $\alpha$ in terms of $\theta$ and $\phi$, and average over the polarization angle, $\phi$.)

[vi] Calculate the total unpolarized scattering cross section

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega}$$

and find its numerical value. (The differential element of solid angle is $d\Omega = \sin \theta d\theta d\phi$.)

[6 marks]

[Total 20 marks]
9. An infrared radiometer is composed of a cylindrical container of cross-sectional area $A$ filled with $n_m$ moles of an ideal monotomic gas. The radiation, of net flux density $F \text{ Wm}^{-2}$, is incident on one end of the cylinder which can be assumed to be a perfect absorber and in perfect thermal contact with the gas. The other end of the cylinder is made of a flexible membrane which allows the gas to expand in response to the heating caused by the absorption of radiation (see Figure 1).

![Figure 1](image)

(i) The radiation is incident for a time $\Delta t$. Assuming that the pressure in the chamber remains constant at the value of the surrounding atmosphere, $p_a$, and neglecting any heat loss to the environment, show that the increase in volume, $\Delta V$, of the gas is:

$$\Delta V = \frac{n_m R F A}{p_a} C \Delta t$$

where $C$ is the heat capacity of the cylinder/gas system and $R$ is the universal gas constant. [3 marks]

(ii) If the heat capacity of the container itself is negligible compared with that of the gas it contains, show that:

$$\Delta V = \frac{2 FA}{5 p_a} \Delta t.$$ [2 marks]

(iii) As the gas expands the membrane assumes a spherical surface so that the volume added to the cylinder is a spherical cap with base area $A$ (see Figure 1). Given that the volume of a spherical cap is approximately $\frac{1}{4} \pi r^3 \theta^3$, where $r$ is the radius of the sphere and $\theta$, assumed $\ll 1$, is the half-angle of the associated spherical cone, show that:

$$r \approx \frac{A^2}{4 \pi \Delta V}$$ [4 marks]

To find the power of the radiation an optical method is used to measure the radius of curvature of the membrane which acts as a convex mirror. A point source of light is placed a distance $f$ behind a thin lens close to the membrane, where $f$ is the focal length of the lens. The light reflected off the membrane is refocussed through the same lens and the position of the image found (see Figure 2).
(iv) (a) At what distance behind the membrane is the virtual image of the light source? [2 marks]

(b) Neglecting the distance between the lens and the membrane, and assuming that \( f \ll r \), show that a real image of the light source will be located a distance

\[
x \simeq \frac{2f^2}{r}
\]

behind the source. [3 marks]

(v) Combining the results of parts (ii)-(iv) show that the flux density of the radiation can be deduced from a measurement of distance \( x \) according to:

\[
F = \frac{5}{16\pi} \frac{p_o A}{\Delta t} \frac{x}{f^2}.
\]

[2 marks]

(vi) At room temperature the membrane is flat; then the instrument is exposed for 0.1sec to a black body at a temperature 10°C above room temperature. If the cylinder has diameter 15mm and the lens a focal length 5cm estimate the displacement \( x \) of the focussed beam. [Assume room temperature and pressure to be 20°C and 10^5 Pa respectively.] [4 marks]

[Total 20 marks]
10. Write an essay about ONE of the following subjects.

(i) Given financial constraints the government should fund only science with clear practical applications.

(ii) Space Weather.

(iii) Is Pluto a planet?

(iv) The physics of fuel cells.


(vi) “Early claims that string theory would provide a Theory of Everything now seem hollow indeed.” (Frank Wilczek)

[Total 20 marks]