Optimal Asset Allocation
with Factor Models for Large Portfolios

M. Hashem Pesaran
University of Cambridge and USC

Paolo Zaffaroni
Imperial College London

July 15, 2008

Abstract

This paper characterizes the asymptotic behaviour, as the number of assets gets arbitrarily large, of the portfolio weights for the class of tangency portfolios belonging to the Markowitz paradigm. It is assumed that the joint distribution of asset returns is characterized by a general factor model, with possibly heteroskedastic components. Under these conditions, we establish that a set of appealing properties, so far unnoticed, characterize traditional Markowitz portfolio trading strategies. First, we show that the tangency portfolios fully diversify the risk associated with the factor component of asset return innovations. Second, with respect to determination of the portfolio weights, the conditional distribution of the factors is of second-order importance as compared to the distribution of the factor loadings and that of the idiosyncratic components. Third, although of crucial importance in forecasting asset returns, current and lagged factors cannot be exploited to forecast the limit portfolio returns. These properties follow since tangency portfolios are asymptotically beta-neutral as the number of assets gets large. Our theoretical results also shed light on a number of issues discussed in the literature regarding the limiting properties of portfolio weights such as the diversifiability property and the number of dominant factors.

JEL Classifications: C32, C52, C53, G11

Key Words: Asset Allocation, Large Portfolios, Factor Models, Diversification, Beta Neutrality.

*A preliminary version of this paper was presented at the Cambridge Finance Workshop, the Gersenzzee CEPR Conference, the Imperial College Financial Econometrics Conference, and at the Bendheim Center at Princeton, University of Pennsylvania, LSE and the University of Brunel. We are grateful to the participants, and in particular to the discussants of our paper Jianqing Fan and Eric Renault for most helpful comments.
1 Introduction

Factor models represent a parsimonious, yet flexible way of modelling the conditional joint probability distribution of asset returns when there are a large number of assets under consideration. Prominent use of factor models initially focused on parameterizing the conditional mean, following the highly influential capital asset pricing model of Sharpe (1964) and Lintner (1965), and the arbitrage pricing theory of Ross (1976). In fact parsimony plays an even more important role when modeling conditional covariance matrix of a large number of asset returns.

Given that the main rationale for using factor models is to deal with portfolios with a large number of assets, this paper characterizes the distribution of portfolio weights, as the number of assets, $N$, increases without bounds, in the case of the commonly used mean-variance efficient portfolios (hereafter MV). Our analysis is confined to myopic asset allocation rules, all particular cases of Markowitz (1952) theory, which are optimal only for a constant investment opportunity set. Focusing on myopic trading strategies is justified from a practical perspective in the case of large portfolios where application of dynamic asset allocation strategies can be prohibitive and is rarely tried in practice. The literature on dynamic asset allocation is often confined to a few broad asset classes, such as Treasury Bills, long term bonds and equities (see, for example, Campbell and Viceira (2002)).

A number of papers have already examined the limiting behavior of MV portfolios when there are a countably infinite number of primitive assets under consideration. Chamberlain (1983) and Chamberlain and Rothschild (1983) studied the implications of no arbitrage for the MV efficient frontier as $N$ tends to infinity. They then considered factor models and extended the arbitrage pricing theory (APT) result of Ross (1976) to the case where asset returns follow an approximate factor structure. The latter extends the exact factor model by permitting certain (limited) degree of correlation across the idiosyncratic component of asset returns. Hansen and Richard (1987) extended the static framework of Chamberlain (1983) and Chamberlain and Rothschild (1983), but did not focus on factor structures. Subsequently, Green and Hollifield (1992) clarified the relationships that exist between diversification and MV efficiency in a general setting. Employing a factor structure, these authors provided a further generalization showing that even the approximate factor structure is too stringent for the APT to hold. Whereas Chamberlain (1983) characterize diversifiability by looking at
the rate at which the square norm of the portfolio weights converge to zero as \( N \) tends to infinity, Green and Hollifield (1992) characterize diversifiability in terms of sup-norm criteria. Sentana (2004) compares the statistical properties of static and dynamic factor representing portfolios, using a dynamic version of the APT.

By and large all of the above papers focus on various aspects and generalizations of the APT under the maintained assumption of an underlying factor structure as \( N \to \infty \). However, once one abstracts from the APT, a number of other interesting issues arise that have hitherto been neglected in the literature. For instance, the precise behavior of the MV portfolio weights as \( N \to \infty \) has been surprisingly overlooked. Likewise, to our knowledge the statistical properties of the limit portfolio return have not been spelled out. It turns out that interesting, and in fact somewhat counter-intuitive, results arises from these investigations, in particular regarding the role played by the conditional distribution of the factors. In this paper we do not make use of no-arbitrage assumption and, therefore, do not investigate implications of the APT in the case of large portfolios, unlike Hubermann (1982), Chamberlain (1983), Chamberlain and Rothschild (1983), Stambaugh (1983), Connor (1984), Ingersoll (1984), Grinblatt and Titman (1987), Green and Hollifield (1992), Sentana (2004) among others. Likewise, our goal here is not to investigate issues related to parameter uncertainty when estimating MV trading strategies (see Brandt (2004) for a complete survey on the econometric issues associated with portfolio choice problems).

We make the maintained hypothesis that the vector of asset returns is distributed according to a dynamic factor model, with a specification of the conditional variance matrix of the idiosyncratic components which is more general than the approximate factor structure of Chamberlain and Rothschild (1983). Under this assumption, the paper establishes three main results:

(a) In the limit the MV portfolios fully diversify the innovations in the common factor components of asset returns. It is well known that MV portfolios do fully diversify the idiosyncratic component of asset returns innovations, but to our knowledge it is not recognized that the same applies to the factor innovations. This is an important feature of MV portfolios with practical implications which we discuss below.

(b) The limit MV portfolio weights (the first-order limit approximation for large \( N \) of the MV portfolio weights) are functionally independent of the conditional distribution of the factors. Notice that this does not imply that the factors themselves are not important but only that their (conditional)
moments are not relevant in so far as the calculation of the MV portfolio weights is concerned. For example, estimation of the factors and their loadings are required for a consistent estimation of the idiosyncratic components.

(c) In the limit the MV portfolio returns are functionally independent of the current and lagged values of the common factors. The factors could play a central role for forecasting asset returns but, as $N$ gets larger, their role vanishes in terms of their contribution to the limit portfolio returns. In other words, at any point in time in the limit as $N \to \infty$, the conditional distribution of the limit returns on MV portfolios are functionally independent of the conditional distribution of the common factor component.

Neither of the above findings strictly implies the other and are all of independent interest. Initially these results may seem rather counter intuitive since one would expect the factor component, being dominant, will be the most important element in the determination of asset returns. But MV portfolios are functions of the inverse of the covariance matrix of asset returns, and the common factor part of asset returns that generate strong cross return dependence will turn into weak cross dependence when the inverse of the variance matrix is considered. By comparison, the idiosyncratic components of asset returns that exhibit weak cross section dependence will begin to play a central role in determination of MV portfolios as $N$ starts to become sufficiently large. Concepts of weak and strong cross section dependence are developed in Pesaran and Tosetti (2007). In particular the concept of weak cross section dependence allows the maximum eigenvalue of the covariance matrix of the idiosyncratic component of asset returns to rise like $o(N)$. Formally, our results follows from a form of asymptotic orthogonality, newly established in this paper, between the inverse of the conditional covariance matrix of asset returns and the matrix of factor loadings. This form of orthogonality directly implies that all MV trading strategies, and in fact all trading strategies that employ the inverse of the asset returns covariance matrix in the same way, are asymptotically beta-neutral. Finally, note that none of these results suggest that the factors themselves are unimportant for the limit MV portfolio weights and the portfolio return, rather that it is the contribution of factors’ conditional distribution to the MV portfolio return that is asymptotically negligible. In fact, knowledge of the factors is essential in practice when deriving the other parts of the factor model, namely the factor loadings and the predictable and non-predictable idiosyncratic components of the asset returns, that are of first-order importance for the MV trading strategies and their associated return.
The above findings also have a number of further implications of interest that we summarize below:

(d-i) The limit MV portfolios are time-invariant unless, depending on the trading strategies, the risk free rate is time-varying and the idiosyncratic component features time-varying conditional heteroskedasticity.

(d-ii) The limit MV portfolio weights are invariant to any orthogonal rotation of the factors.

(d-iii) Primitive conditions required for full-diversification in the sup-norm sense of Green and Hollifield (1992) are established.

(d-iv) Analytical characterizations of the occurrence of negative portfolio weights and of the related issue of factor dominance, in the sense of Green and Hollifield (1992) and Jagannathan and Ma (2003), are provided.

The remainder of the paper is organized as follows. Section 2 introduces the concepts, sets out the dynamic factor model, and discusses its properties. The main findings are illustrated, as an example, with respect to a single factor model in Section 3. Section 4 presents the main results in the general with respect to the commonly used trading strategies: the global minimum-variance and the maximum expected quadratic utility portfolios. Section 5 elaborates and discusses the implications of the theoretical results. Section 6 extends the results to two other tangency portfolios, namely the minimum-variance and the maximum expected return portfolios. Section 7 concludes. Mathematical proofs are collected in an appendix.

2 Factor model: definitions and assumptions

We assume the $N$-dimensional vector $\mathbf{r}_t = (r_{1t}, r_{2t}, ..., r_{Nt})'$ of asset returns can be characterized by the following linear dynamic factor model

$$
\mathbf{r}_t = \alpha_{t-1} + \mathbf{Bf}_t + \mathbf{\varepsilon}_t,
$$

where $\mathbf{f}_t$ is the $k \times 1$ vector of possibly latent common factors, $\mathbf{B} = (\beta_1, ..., \beta_N)'$ is an $N \times k$ matrix of factor loadings, $\mathbf{\varepsilon}_t$ is an $N \times 1$ vector of idiosyncratic
components, and the $N \times 1$ vector $\alpha_{t-1}$ represents the part of the conditional mean of the $r_t$ that does not depend on the common factors. Throughout it will be assumed that $k$ remains fixed as $N \to \infty$. We identify the factor model by means of the following assumptions:

Assumption 1 (conditional mean returns) The vector of latent factors $f_t$ can be decomposed into its predictable component, $\nu_{t-1}$, and the remainder $u_t$ as

$$f_t = \nu_{t-1} + u_t,$$

where $\nu_{t-1} = E(f_t \mid Z_{t-1})$, with $Z_{t-1}$ being the sigma-algebra induced by a $N \times g$ matrix of observed variates $\{Z_{t-s}, s > 0\}$.

$$\alpha_{t-1} = E(r_t - Bf_t \mid Z_{t-1}),$$

where $\alpha_{t-1}$ and $u_t$ are independently distributed for all $t$ and $s$.

Under this assumption the conditional mean of asset returns is given by

$$E(r_t \mid Z_{t-1}, B) \equiv \mu_{t-1} = \alpha_{t-1} + Bu_{t-1},$$

and the innovation in the common components $u_t$ is a martingale difference process with respect to $Z_{t-1}$.

It is worth noting that the decomposition in Assumption 1 can also be defined with respect to the sigma-algebra spanned by the unobserved information set $f_{t-s}, s > 0$. This will not affect our main conclusions, but will raise a number of additional difficulties with respect to the empirical implementation of the model.

Assumption 1 rules out a dynamic factor representation of asset returns (see Forni, Hallin, Lippi, and Reichlin (2000) and Stock and Watson (2002)) such as $r_{it} = \alpha_{i,t-1} + \beta_i(1 - c_i L)^{-1}u_t + \varepsilon_{it}$, where $L$ is a lag operator and $c_i$ differs across $i$. This does not seem a particularly important limitation in the case of asset pricing models where the returns are only tenuously serially correlated.

In practice, specification and estimation of $\mu_{t-1}$ could be a major empirical undertaking, particularly in the case of large portfolios. But given the focus of our analysis, in what follows we take the specification of $\mu_{t-1}$, especially its $\alpha_{t-1}$ component, as given. It is important, nevertheless, to separate
\( \alpha_{t-1} \) from \( \nu_{t-1} \) since the latter, as we shall see, does not enter the limit MV portfolios as \( N \) gets large.

Conditions (2), (3) and (4) together imply
\[
\mathbf{r}_t = \mu_{t-1} + \mathbf{B} \mathbf{u}_t + \mathbf{\varepsilon}_t.
\] (6)

Hereafter, we shall refer to (6) as the factor model with \( \mathbf{u}_t \) being the \( k \times 1 \) vector of factor innovations without further reference to \( \mathbf{f}_t \).

Regarding the factor loadings, we consider the case where the elements of \( \mathbf{B} \) are random variates satisfying the following limit condition:

**Assumption 2** (factor loadings) As \( N \to \infty \)
\[
\mathbf{B}' \mathbf{e} \to_p \bar{\beta} \neq 0,
\] (7)
where \( \mathbf{e} = (1, ..., 1)' \) is an \( N \times 1 \) vector of ones, and \( \to_p \) denotes convergence in probability.

From (7) it follows that \( \bar{\beta} \) represents the mean vector \( \mathbb{E}(\beta_i) \). Assumption 2 is an ergodicity assumption over the cross section. It is much weaker than the i.i.d. assumption typically made when considering random factor loadings. For instance, a strong sufficient condition for (7) to hold is when the factor loadings have finite second-order moments and absolutely summable cross covariances but, in fact, Assumption 2 is compatible with a much more substantial degree of (cross-sectional) dependence among the elements of \( \beta_i \).

The results presented in this paper can be generalized further to the case of heterogeneous yet non-random \( \beta_i \).

**Assumption 3** (innovations) At any given point in time \( t \)
\[
\mathbf{u}_t \mid Z_{t-1} \sim (0, \Omega_{t-1}), \quad \mathbf{\varepsilon}_t \mid Z_{t-1} \sim (0, \mathbf{H}_{t-1}),
\] (8)
\[
\mathbf{\varepsilon}_t \text{ and } \mathbf{u}_t \text{ are mutually independent},
\] (9)
where \( \Omega_{t-1} \) and \( \mathbf{H}_{t-1} \) are positive definite matrices, respectively, of dimension \( k \times k \) and \( N \times N \) for a fixed \( k \) and any finite \( N \).

The results that follow do not depend on a particular specification of the volatility model characterizing the asset returns. Moreover, the factors can either be observable or non-observable. As a consequence, \( \Omega_{t-1} \) and \( \mathbf{H}_{t-1} \) could belong to the multivariate stochastic volatility class as well as to the
generalized autoregressive conditional heteroskedasticity class of volatility models. Particular examples, to which Assumption 3 applies, are discussed below.

To derive the limiting behavior (as $N \to \infty$) of the various tangency portfolio weights to be considered below, we further require the following assumption:

**Assumption 4 (limit conditions)** At any given point in time $t$ as $N \to \infty$

$$\frac{(B - e\bar{β}')(H^{-1}_t(B - e\bar{β}'))}{N} \xrightarrow{p} A_t > 0,$$

$$\frac{B'H^{-1}_tH^{-1}_tB}{N} \xrightarrow{p} C_t \geq 0,$$

and

$$\frac{e'H^{-1}_t e}{N} \xrightarrow{p} a_t > 0,$$

$$\frac{e'H^{-1}_t\alpha_t}{N} \xrightarrow{p} c_t,$$

$$\frac{\alpha'_tH^{-1}_t\alpha_t}{N} \xrightarrow{p} d_t > 0,$$

where hereafter $> 0$ and $\geq 0$ means, respectively, positive definitive and positive semi-definite.

Moreover $a_t, c_t, d_t, A_t, C_t$ are $O_p(1)$ (element by element) such that

$$d_t a_t - c_t^2 > 0 \text{ almost surely}$$

and

**B is independently distributed from both $H_t$ and $\alpha_t$.**

(16)

The common feature of the limits presented in Assumption 4 is that they involve, possibly weighted, averages of the elements of $H^{-1}_t$. In particular, they impose implicitly an upper bound on the speed with which the maximum eigenvalue of $H^{-1}_t$ could diverge to infinity. (Recall that the largest eigenvalue of $H^{-1}_t$ coincide with the smallest eigenvalue of $H_t$, by construction.) This is clearly seen from condition (12): assuming for illustrative
purposes that $H_t^{-1}$ is diagonal, with $h_{ii,t}^{-1}$ in the $(i,i)$th entry, then (12) allows $\max_{1 \leq i \leq N} h_{ii,t}^{-1} = o_p(N)$. Condition (11) requires a further constraint on the speed of divergence of $\max_{1 \leq i \leq N} h_{ii,t}^{-1}$ which can now be at most $o_p(N^{\frac{1}{2}})$. Even this case is much weaker than $\max_{1 \leq i \leq N} h_{ii,t}^{-1} \leq C < \infty$, for some constant $C$, implied by the definition of approximate factor structure (see Chamberlain and Rothschild (1983)). Green and Hollyfield (1992) were the first to note that, insofar as optimal asset allocation is concerned, a degree of cross-sectional dependence stronger than the one implied by the approximate factor structure is permitted. When $H_t^{-1}$ is non-diagonal, the previous discussion applies to its largest eigenvalues.

Conditions (10) and (11) require the existence of the second-order moments of the factor loadings and impose certain constraints on the degree of cross-sectional dependence of the $\beta_i$. Note that when (7), (12) and (16) hold, then (10) is equivalent to saying that $N^{-1}B'H_t^{-1}B$ has a positive definite limit. When $\beta_i$ are i.i.d. and $H_t$ is diagonal, then $A = a_t \text{cov}(\beta_i)$. Concerning (16), note that $H_t$ and $\alpha_t$ need not be, and in general will not be, mutually independent. Conditions (13) and (14) also require the elements of $\alpha_t$ not to grow, if at all, too fast as compared with $N$. The limit $c_t$ in condition (13) is bounded, in absolute value, by $(a_t d_t)^{\frac{1}{2}}$. The limit $d_t$ in condition (14) is finite whenever (12) holds and $\alpha_t' \alpha_t/N$ has a finite limit. Condition (15) is not needed in the case where $\alpha_t$ is a non-degenerate random variable.

For some results, in particular to derive the limit distribution of the MV portfolio weights, a stronger version of Assumption 4 is needed as set out below:

**Assumption 5** (further limit conditions)

For any $i$ and at any given point in time $t$, as $N \to \infty$

\begin{align*}
B' H_t^{-1} e_i^{(N)} &\to_p \xi_{1it}, \\
e' H_t^{-1} e_i^{(N)} &\to_p \xi_{2it}, \\
\alpha_i' H_t^{-1} e_i^{(N)} &\to_p \xi_{3it},
\end{align*}

with $\|\xi_{jit}\| = O_p(1)$, for $j=1,2,3$, where $e_i^{(N)}$ is the $i$th column of the identity matrix $I_N$ and $\|\cdot\|$ denotes the Euclidean norm.
\[
N^\frac{1}{2} \left( N^{-1} \text{vech}(\B H_t^{-1} \B) - \text{vech}(\A_t + a_t \bar{\beta}) \right) \\
N^{-1} \B H_t^{-1} \e - a_t \bar{\beta} \\
N^{-1} \B H_t^{-1} \alpha_t - c_t \bar{\beta} \\
N^{-1} \e' H_t^{-1} \e - a_t \right) \rightarrow_d N(0, \V_t), \\
(20)
\]

for some positive semi-definite matrix \( \V_t \), where \( \rightarrow_d \) denotes convergence in distribution and \( \text{vech}(\A) \) stacks the diverse elements of a symmetric matrix \( \A \) into a column vector.

Conditions (17)-(18)-(19), impose a finite upper bound to each of the columns of \( \H_t^{-1} \) and are therefore much stronger than (10)-(12)-(13) that are expressed in terms of averages. In particular, (18) is satisfied by an approximate factor structure. Condition (20) is somewhat weaker than the other parts of Assumption 5 although, again, it allows for a smaller degree of cross-sectional dependence than the one permitted by Assumption 4. In particular, note that (20) rules out that \( \alpha_t \) contains a common factor structure. This is not restrictive as it appears, since any common components of \( \alpha_t \) can be included in \( \nu_{t-1} \).

In view of (8), the factor structure (6) implies the well-known form of the asset return conditional variance-covariance matrix:

\[
E [(r_t - \mu_{t-1})(r_t - \mu_{t-1})' | Z_{t-1}, \B] = \Sigma_{t-1} = B \Omega_{t-1} B' + H_{t-1}. \\
(21)
\]

Thus model (6) nests the various factor models with time-varying conditional second moment proposed in the econometrics literature (see among many others Diebold and Nerlove (1989), King, Sentana, and Wadhwani (1994), Chib, Nardari, and Shephard (2002), Fiorentini, Sentana, and Shephard (2004), Connor, Korajczyk, and Linton (2006), Doz and Renault (2006)). These papers, which focus on estimation of volatility factor models, in particular when \( u_t \) is not observable, all assume constant conditional first-order moments. On the other hand, the finance literature dealing with factor models-based asset allocation assumes homoskedastic factors whereby \( \Omega_{t-1} = \Omega \), often normalized to be equal to the identity matrix (see among many others Pesaran and Timmermann (1995) and Kandel and Stambaugh (1996)). A few contributions analyze asset allocation problems allowing for volatility dynamics but impose constant conditional means (see for instance Aguilar and West (2000) and Fleming, Kirby, and Ostdiek (2001)). Only recently, a limited number
of studies have considered time variations in both the first and second conditional moments of asset returns (see for instance Johannes, Polson, and Stroud (2002) and Han (2006)). Model (6) nests all of the above specifications.

3 The single factor case

Here we illustrate our results using a single factor model \((k = 1)\) where (6) becomes

\[
\mathbf{r}_t = \mu_{t-1} + \beta \mathbf{u}_t + \mathbf{e}_t, \tag{22}
\]

and Assumptions 1, 2 and 3 hold. Therefore now \(u_t\) is a scalar martingale difference process with conditional variance \(\omega_{t-1} > 0\), and \(\beta\) is a \(N \times 1\) vector of factor loadings with mean \(\beta \mathbf{e} \neq \mathbf{0}\), and the variance matrix \(\sigma_\beta^2 \mathbf{I}_N > 0\), where \(\beta\) and \(\sigma_\beta^2\) are scalar. Let us also assume that the idiosyncratic errors \(\epsilon_{it}\) are cross-sectionally uncorrelated, implying a diagonal \(\mathbf{H}_{t-1}\), with conditional variances \(h_{ii,t-1} > 0\), a.s. The conditional covariance matrix of \(\mathbf{r}_t\) will then be

\[
\Sigma_{t-1} = \omega_{t-1} \beta \beta' + \mathbf{H}_{t-1}.
\]

Note that an observationally equivalent model can be obtained if one assumes homoskedastic factor, viz. \(E(u_t^2 | \mathbf{Z}_{t-1}) = 1\), but allow time-varying factor loadings of the form, \(\beta_t = \omega_t^{\frac{1}{2}} \beta\).

Consider now two types of portfolios: the global minimum-variance (henceforth \(\text{gmv}\)) portfolio and the maximum expected utility (henceforth \(\text{meu}\)) portfolio.

3.1 The limit behaviour of the \(\text{gmv}\) portfolio

The \(\text{gmv}\) portfolio weights, \(\mathbf{w}_{t}^{\text{gmv}} = (w_{1t}^{\text{gmv}}, ..., w_{Nt}^{\text{gmv}})'\), are the solution to the problem:

\[
\mathbf{w}_{t}^{\text{gmv}} = \arg\min_{\mathbf{w}} (\mathbf{w}' \Sigma_{t} \mathbf{w}) , \text{ such that } \mathbf{w}' \mathbf{e} = 1, \tag{23}
\]

yielding

\[
\mathbf{w}_{t-1}^{\text{gmv}} = \frac{\Sigma_{t-1}^{-1} \mathbf{e}}{\mathbf{e}' \Sigma_{t-1}^{-1} \mathbf{e}}. \tag{24}
\]

It is well known that this portfolio does not belong to the efficient frontier, except when the conditional expected returns \(\mu_{i,t-1}\) are the same across \(i\).
but, with some abuse of notation, we will view it as belonging to the set of MV trading strategies. Nevertheless, this portfolio is still of interest since its implementation does not require the estimation of expected returns. Jagannathan and Ma (2003) show that, in terms of asset allocation, its out-of-sample performance is comparable with the performance of other tangency portfolios.

In the case of the single factor model the optimal portfolio weight for the \( i \)th asset is given by

\[
w_{it}^{gmv} = N^{-1} h_{it}^{-1} \phi_{N,t}^{-1} \left( 1 - \beta_i \kappa_{N,t} \right),
\]

where

\[
\phi_{N,t} = N^{-1} \sum_{j=1}^{N} h_{jj,t}^{-1} - \frac{(N^{-1} \sum_{j=1}^{N} \beta_j h_{jj,t}^{-1})^2}{(N^{-1} \omega_t^{-1} + N^{-1} \sum_{j=1}^{N} \beta_j^2 h_{jj,t}^{-1})}, \tag{25}
\]

\[
\kappa_{N,t} = \frac{(N^{-1} \sum_{j=1}^{N} \beta_j h_{jj,t}^{-1})}{(N^{-1} \omega_t^{-1} + N^{-1} \sum_{j=1}^{N} \beta_j^2 h_{jj,t}^{-1})} \tag{26}
\]

Under Assumption 4, in particular given (16),

\[
\frac{\mathbf{e}' \mathbf{H}_t^{-1} \mathbf{e}}{N} = \frac{\sum_{j=1}^{N} h_{jj,t}^{-1}}{N} \rightarrow_p \mathbf{a}_t, \quad \frac{\beta' \mathbf{H}_t^{-1} \mathbf{e}}{N} = \frac{\sum_{j=1}^{N} \beta_j h_{jj,t}^{-1}}{N} \rightarrow_p \mathbf{a}_t \bar{\beta}, \tag{27}
\]

\[
\frac{\beta' \mathbf{H}_t^{-1} \beta}{N} = \frac{\sum_{j=1}^{N} \beta_j^2 h_{jj,t}^{-1}}{N} \rightarrow_p \mathbf{a}_t (\sigma_\beta^2 + \bar{\beta}^2),
\]

and since \( \omega_t^{-1} = O_p(1) \) we have

\[
\phi_{N,t} \rightarrow_p \frac{\mathbf{a}_t \sigma_\beta^2}{\sigma_\beta^2 + \bar{\beta}^2}, \quad \kappa_{N,t} \rightarrow_p \frac{\bar{\beta}}{\sigma_\beta^2 + \bar{\beta}^2}.
\]

Hence, it readily follows that

\[
N w_{it}^{gmv} \rightarrow_p \frac{h_{it}^{-1}}{\alpha_t} \left( 1 - \frac{\bar{\beta}}{\sigma_\beta^2} (\beta_i - \bar{\beta}) \right). \tag{28}
\]

which clearly shows that the limit \( gmv \) portfolio weights are functionally independent of the factor conditional variance, \( \omega_t \). The limit \( gmv \) portfolio
weights are also asymptotically beta-neutral in the sense that

$$
\beta' w_{t-1}^{gmv} = \sum_{i=1}^{N} \beta_i w_{it}^{gmv} = N^{-1} \sum_{i=1}^{N} \beta_i h_{ii}^{-1} \left(1 - \frac{\overline{\beta}}{\sigma_\beta^2} (\beta_i - \overline{\beta}) \right) (1 + o_p(1)) \to_p 0.
$$

(29)

It is also worth noting that the limit portfolio weights are time-varying only if $h_{ii,t-1}$ are time-varying. Moreover, notice that if we assume, for some positive stochastic process $\theta_t > 0$,

$$
h_{ii,t} = h_{ii} \theta_t > 0,
$$

(30)

then despite the time varying nature of $h_{ii,t}$, the limit gmv portfolio weights will be time invariant since

$$
N w_{it}^{gmv} \to_p \frac{1}{a} \left(1 - \frac{\overline{\beta}}{\sigma_\beta^2} (\beta_i - \overline{\beta}) \right),
$$

where $a = p \lim_{N \to \infty} \left(N^{-1} \sum_{j=1}^{N} h_{jj}^{-1}\right)$.

Consider now the limit properties of the portfolio return $\rho_t^{gmv} = r'_t w_{t-1}^{gmv}$. By (22) this can be written as

$$
\rho_t^{gmv} = \mu'_t w_{t-1}^{gmv} + (\beta' w_{t-1}^{gmv}) u_t + \varepsilon'_t w_{t-1}^{gmv}.
$$

(31)

Since we take the limit as $N \to \infty$, we can directly use the limit approximation yielding, for the third term in $\rho_t^{gmv}$,

$$
\varepsilon'_t w_{t-1}^{gmv} = O_p \left(\frac{1 + \overline{\beta}^2/\sigma_\beta^2}{a_{t-1}^{1/2} N^{1/2}}\right) = o_p(1),
$$

establishing that the gmv portfolio does diversify away the idiosyncratic risk.

In view of (29) the second term of (31) also vanishes as $N \to \infty$, and hence there will be no contribution from the common source of risk $u_t$ to the limit portfolio return. Therefore, the only term of $\rho_t^{gmv}$ that is not vanishing is

$$
\mu'_t w_{t-1}^{gmv} = \alpha'_t w_{t-1}^{gmv} + (\beta' w_{t-1}^{gmv}) \nu_t.
$$

However, we have just seen that $\beta' w_{t-1}^{gmv} = o_p(1)$ implying that there is no contribution from the predictable common component $\nu_{t-1}$ to the limit
portfolio return. Instead, one simply needs to consider the limit of

\[ \alpha_{t-1} w_{t-1}^{gmv} = \left( \frac{1}{Na_{t-1}} \sum_{j=1}^{N} h_{j,i,t-1}^{-1} \left[ 1 - \frac{\beta}{\sigma_{\beta}} (\beta_j - \bar{\beta}) \right] \alpha_{j,t-1} \right) \left( 1 + o_{p}(1) \right) \]

\[ = \left( \frac{1}{Na_{t-1}} \sum_{j=1}^{N} h_{j,i,t-1}^{-1} \alpha_{j,t-1} - \frac{\beta}{Na_{t-1} \sigma_{\beta}} \sum_{j=1}^{N} h_{j,i,t-1}^{-1} (\beta_j - \bar{\beta}) \alpha_{j,t-1} \right) \left( 1 + o_{p}(1) \right) \]

\[ = \left( \frac{1}{Na_{t-1}} \sum_{j=1}^{N} h_{j,i,t-1}^{-1} \alpha_{j,t-1} \right) \left( 1 + o_{p}(1) \right) = \frac{c_{t-1}}{a_{t-1}} \left( 1 + o_{p}(1) \right), \]

where by Assumption 4

\[ \frac{\alpha' H^{-1}_t e}{N} = \frac{1}{N} \sum_{j=1}^{N} h_{j,i,t}^{-1} \alpha_{j,t} \to_p c_t. \] (32)

Summarizing, we have

\[ \rho_{t}^{gmv} \to_p c_{t}/a_{t-1}, \] (33)

namely that the limit gmv portfolio return is \( Z_{t-1} \)-adapted since both the idiosyncratic and the common innovation are fully diversified. Moreover, in the light of the asymptotic beta-neutrality property, both the predictable and the non-predictable common components have an asymptotically negligible contribution to the outcomes. The only parts of the asset return distribution relevant to the limit gmv portfolio return are via the terms \( a_t \) and \( c_t \), that involve averages of asset-specific means and volatilities, \( \alpha_{it} \) and \( h_{ii,t}^{-1} \), over \( i \). Finally, if \( h_{ii,t} \) are mutually independent, then the limit gmv portfolio return is given by the limit of the sample mean of the \( \alpha_{it} \), namely in the limit only the asset-specific returns matter.

### 3.2 The limit behaviour of the meu portfolio

Suppose now that besides the \( N \) risky assets, investors can also allocate their funds to a risk free asset with a time-varying rate of return, \( r_{0t} \), which is known at the start of trading period \( t \). The maximum expected utility (meu) portfolio, based on a mean-variance utility function, is defined by the solution to the following optimization problem

\[ w_{t-1}^{mu} = \arg\max_{w} \left( w' \mu_{t-1} + (1 - w'e)r_{0t-1} - \frac{\gamma_{t-1}}{2} w' \Sigma_{t-1} w \right), \] (34)
where \( w_{meu}^t = (w_{meu}^{t1}, \ldots, w_{meu}^{tN})' \), \( 0 < \gamma_{t-1} < \infty \) is, possibly time-varying, coefficient of risk aversion. The solution is

\[
\begin{align*}
  w_{meu}^{t1} &= \frac{1}{\gamma_{t-1}} \Sigma_{t-1}^{-1} (\mu_{t-1} - \text{er}_{0t-1}). 
\end{align*}
\]

(35)

In the case of the single factor model, the \( meu \) portfolio weight for the \( i^{th} \) asset is given by

\[
\begin{align*}
  w_{meu}^{it} &= \gamma_t^{-1} h_{it}^{-1} \left[ \mu_{it} - \beta_i \delta_{t,N} - r_{0t} \left( 1 - \beta_i \kappa_{t,N} \right) \right], \\
  &= \gamma_t^{-1} h_{it}^{-1} \left[ \alpha_{it} + \beta_i (\nu_t - \delta_{t,N}) - r_{0t} \left( 1 - \beta_i \kappa_{t,N} \right) \right], \\
  &= \gamma_t^{-1} h_{it}^{-1} \left[ \alpha_{it} - r_{0t} - \beta_i \left( \delta_{t,N} - \nu_t - \beta_i \kappa_{t,N} \right) \right],
\end{align*}
\]

where

\[
\begin{align*}
  \delta_{t,N} &= \sum_{j=1}^{N} \beta_j h_{jj,t}^{-1} \mu_{jt} - \frac{c_t}{\beta_t (\sigma_{\beta}^2 + \beta^2)} + \nu_t, \\
  \text{with} \ c_t \text{ defined as before by (32). Recall also that under Assumption 4}
\end{align*}
\]

\[
N^{-1} \sum_{j=1}^{N} \beta_j h_{jj,t}^{-1} \alpha_{jt} \to_p \bar{c}_t \beta.
\]

Hence, using the above results and recalling that \( \kappa_{N,t} \to_p \beta/(\sigma_{\beta}^2 + \beta^2) \) we have

\[
\begin{align*}
  w_{meu}^{it} &\to_p \gamma_t^{-1} h_{it}^{-1} \left[ (\alpha_{it} - r_{0t}) - \beta_i \frac{\bar{\beta} (c_t / \sigma_t - r_{0t})}{\sigma_{\beta}^2 + \beta^2} \right] \\
  &= \gamma_t^{-1} h_{it}^{-1} \left[ (1 + \beta^2 / \sigma_{\beta}^2) (\alpha_{it} - r_{0t}) - \beta_i \frac{c_t}{\sigma_{\beta}^2} (\sigma_t - r_{0t}) \right],
\end{align*}
\]

where \( b = 1 + \beta^2 / \sigma_{\beta}^2. \)

Thus, the limit \( meu \) portfolio weights are functionally independent of both \( \nu_t \) and \( \omega_t. \) In contrast to the previous case, when (30) holds the \( meu \) portfolio weights will still be time varying and, in fact, will be so even for homoskedastic \( \varepsilon_{it}. \) Further, notice that each portfolio weight will in general be different from zero even asymptotically in \( N. \)

Consider now the portfolio return

\[
\rho_t^{meu} = r_t' w_t^{meu} + (1 - e' w_t^{meu}) r_{0t-1} = \mu_{t-1} w_t^{meu} + (\beta' w_t^{meu}) u_t + \epsilon_t' w_t^{meu} + (1 - e' w_t^{meu}) r_{0t-1}
\]

15
When normalizing $\rho_{t}^{\text{meu}}$ by $N^{-1}$ for the fourth term one obtains

$$
\frac{\mathbf{e}'\mathbf{w}_{t-1}^{\text{meu}}}{N} = \frac{1}{\gamma_{t-1}b} \left\{ \left(1 + \frac{\beta^{2}}{\sigma_{\beta}^{2}}\right) \left[ \frac{1}{N} \sum_{j=1}^{N} h_{j,1}^{-1} \alpha_{j,t-1} - r_{0,t-1} \frac{1}{N} \sum_{j=1}^{N} h_{j,1}^{-1} \right]
\right.
\left. - \frac{\beta}{\sigma_{\beta}^{2}} \left( c_{t-1}^{1} - r_{0,t-1} a_{t-1} \right) \frac{1}{N} \sum_{j=1}^{N} h_{j,1}^{-1} \beta_{j} \right\} + o_{p}(1),
$$

$$
= \frac{1}{\gamma_{t-1}b} \left[ \left(1 + \frac{\beta^{2}}{\sigma_{\beta}^{2}}\right)\left( c_{t-1}^{1} - r_{0,t-1} a_{t-1} \right) - \left( c_{t-1}^{1} - r_{0,t-1} \right) \frac{\beta^{2}}{\sigma_{\beta}^{2}} a_{t-1} \right] + o_{p}(1)
$$

$$
= \left( \frac{c_{t-1}^{1} - r_{0,t-1} a_{t-1}}{\gamma_{t-1}b} \right) + o_{p}(1). \tag{36}
$$

For the third term of $\rho_{t}^{\text{meu}}/N$

$$
\frac{\mathbf{e}'\mathbf{w}_{t-1}^{\text{meu}}}{N} = O_{p}\left( (N^{-2} \sum_{j=1}^{N} h_{j,t-1} \mathbf{w}_{j,t-1}^{\text{meu}})^{2} \right) \tag{37}
$$

$$
= O_{p} \left( \left[ a_{t-1} d_{t-1} - c_{t-1}^{2} + (1 - b^{-1})(c_{t-1} - a_{t-1} r_{0,t-1})^{2} \right] \frac{1}{a_{t-1}^{2} N^{2}} \right) = o_{p}(1),
$$

and for the second term

$$
\frac{\mathbf{w}_{t-1}^{\text{meu}}}{N} = O_{p} \left( \frac{1}{\gamma_{t-1}N} \sum_{j=1}^{N} h_{j,t-1}^{-1} (\alpha_{j,t-1} - r_{0,t-1}) \beta_{j} - \frac{\beta}{\sigma_{\beta}^{2}} \left( c_{t-1} - a_{t-1} r_{0,t-1} \right) \frac{\beta^{2}}{\gamma_{t-1} a_{t-1} b N} \sum_{j=1}^{N} h_{j,t-1}^{2} \right) = o_{p}(1).
$$

Therefore, even the meu portfolio is beta-neutral since both the idiosyncratic and common sources of risk in $\rho_{t}^{\text{meu}}/N$ are diversified away as $N$ gets large.

It remains to consider

$$
\frac{\mathbf{w}_{t-1}^{\text{meu}}}{N} = \left( \frac{\mathbf{w}_{t-1}^{\text{meu}}}{N} \right) + \nu_{t} \left( \frac{\mathbf{w}_{t-1}^{\text{meu}}}{N} \right),
$$

but, again, since $\beta^{\prime} \mathbf{w}_{t-1}^{\text{meu}}/N = o_{p}(1)$ there is no contribution of the predictable common component $\nu_{t-1}$ to the limit portfolio return. One is left to consider
the limit of
\[
\frac{\alpha'_{t-1} w^{\text{mcu}}_{t-1}}{N} = \frac{1}{b \gamma_{t-1}} \times 
\]
\[
N^{-1} \sum_{j=1}^{N} h_{j,j,t-1}^{-1} \left( (1 + \frac{\tilde{\beta}^2}{\sigma^2}) (\alpha_{j,t-1} - r_{0t-1}) \alpha_{j,t-1} - \frac{\tilde{\beta} \beta_j \alpha_{j,t-1}}{\sigma^2} \left( \frac{c_{t-1}}{a_{t-1}} - r_{0t-1} \right) \right) + o_p(1),
\]
\[
= \frac{1}{b \gamma_{t-1}} \left( (1 + \frac{\tilde{\beta}^2}{\sigma^2}) (d_{t-1} - c_{t-1} r_{0t-1}) - \frac{\beta^2}{\sigma^2} \left( \frac{c_{t-1}}{a_{t-1}} - r_{0t-1} \right) c_{t-1} \right) + o_p(1),
\]
\[
= \frac{1}{b \gamma_{t-1}} \left( d_{t-1} - c_{t-1} r_{0t-1} + \frac{\beta^2}{a_{t-1} \sigma^2} (a_{t-1} d_{t-1} - c_{t-1}^2) \right) + o_p(1), 
\]
(38)

where by Assumption 4
\[
\frac{\alpha'_{t-1} H^{-1}_{t} \alpha_{t}}{N} = \frac{1}{N} \sum_{j=1}^{N} h_{j,j,t}^{-1} \alpha_{j,t}^2 \rightarrow_p d_t > 0.
\]

Thus, taking the difference between (38) and (36) yields
\[
\frac{\rho^{\text{mcu}}_{t}}{N} \rightarrow_p e_{t-1} \gamma_{t-1} b,
\]
where
\[
e_{t} = d_{t} - 2c_{t} r_{0t} + a_{t} r_{0t}^2 + \frac{\beta^2}{\sigma^2} (d_{t} - c_{t}^2).
\]

Therefore, like the gmv limit portfolio return, the limit of \(\frac{\rho^{\text{mcu}}_{t}}{N}\) is \(Z_{t-1}\)-adapted since both the idiosyncratic and the common innovation are diversified away. Moreover the predictable common component has an asymptotically negligible contribution to portfolio returns.

In the following section we establish these results in the general multi-factor setting, with \(k > 1\), and non-diagonal \(H_t\), where we also show that \(e_{t-1} \rightarrow 0\), a.s.

4 The general multi-factor case

We begin with the gmv portfolio weights defined by (24) and in what follows we suppose that \(r_t\) is generated according to the multi-factor model (6),
Assumptions 1, 2 and 3 hold, and all the limits are taken for each $t$ and as $N \rightarrow \infty$. In this section we also allow the idiosyncratic errors, $\varepsilon_{it}$, to be (weakly) cross-sectionally correlated, namely we allow for a non-diagonal $H_{t-1}$ matrix.

**Theorem 1** (global minimum-variance portfolio)

(i) Let

$$\hat{\omega}_{it}^{\text{gmv}} = N^{-1}e_i^{(N)r}H_{t-1}^{-1}\left[\mathbf{e} + a_t(e\tilde{\beta} - \mathbf{B})A_t^{-1}\tilde{\beta}\right],$$

and recall that $e_i^{(N)r}$ is a $N \times 1$ row vector of zeros except for its $i^{th}$ element which is unity. When Assumptions (7), (10), (12) and (16)

$$N(\omega_{it}^{\text{gmv}} - \hat{\omega}_{it}^{\text{gmv}}) \rightarrow_p 0.$$  

(ii) When, in addition to the assumptions made in (i), (17)-(18)-(20) hold

$$\omega_{it}^{\text{gmv}} = \hat{\omega}_{it}^{\text{gmv}} + N^{-3/2}z_{it}^{\text{gmv}} + N^{-2}b_t\left[\mathbf{e}_i^{(N)r}H_{t-1}^{-1}\mathbf{B}\left(A_t + a_t\bar{\beta}\bar{\beta}'\right)^{-1}\Omega_t^{-1}\left(A_t + a_t\bar{\beta}\bar{\beta}'\right)^{-1}\tilde{\beta}\right] + o_p(N^{-2}),$$

in which

$$b_t = (1 + a_t\bar{\beta}'A_t^{-1}\bar{\beta}),$$

and $z_{it}^{\text{gmv}}$ is a mixture of normally distributed random variables that are only functions of $\mathbf{B}$ and $H_t$.

(iii) When, in addition to the assumptions made under (i), relations (11) and (13) hold:

$$\rho_{it}^{\text{gmv}} = r_t'\omega_{it}^{\text{gmv}} \rightarrow_p \frac{c_{t-1}}{a_{t-1}},$$

and

$$N^{-\frac{1}{2}}\left(\begin{array}{c}
\mu_{\rho,t-1}^{\text{gmv}} \\
\sigma_{\rho,t-1}^{\text{gmv}}
\end{array}\right) \rightarrow_p \frac{c_{t-1}}{\sqrt{a_{t-1}}},$$

where $\mu_{\rho,t-1}^{\text{gmv}} = E(\rho_{it}^{\text{gmv}} \mid Z_{t-1})$, and $\sigma_{\rho,t-1}^{\text{gmv}} = \sqrt{\text{var}(\rho_{it}^{\text{gmv}} \mid Z_{t-1})}$.

**Remark 1(a)** The gmv portfolio weight of the $i^{th}$ asset is, asymptotically in $N$, equivalent to $\hat{\omega}_{it}^{\text{gmv}}$. Inspecting (39) it emerges that $\omega_{it}^{\text{gmv}}$ is functionally independent from the factors covariance matrix, $\Omega_t$. Instead, it is a function
of the factor loadings $B_t$, of their first moments $\bar{\beta}$, of the mixed moment $A_t$ and of the (inverse of the) idiosyncratic component covariance matrix, $H_t$.

**Remark 1(b)** From (41) it is also easily seen that the effect of $\Omega_t$ on the dispersion of the $w_{it}^{gmv}$ around $\hat{w}_{it}^{gmv}$ vanishes at a sufficiently rapid rate such that even the asymptotic distribution of $w_{it}^{gmv}$ does not depend on $\Omega_t$ as $N$ tends to infinity.

**Remark 1(c)** The $gmv$ portfolio becomes fully diversified with respect to the idiosyncratic as well as the factor components of asset return innovations as $N \to \infty$. Moreover, the limit portfolio return is $\mathcal{Z}_{t-1}$-adapted as well as independent of the factor component of asset returns conditional mean $\nu_{t-1}$.

**Remark 1(d)** The *ex ante* Sharpe ratio, defined by $\mu_{gmv}^{\rho,t} - \sigma_{gmv}^{\rho,t}$, diverges at the rate of $N^{-1/2}$, unless $c_{t-1} = 0$. But it is not guaranteed that the *ex ante* Sharpe ratio in the case of $gmv$ will diverge to plus infinity. The outcome depends on sign of $c_{t-1}$ which is not guaranteed to be positive. This arises since $gmv$ portfolio does not make use of expected mean returns.

Consider now the $meu$ portfolio given by (35), and as before suppose that $r_t$ follows the multi-factor model (6), and Assumptions 1,2 and 3 hold. Then we have

**Theorem 2 (maximum expected utility portfolio)**

(i) Let

$$w_{it}^{meu} = \frac{e^{(N)}_t H_t^{-1}}{\gamma_t b_t} \left\{ \left( \alpha_t - e r_{0t} \right) + \left[ a_t (\alpha_t - e r_{0t}) \tilde{\beta} - (c_t - a_t r_{0t}) B \right] A_t^{-1} \tilde{\beta} \right\}. $$

When conditions (7), (10), (11), (12), (13), (16), (17) and (19) hold:

$$w_{it}^{meu} - \hat{w}_{it}^{meu} \to_p 0. $$

(ii) When, in addition to the conditions in (i), (20) also holds:

$$w_{it}^{meu} = \hat{w}_{it}^{meu} + N^{-1/2} z_{it}^{meu} +$$

$$N^{-1} \left\{ \gamma_t^{-1} e^{(N)}_t H_t^{-1} B \left( A_t + a_t \tilde{\beta} \tilde{\beta}^t \right)^{-1} \Omega_{t-1}^{-1} \nu_t + \left( A_t + a_t \tilde{\beta} \tilde{\beta}^t \right)^{-1} \Omega_{t-1}^{-1} \right\} + o_p(N^{-1}),$$

where $z_{it}^{meu}$ is a mixture of normally distributed random variables that are only functions of $\gamma_t$, $r_{0t}$, $\alpha_t$, $B$, and $H_t$.
(iii) When, in addition to the conditions in (i), (14) also holds
\[
\rho_t^{\text{meu}} = r_t^l w_{t-1}^{\text{meu}} + (1 - e_t^{\text{meu}}) r_{0t-1},
\]
satisfies
\[
N^{-1} \rho_t^{\text{meu}} \xrightarrow{p} \frac{e_{t-1}}{\gamma_{t-1} b_{t-1}}, \tag{48}
\]
\[
N^{-\frac{1}{2}} \left( \frac{\mu_{\rho,t-1}^{\text{meu}} - r_{0t-1}}{\sigma_{\rho,t-1}^{\text{meu}}} \right) \xrightarrow{p} \sqrt{e_{t-1}}, \tag{49}
\]
where \( \mu_{\rho,t-1}^{\text{meu}} = E(\rho_t^{\text{meu}} | Z_{t-1}) \), \( \sigma_{\rho,t-1}^{\text{meu}} = \sqrt{\text{var}(\rho_t^{\text{meu}} | Z_{t-1})} \),
\[
e_t = d_t - 2r_{0t} c_t + a_t r_{0t}^2 + (a_t d_t - c_t^2) \bar{\beta}^\prime A_t^{-1} \bar{\beta}, \tag{50}
\]
and \( e_{t-1} > 0 \) almost surely.

Remark 2(a) At a given point in time \( t \), the meu portfolio weight of the \( i \)th asset is asymptotically equivalent to \( \bar{w}_{it}^{\text{meu}} \) and does not converge to zero. Moreover, \( \bar{w}_{it}^{\text{meu}} \) is functionally independent from the factors covariance matrix, \( \Omega_t \), as well as from the factors conditional mean, \( \nu_t \).

Remark 2(b) There is no contribution from either \( \Omega_t \) and \( \nu_t \) to the asymptotic distribution of \( w_{it}^{\text{meu}} \) around \( \bar{w}_{it}^{\text{meu}} \).

Remark 2(c) The meu portfolio does not achieve diversification of the idiosyncratic and the factors component of asset return innovations. Moreover, the part of the portfolio return involving the factors component is of the same order of magnitude, in \( N \), as the part involving the idiosyncratic component. Diversification of both components is achieved if one considers \( N^{-1} w_{it}^{\text{meu}} \). For the same reasons, convergence of the portfolio return \( \rho_t^{\text{meu}} \) is achieved when normalizing by \( N \) and its limit is \( Z_{t-1} \)-adapted. In particular, the limiting value of \( N^{-1} \rho_t^{\text{meu}} \) will be a function of \( \alpha_{t-1} \), but not of \( \nu_{t-1} \).

Remark 2(d) The ex ante Sharpe ratio diverges at the rate \( N^{\frac{1}{2}} \), and the limit is always positive. Note that limit of the normalized Sharpe ratio is independent of the coefficient of risk aversion, \( \gamma_{t-1} \).

Analog results can be derived for the minimum-variance (mv) and the mean expected (me) tangency portfolios, as discussed in Section 6.
5 Discussion of results

5.1 Contribution of factors to portfolio return

The above theorems (their part (iii)) establish the limit portfolio return, normalized with a suitable scaling factor, for various MV trading strategies. In particular, \( \rho_{\text{gmv}} \) has a well defined limit whereas \( \rho_{\text{meu}} \) requires the scaling factor \( N^{-1} \). The scaling factor is necessary since the meu portfolio weights do not converge to zero but are in fact \( O_p(1) \).

Inspecting the results, it is evident that the limit MV portfolio returns are \( Z_{t-1} \)-adapted, that is they are neither functions of the idiosyncratic innovations, \( \varepsilon_t \), nor the common innovations, \( u_t \). The first result is well known, namely that the contribution of the idiosyncratic innovations to the portfolio return vanishes in mean square as \( N \to \infty \). One of the novel results of this paper is to show that MV trading strategies also succeed in diversifying the effects of the common innovations, \( u_t \). This result is driven by the fact that the MV trading strategies make use of the inverse of the conditional covariance matrix \( \Sigma_{t-1} \) in a convenient way. In particular, the MV portfolio weights have the form \( \Sigma_{t-1}^{-1} \delta_t \), for some \( N \times 1 \) vector \( \delta_t = \delta(Z_{t-1}) \), meaning that is function of \( Z_{t-1} \), the exact form of which depends on the type of trading strategy under consideration. As a consequence, the portfolio return can be decomposed as:

\[
\delta_{t-1}' \Sigma_{t-1}^{-1} r_t = \delta_{t-1}' \Sigma_{t-1}^{-1} \alpha_{t-1} + \delta_{t-1}' \Sigma_{t-1}^{-1} B \nu_{t-1} + \delta_{t-1}' \Sigma_{t-1}^{-1} B u_t + \delta_{t-1}' \Sigma_{t-1}^{-1} \varepsilon_t.
\]

Lemma A in the appendix establishes that \( \| \Sigma_{t-1}^{-1} B \|_2 = O_p(N^{-1}) \), so that \( \Sigma_{t-1}^{-1} \) and \( B \) are asymptotically orthogonal, and therefore the contribution of the common factor innovation, \( \delta_{t-1}' \Sigma_{t-1}^{-1} B u_t \), to the return portfolio \( \delta_{t-1}' \Sigma_{t-1}^{-1} r_t \) is of smaller order than the mean term \( \delta_{t-1}' \Sigma_{t-1}^{-1} \alpha_{t-1} \), as \( N \) gets large. In other words, by lemma A any portfolio of the form \( \Sigma_{t-1}^{-1} \delta_{t-1} \) is asymptotically beta-neutral. Obviously, the contribution of the idiosyncratic term, \( \delta_{t-1}' \Sigma_{t-1}^{-1} \varepsilon_t \), is also of smaller order. Therefore

\[
\delta_{t-1}' \Sigma_{t-1}^{-1} r_t = \delta_{t-1}' \Sigma_{t-1}^{-1} \alpha_{t-1}(1 + o_p(1)) \text{ as } N \to \infty.
\]

This implies that, subject to a suitable normalization, the contributions of \( u_t \) and \( \varepsilon_t \) to the limit portfolio return converges to zero, the only difference between the two being that convergence occurs in first mean in the case of the terms involving \( u_t \), and in mean square in the case of the terms in \( \varepsilon_t \).
Given the asymptotic orthogonality of $\Sigma^{-1}_t$ and $B$ it also happens that the contribution of the factors to the returns conditional mean, namely $\delta'_{t-1} \Sigma^{-1}_{t-1} B$ $\nu_{t-1}$, typically involving lagged factors $f_{t-1}, f_{t-2}, \ldots$, is also of smaller order. Therefore the limit portfolio return will be given simply by the limit of $\delta'_{t-1} \Sigma^{-1}_{t-1} \alpha_{t-1}$, where this limit is $\mathcal{Z}_{t-1}$-adapted.

Note that our focusing on MV trading strategies is less restrictive than it might appear at first since our results apply to other trading strategies as long as they can be written as $\Sigma^{-1}_{t-1} \delta_{t-1}$. This for instance holds for certain dynamic trading strategies where the portfolio weights can be written as the sum of the MV component and an inter-temporal hedging component, both of which employ the inverse of the covariance matrix in the suitable way (see Campbell and Viceira (2001), Campbell, Chan, and Viceira (2003) among others).

The above results, on the vanishing importance of the common component of both the innovation and the conditional mean as $N$ gets large, can also be understood by considering the following approximation of $\Sigma^{-1}_t$

$$\Sigma^{-1}_t = H_t^{-1} - H_t^{-1} B (B' H_t^{-1} B)^{-1} B' H_t^{-1}. \quad (51)$$

Now define the class of MV portfolio weights

$$\tilde{w}_{t-1} = \tilde{\Sigma}^{-1}_{t-1} \tilde{\xi}_{t-1}, \quad (52)$$

setting $\tilde{\xi}_{t-1} = \tilde{\delta} (\tilde{Z}_{t-1}, 0)$ where $Z_{t-1} = \tilde{Z}_{t-1} \cap \nu_{t-1}$ implying $\tilde{\delta}_{t-1} = \delta (\tilde{Z}_{t-1}, \nu_{t-1}) = \delta (\mathcal{Z}_{t-1})$. Note that, by construction $\tilde{w}_{t-1}$ is functionally independent of both $\Omega_{t-1}$ and $\nu_{t-1}$. Also It is also easily seen that

$$\tilde{\Sigma}^{-1}_t B = 0 \text{ for any finite } N, \text{ and } \| \tilde{\Sigma}^{-1}_t - \Sigma^{-1}_t \| = O_p (N^{-1}).$$

Hence, not only $\tilde{w}_{t-1}$ is approximately equivalent to $\Sigma^{-1}_{t-1} \delta_{t-1}$ in the sense just described, but more importantly it yields the portfolio return

$$\tilde{w}'_{t-1} r_t = \tilde{w}'_{t-1} (\alpha_{t-1} + B \nu_{t-1}) + \tilde{w}'_{t-1} B u_t + \tilde{w}'_{t-1} \varepsilon_t = \tilde{w}'_{t-1} \alpha_{t-1} + \tilde{w}'_{t-1} \varepsilon_t,$$

which is functionally independent of both $\nu_{t-1}$ and $u_t$ due to beta-neutrality of $\tilde{w}_{t-1}$, for any finite $N$. In short the two portfolios $w_{t-1}$ and $\tilde{w}_{t-1}$ are equivalent asymptotically, namely $\tilde{w}'_{t-1} r_t = w'_{t-1} \alpha_{t-1} (1 + o_p(1))$ and $\tilde{w}'_{t-1} \alpha_{t-1} - w'_{t-1} \alpha_{t-1} = o_p(1)$ as $N \rightarrow \infty$.

We have seen that different MV trading strategies imply different rates at which the corresponding portfolio weights converge, if any, to zero. However,
the *ex ante* Sharpe ratio \((\mu_{s,t-1} - r_{0,t-1})/\sigma_{s,t-1}\), corresponding to the MV strategy of type \(s\), diverges as \(N\) tends to infinity, and at the same rate of \(\sqrt{N}\). The main difference is that whereas for the *meu* strategy it can only diverge to plus infinity, this is not guaranteed by the *gmv* strategy, for which divergence towards minus infinity can occur. This partly reflects the suboptimal nature of the *gmv* strategy that does not make use of the return predictions, \(\alpha_{t-1}\).

### 5.2 Contribution of factors to portfolio weights

The conditional distribution of the factors, \(f_t\), is irrelevant, as far as the form of the limiting MV portfolio weights \(w^*_t\) is concerned. In fact, the factors conditional mean \(\nu_{t-1}\) and conditional covariance matrix \(\Omega_{t-1}\) do not appear in the first-order limit approximations set out in (39) and (45). This outcome is a direct consequence of lemma A proved in the Appendix. An immediate implication is that when evaluating the MV portfolio weights empirically one can avoid specifying, let alone estimating, the conditional mean and the conditional covariance matrix of the common factors. Estimates of these quantities are needed only in so far as they help in estimation of \(\alpha_{t-1}\) and \(H_{t-1}\). For a finite \(N\), this estimation strategy clearly would involve an approximation error since the finite-\(N\) expression of the MV weights will necessarily be a function of \(\Omega_{t-1}\) and \(\nu_{t-1}\). However, such approximation error decreases to zero as \(N\) increases and, at the same time, using either the limit portfolio formulae or the approximate expression (52), is likely to be robust to the consequences of incorrectly specifying, or poorly estimating \(\Omega_{t-1}\) and \(\nu_{t-1}\).

Part (i) of Theorems 1 and 2 can be interpreted as a consistency result, showing the form of the limit approximations, as \(N \to \infty\), of the MV portfolio weights. Part (ii) of these theorems considers if the conditional distribution of \(f_t\) plays a role with respect to the dispersion of the finite-\(N\) portfolios around their limit approximation. Under suitable regularity conditions, the MV portfolio weights have an asymptotic distribution, centered around the limit portfolio weights, which is distributed independently of the conditional moments of \(f_t\). In other words, the contribution of these moments to the (finite-\(N\)) MV portfolio weights vanishes at a suitably fast rate, faster than the rate required to obtain the asymptotic distribution of the portfolio weights.

The result in part (i) of the above theorems hold not only point-wise for
each $i = 1, 2, ..., N$ but also jointly for the entire vector of portfolio weights. In fact, it can be shown that $\| w_{gmv}^t - \bar{w}_{gmv}^t \| = o_p(N^{-1})$ and $\| w_{meu}^t - \bar{w}_{meu}^t \| = o_p(1)$.

Another important consequence of part (i) of these theorems is that the limiting portfolio weights will not be time-varying unless $H_t$ is, that is only if the idiosyncratic component $\varepsilon_t$ features dynamic conditional heteroskedasticity. The mean-variance portfolio $meu_t$ will be time-varying both due to possible time variation in $H_t$, and in the risk-free rate $r_0$. If we relax our assumptions, say allowing $B_t$ to be time-varying $B_t$, then for instance the $gmv$ portfolio weights (39) become, under regularity conditions similar to the ones spelled out in Theorem 1,

$$N w_{gmv}^t - \frac{1}{a_t} e_t^{(N)'H_t} [e + a_t(e_\bar{\beta}_t - B_t)A_t^{-1}\bar{\beta}_t] \to_p 0 \text{ as } N \to \infty.$$  

For this case to be genuinely interesting, $B_t$ needs to be independent from the factors $f_t$ though. This rules out the case $B_t = B\Omega_t^{\frac{1}{2}}$, which, as far as the dynamics of $r_t$ is concerned, is observationally equivalent to (1). If instead one alternatively assumes the parameter-free form $\Omega_t = I_k$, our result continues to apply since the limit portfolios continue to be functionally independent of any parametric aspect of $\Omega_t$.

Factor models are inherently undetermined since (6) yields the same vector $r_t$ given a non-singular $k \times k$ matrix $C$ and replacing $B$ and $u_t$ by $BC$ and $C^{-1}u_t$, respectively. Determination of $C$ is crucial for identification and estimation of model (6). This is particularly relevant in our context since besides the factor loadings, the matrix $C$ induces also a rotation of $\Omega_t$ and $\nu_t$ and, due to their time-variation, the risk of possible lack of identification is even more pronounced. However, this issue is of second-order importance since the limit portfolio weights do not dependent on the conditional mean and covariance matrix of $f_t$. One can easily verify this by replacing $B$, $A_t^{-1}$ and $\bar{\beta}$ with $BC$, $C^{-1}A_t^{-1}C'^{-1}$, and $C'\bar{\beta}$, respectively into (39) and (45).

5.3 Portfolio diversification

Under our assumptions

$$w_{gmv}^t - \bar{w}_{gmv}^t \to_p 0 \text{ as } N \to \infty,$$

where $N w_{gmv}^t = O_p(1)$, for given $i$ and $t$, and are different from zero almost surely. Therefore, the $gmv$ portfolio is diversified in the sense that each
coefficient \( w_{it}^{gmv} \) becomes arbitrarily small as \( N \) grows.

More formally, if \( \sup_{1 \leq i \leq N} | w_{it}^{gmv} | = o_p(1) \) for each \( t \), then we achieve full diversification in the sup-norm sense of Green and Hollifield (1992). Using the limit approximation \( \hat{w}_{it}^{gmv} \) it turns out to be much easier to find sufficient conditions for full diversification. For instance, using results of Theorem 1, one obtains

\[
\sup_{1 \leq i \leq N} \left( |h_{(i)t}^i e| + \sum_{j=1}^{k} |h_{(i)t}' \beta^{(j)}| \right) = o_p(N) \tag{53}
\]

where \( \beta^{(j)} = B \epsilon_{(k)}^j \) and \( h_{(i)t} = H_t^{-1} e_i^{(N)} \). If full diversification at rate \( N^{-1} \) is required, the left hand side of the previous expression must be \( o_p(1) \). In turn this is satisfied whenever \( \sup_{1 \leq i \leq N} \sup_{1 \leq j \leq k} |\beta_{ij}| = O_p(1) \) and \( |h_{(i)t}' e| = O_p(1) \).

In contrast, the \textit{meu} portfolio is not fully diversifiable in the sense that its weights do not converge to zero and instead \( \hat{w}_{it}^{meu} = O_p(1) \). Therefore, as a consequence, the limit portfolio \( \rho_t^{meu} \) requires the normalization \( N^{-1} \) in order to obtain a well-defined limit.

Thus, the common practice of building (optimal) portfolios imposing the restriction that the portfolio weights are smaller than a given predetermined quantity, appears justified for the \textit{meu} portfolio. In fact, there is no guarantee that the weights will be smaller the larger the number of assets under consideration. On the other hand, under conditions such as (53) or variations of, the \textit{gmv} portfolio weights gets arbitrarily small, for a sufficiently large \( N \).

The definition of complete diversifiability of Chamberlain and Rothschild (1983) instead requires, for the \textit{s} trading strategy, \( \sum_{i=1}^{N} (\hat{w}_{it}^s)^2 = o_p(N^2) \) for each \( t \), and sufficient conditions can be easily derived. For instance, for the \textit{gmv} portfolio it is required

\[
\mathbf{e}' H_t^{-1} H_t^{-1} \mathbf{e} = o_p(N^2), \quad \mathbf{B}' H_t^{-1} H_t^{-1} \mathbf{B} = o_p(N^2).
\]

Notice that the second condition is implied by (11). This definition of complete diversifiability requires stronger conditions than the notion based on the sup-norm discussed earlier.
5.4 Short-selling and factor dominance

When $H_t$ is diagonal, it easily follows that $A_t = a_t \Sigma_\beta$, where $\Sigma_\beta$ is the covariance matrix of the $\beta_i$, yielding for the gmv portfolio weights

$$Nw_{it}^{gmv} \rightarrow_p \frac{h_{ii,t}^{-1}}{a_t} \left[ 1 - \bar{\beta}' \Sigma_\beta^{-1} (\beta_i - \bar{\beta}) \right].$$

Moreover, if $\Sigma_\beta$ is diagonal, with $\sigma_{\beta j}$ being its $(j, j)^{th}$ entry, (54) simplifies further to

$$Nw_{it}^{gmv} \rightarrow_p \frac{h_{ii,t}^{-1}}{a_t} \left[ 1 - \left( \frac{\bar{\beta}_1}{\sigma_{\beta 1}} \right)^2 \left( \frac{\beta_{11} - \bar{\beta}_1}{\beta_1} \right) - \ldots - \left( \frac{\bar{\beta}_k}{\sigma_{\beta k}} \right)^2 \left( \frac{\beta_{ik} - \bar{\beta}_k}{\beta_k} \right) \right],$$

(55)

where $\bar{\beta}_j$ and $\beta_{ij}$ are the $j^{th}$ element of $\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_k)'$ and $\beta_i = (\beta_{i1}, \ldots, \beta_{ik})'$, respectively.

Green and Hollifield (1992) argue that the possibility of short-selling, in the sense of a repeated finding of negative optimal portfolio weights, is related to the presence of one dominant factor. Our result sheds some light on this. One can see from (55) that the limit portfolio weights only depend on factor loadings if the mean of these loadings is non-zero (i.e. if $\bar{\beta}_1 \neq 0$). Such factors are regarded as dominant by Jagannathan and Ma (2003).

More generally, a negative weight can arise whenever the factor loading assumes values smaller than their cross-sectional average. This effect is magnified, the larger is the Sharpe ratio of the factor loading, defined by $\bar{\beta}_j / \sigma_{\beta j}$. A large dispersion implies a smaller chance of finding negative weights, corroborating the findings based on simulations reported by Jagannathan and Ma (2003). On the other hand, note also that the larger the number of dominant factors under consideration (in the sense of Jagannathan and Ma (2003)), the less likely it is that a negative weight would be encountered. Similar outcomes obtain for non-diagonal $H_t$. This reinforces Green and Hollifield (1992)’s conjecture about the presence of a single dominant factor whenever large negative weights are observed.

Under the same conditions as above, for the meu portfolio weights one
obtains
\[
 w_{it}^{\text{mv}} \rightarrow_p \frac{h_{it}^{-1}}{\gamma_t} \left[ \alpha_{it} - r_{0t} + \left( \frac{\tilde{\beta}_1}{\sigma_{\beta 1}} \right)^2 \left( \alpha_{it} - r_{0t} \right) - \left( c_t - a_t r_{0t} \right) \frac{\beta_{i1}}{\alpha_i \beta_{i1}} \right] 
+ \left( \frac{\tilde{\beta}_k}{\sigma_{\beta k}} \right)^2 \left( \alpha_{it} - r_{0t} \right) - \left( c_t - a_t r_{0t} \right) \frac{\beta_{ik}}{\alpha_i \beta_{ik}} \right]^{(56)}
\]

Therefore, as with the gmv portfolio weights one can see that a negative weight is more likely for the asset for which \( \alpha_{it} - r_{0t} < 0 \).

Assuming \( c_t > a_t r_{0t} \), a negative weight is more likely to arise whenever the factor loading assumes values smaller than their cross-sectional average and this effect is magnified, the larger is the Sharpe ratio of the factor loading.

Finally, the larger the number of dominant factors under consideration, the less likely that a negative weight would be encountered.

6 Other optimization strategies

We now present results for two other MV tangency portfolios considered in the literature, namely the minimum variance and the maximum expected return portfolios. We present two corresponding theorems without proof, and comment on the results afterwards.

The \( \text{mv} \) portfolio weights \( w_{it}^{\text{mv}} = (w_{1t}^{\text{mv}}, \ldots, w_{Nt}^{\text{mv}})' \) are defined by

\[
w_{t-1}^{\text{mv}} = \arg\min_w (w' \Sigma_{t-1} w), \quad \text{such that } w' \mu_{t-1} + (1 - w'e)r_{0t-1} = \mu_p,
\]

where \( \mu_p \) is the targeted expected portfolio return assumed to exceed \( r_{0t-1} \) \( (\mu_p > r_{0t-1}) \), yielding

\[
w_{t-1}^{\text{mv}} = \frac{\mu_p - r_{0t-1}}{(\mu_{t-1} - e r_{0t-1})' \Sigma_{t-1}^{-1} (\mu_{t-1} - e r_{0t-1})} \Sigma_{t-1}^{-1} (\mu_{t-1} - e r_{0t-1}). \quad (57)
\]

**Theorem 3** (minimum variance portfolio)

(i) Let

\[
w_{it}^{\text{mv}} = N^{-1} \left( \frac{\mu_p - r_{0t}}{e_t} \right) e_t' H_t^{-1} \left\{ (\alpha_t - e r_{0t}) + [a_t (\alpha_t - e r_{0t}) \tilde{\beta} - (c_t - a_t r_{0t}) B] A_t^{-1} \tilde{\beta} \right\}. \quad (58)
\]
When conditions (7), (10), (11), (12), (13), (14), (16), (17) and (19) hold:

\[ N(w_{mv}^{it} - \hat{w}_{mv}^{it}) \rightarrow p 0. \]

(ii) When, in addition to the conditions in (i), (20) holds:

\[ w_{mv}^{it} = \hat{w}_{mv}^{it} + N^{-3/2} z_{mv}^{it} \]
\[ + N^{-2} \left\{ \left( \frac{\mu_{\rho} - r_{0t}}{e_t} \right) e_t^{(N)/} H_t^{-1} B \left( A_t + a_t \beta \right) \right\}^{-1} \left[ \nu_t + \left( A_t + a_t \beta \right)^{-1} \beta (c_t - a_t r_{0t}) \right] \]
\[ + o_p(N^{-2}) \]

where \( z_{mv}^{it} \) is a mixture of normally distributed random variables that are only functions of \( \mu_{\rho}, r_{0t}, \alpha_t, B, \) and \( H_t. \)

(iii) When the conditions in (i) hold

\[ \rho_{mv}^{t} = r_t' w_{mv}^{t-1} + (1 - e' w_{mv}^{t-1}) r_{0t-1}, \]

satisfies

\[ N^{-\frac{1}{2}} \left( \frac{\mu_{\rho,t}^{mv}}{\sigma_{\rho,t-1}^{mv} - r_{0t-1}} \right) \rightarrow p \mu_{\rho}, \]

(59)

\[ \left[ \frac{\sigma_{\rho,t-1}^{mv}}{\sigma_{\rho,t-1}^{mv}} \right] \rightarrow p \sqrt{\epsilon_{t-1}}, \]

(60)

setting \( \mu_{\rho,t-1}^{mv} = E(\rho_{t}^{mv} | Z_{t-1}) \) and \( \sigma_{\rho,t-1}^{mv} = \sqrt{\text{var}(\rho_{t}^{mv} | Z_{t-1})}. \)

For the maximum expected return portfolio, \( w_{me}^{it} = (w_{me}^{1t}, ..., w_{me}^{Nt})' \), we have

\[ w_{me}^{it} = \arg\max_w w' \mu_{t-1} + (1 - w' e)r_{0t-1}, \text{ such that } w' \Sigma_{t-1} w = \sigma_{\rho}^2, \]

where \( \sigma_{\rho}^2 \) is the targeted portfolio variance, yielding the maximum expected return portfolio

\[ w_{me}^{it} = \left[ \sigma_{\rho}^2 (\mu_{t-1} - e r_{0t-1}) \Sigma_{t-1}^{-1} (\mu_{t-1} - e r_{0t-1}) \right]^{\frac{1}{2}} \Sigma_{t-1}^{-1} (\mu_{t-1} - e r_{0t-1}). \]
Theorem 4 (maximum expected return portfolio)

(i) Let

\[
\hat{w}_{it}^me = N^{-\frac{1}{2}} \frac{\sigma_p}{\sqrt{e_t \epsilon_t}} e_{it}^{(N)} H_t^{-1} \left\{ \left( \alpha_t - e r_{0t} \right) + [a_t(\alpha_t - er_{0t})\beta' - (c_t - a_t r_{0t})B] A_t^{-1} \beta \right\}.
\]

(61)

When conditions (7), (10), (11), (12), (13), (14), (16), (17) and (5.2 due) hold:

\[
N^{\frac{1}{2}} (\hat{w}_{it}^me - \hat{w}_{it}^{me}) \to_p 0.
\]

(ii) When, in addition to the conditions in (i), (20) hold:

\[
\begin{align*}
\hat{w}_{it}^me & = \hat{w}_{it}^me + N^{-\frac{3}{2}} z_{it}^me \\
& + N^{-\frac{3}{2}} \frac{\sigma_p}{\sqrt{e_t}} e_{it}^{(N)} H_t^{-1} B \left( A_t + a_t \bar{\beta} \bar{\beta}' \right) \Omega_t^{-1} \left[ \nu_t + \left( A_t + a_t \bar{\beta} \bar{\beta}' \right) \bar{\beta} (c_t - a_t r_{0t}) \right] \\
& + o_p(N^{-\frac{3}{2}}),
\end{align*}
\]

where \( z_{it}^me \) is a mixture of normally distributed random variables, function of \( \sigma_p^2, r_{0t}, \alpha_t, B, \) and \( H_t, \) only.

(iii) When the conditions in (i) hold:

\[
\rho_t^{me} = r_t' w_t^{me} + (1 - e' w_t^{me}) r_{0t-1},
\]

satisfies

\[
\begin{align*}
N^{-\frac{1}{2}} \rho_t^{me} & \to_p \sigma_p \sqrt{e_{t-1}}, \\
N^{-\frac{1}{2}} \left( \frac{\mu_{me, t-1}^{me} - r_{0t-1}}{\sigma_{me, t-1}} \right) & \to_p \sqrt{e_{t-1}},
\end{align*}
\]

where \( \mu_{me, t-1}^{me} = E(\rho_t^{me} | Z_{t-1}) \), and \( \sigma_{me, t-1}^{me} = \sqrt{\text{var}(\rho_t^{me} | Z_{t-1})} \).

Remark 3-4(a) The mv and me portfolio weights of the \( i \)th asset are, asymptotically in \( N \), equivalent to \( \hat{w}_{it}^{mv} \) and \( \hat{w}_{it}^{me} \), respectively. Moreover, the latter are functionally independent of \( \Omega_t \) and \( \nu_t \).

Remark 3-4(b) The asymptotic distributions of \( w_{it}^{mv} \) and \( w_{it}^{me} \), respectively around \( \hat{w}_{it}^{mv} \) and \( \hat{w}_{it}^{me} \), do not depend on \( \Omega_t \) and/or \( \nu_t \).

Remark 3-4(c) The mv and me portfolios achieve full diversification of both the idiosyncratic and common factor components of asset return innovations, the latter when normalized by \( N^{\frac{1}{2}} \). Under the same conditions, the corresponding limit portfolio returns are \( Z_{t-1} \)-adapted.
Remark 3-4(d) The \textit{ex ante} Sharpe ratio diverges to plus infinity at rate $N^{\frac{3}{2}}$. The limit of the normalized Sharpe ratio is independent of $\mu_p$ and of $\sigma^2_p$ for \texttt{mv} and \texttt{me} trading strategies, respectively. In particular, once normalized by $N^{-\frac{3}{2}}$, the limit is the same and coincide with the one obtained for the \texttt{meu} portfolio return. This follows since all the three MV tangency portfolio weights are proportional to one another. This important property is not shared by the \texttt{gmv} portfolio.

Remark 3-4(e) Part (i) of the above theorems hold also jointly for the entire vector of portfolio weights, that is $\|w^\text{mv}_t - \hat{w}^\text{mv}_t\| = o_p(N^{-1})$ and $\|w^\text{me}_t - \hat{w}^\text{me}_t\| = o_p(N^{-\frac{3}{2}})$.

Remark 3-4(f) As for the other optimization strategies, both (58) and (61) do not depend on any particular rotation of the factors and factor loadings.

Remark 3-4(g) Both the \texttt{mv} and the \texttt{me} portfolios are fully diversifiable, although at different rates of $N^{-1}$ and $N^{-1/2}$, respectively, achieved whenever $\sup_{1 \leq i \leq N} \|h_{(i)}'\| = O_p(1)$ and the left hand side of (53) is $O_p(1)$.

7 Final remarks

In this paper we have provided a number of theoretical results for the MV portfolios as the number of assets in the portfolio gets large. These results are a consequence of the asymptotic beta-neutrality satisfied by MV portfolios. In particular, under fairly general conditions we have shown that to a first order approximation the portfolio weights and the associated \textit{ex ante} Sharpe ratios do not depend on the means and the variance-covariances of the common factors. This result has a number of important practical implications.

It is well known that under the assumption of correct model specification, factor model-based optimal portfolios weights leads to more efficient estimates of the corresponding portfolio variance, as compared to the familiar sample moment plug-in estimates (see the empirical results of Chan, Karceski, and Lakonishok (1999) and the theoretical results of Fan, Fan, and Lv (2007)). However, the asymptotic independence of MV portfolio weights from the common factors’ conditional distribution, established in this paper, suggests that in the case of large portfolios it might be prudent to side-step the tasks of specification and estimation of the conditional distribution of the factors and instead use the formulae for the portfolio weights advanced in this paper. In this way it might be possible to avoid the adverse effects of
model and parameter uncertainties that surround the specification of the un-
observed common factor models. But before this issue can be examined one
also needs to consider the extent to which the properties of the limit port-
folios are still valid when the remaining unknown parameters are replaced
by their estimates. Double asymptotic results will need to be established,
where both the cross-section and the time series dimension diverge to infinity,
unlike this paper whose results hold at each point in time.

Another natural direction is to investigate the implication of no-arbitrage
on our results, and as a consequence their relationship with the APT of
Ross (1976). In fact, when APT hold even the asset-specific predictable
component of asset returns will be, approximately, an affine function of the
factor loadings. Therefore, in view of the asymptotic beta-neutrality of the
MV strategies, our results will need to be modified accordingly, in particular
when looking at the behaviour of the limit portfolio returns. The basic
message of our paper would be unchanged though, in particular regarding the
vanishing importance of the factors’ conditional distribution for MV portfolio
weights and return.

Appendix A: mathematical proofs

We start with a Lemma where we show that for a given \( t \) and as \( N \to \infty \),
\( \Sigma^{-1}_{t-1} \) and \( B \) are asymptotically orthogonal. This result turns out to be critical
for characterizing the behavior of optimal portfolios as \( N \) gets large.

**Lemma A** Let \( P_t \) be a sequence of random positive definitive matrices such
that

\[
\frac{B'\Sigma_t^{-1}B}{N} \to_p P_t > 0 \text{ as } N \to \infty.
\]  

(65)

Recalling that \( e_i^{(N)} \) is the \( i \)th column of the identity matrix \( I_N \), then for any
\( t, i \) and \( j \)

\[
e_i^{(N)} \Sigma_t^{-1} \beta^{(j)} \to_p 0 \text{ as } N \to \infty, \quad 1 \leq j \leq k;
\]

(66)

where \( \beta^{(j)} \) denotes the \( j \)th column of \( B = (\beta^{(1)} \ldots \beta^{(k)}) \).

Under (65) and

\[
\frac{B'\Sigma_t^{-1}H_t^{-1}B}{N} \to_p Q_t \geq 0,
\]  

(67)

where \( Q_t \) denotes a sequence of random positive semi-definitive matrices, for
any \( t \)

\[
\| \Sigma_t^{-1} \beta^{(j)} \|^2 = O_p(N^{-1}), \quad 1 \leq j \leq k, \quad \text{as } N \to \infty.
\]  

(68)

31
Proof of Lemma A. The results follow from the identity

\[ \Sigma_t^{-1} = H_t^{-1} - H_t^{-1}B(N^{-1}\Omega_t^{-1} + N^{-1}B'H_t^{-1}B)^{-1}N^{-1}B'H_t^{-1}. \] (69)

Pre-multiplying both sides by \(e_i^{(N)}\) and post-multiplying both sides by \(\beta^{(j)}\) yields (66).

We deal with (68) more explicitly. First note that \((e_j^{(k)})^t\) denotes the \(j^{th}\) column of the identity \(I_k\) matrix)

\[
\begin{align*}
&\quad (N^{-1}\Omega_t^{-1} + N^{-1}B'H_t^{-1}B)^{-1}N^{-1}B'H_t^{-1}\beta^{(j)} - e_j^{(k)} \\
&= (N^{-1}\Omega_t^{-1} + N^{-1}B'H_t^{-1}B)^{-1}N^{-1}B'H_t^{-1}\beta^{(j)} - (N^{-1}B'H_t^{-1}B)^{-1}N^{-1}B'H_t^{-1}\beta^{(j)} \\
&= N^{-1} \left[-(N^{-1}\Omega_t^{-1} + N^{-1}B'H_t^{-1}B)^{-1}\Omega_t^{-1}(N^{-1}B'H_t^{-1}B)^{-1}N^{-1}B'H_t^{-1}\beta^{(j)} \right] \\
&= N^{-1}g_t^{(j)},
\end{align*}
\]

where notice that \(g_t^{(j)}\) is a \(k \times 1\) vector with a finite norm.

Therefore, substituting the latter expression into (69) and recalling that \(Be_j^{(k)} = \beta^{(j)}\) it follows that

\[ \Sigma_t^{-1}\beta^{(j)} = H_t^{-1}\beta^{(j)} - H_t^{-1}B(e_j^{(k)} + N^{-1}g_t^{(j)}), \]

yielding

\[
\|
\Sigma_t^{-1}\beta^{(j)}
\|^2 = \beta^{(j)}H_t^{-1}H_t^{-1}\beta^{(j)} + N^{-1}g_t^{(j)}B'H_t^{-1}H_t^{-1}\beta^{(j)} - \beta^{(j)}H_t^{-1}H_t^{-1}\beta^{(j)} \\
- N^{-1}g_t^{(j)}B'H_t^{-1}H_t^{-1}\beta^{(j)} - N^{-1}\beta^{(j)}H_t^{-1}B\beta^{(j)} - N^{-2}g_t^{(j)}B'H_t^{-1}H_t^{-1}Bg_t^{(j)} \\
- \beta^{(j)}H_t^{-1}H_t^{-1}\beta^{(j)} + \beta^{(j)}H_t^{-1}H_t^{-1}\beta^{(j)} + N^{-1}\beta^{(j)}H_t^{-1}H_t^{-1}B\beta^{(j)} \\
- N^{-1}g_t^{(j)}(N^{-1}B'H_t^{-1}H_t^{-1}B)g_t^{(j)} = O_p(N^{-1}g_t^{(j)}Q_tg_t^{(j)}). \quad \Box
\]

Proof of Theorem 1. All the limits below are based on \(N \to \infty\).

(i) For \(N < \infty\), set \(w_t^{\text{gmv}} = C_{t,N}/D_{t,N}\) where

\[ \begin{align*}
C_{t,N} &= H_t^{-1}e - H_t^{-1}B(\Omega_t^{-1} + B'H_t^{-1}B)^{-1}B'H_t^{-1}e, \\
D_{t,N} &= e'H_t^{-1}e - e'H_t^{-1}B(\Omega_t^{-1} + B'H_t^{-1}B)^{-1}B'H_t^{-1}e,
\end{align*} \]

which easily follow from the identity (69).
For $\tilde{B} = B - e\tilde{b}'$

$$N^{-1}B'H_t^{-1}B = N^{-1}\tilde{b}'e'H_t^{-1}e + N^{-1}\tilde{B}'H_t^{-1}\tilde{B} + N^{-1}\tilde{b}'e'H_t^{-1}B + N^{-1}\tilde{B}'H_t^{-1}e\tilde{b}' ,$$

so that collecting terms

$$N^{-1}B'H_t^{-1}B \rightarrow_p A_t + a_t\tilde{b}\tilde{b}' ,$$

since $N^{-1}\tilde{B}'H_t^{-1}e \rightarrow_p 0$ by $E(\tilde{b}) = 0$. Similarly

$$N^{-1}B'H_t^{-1}e \rightarrow_p a_t\tilde{b} .$$

Hence, using the identity

$$(A_t + a_t\tilde{b}\tilde{b}')^{-1} = \left(A_t^{-1} - \frac{a_t}{(1 + a_t\tilde{b}'A_t^{-1}\tilde{b})}A_t^{-1}\tilde{b}\tilde{b}'A_t^{-1}\right),$$

which yields $\tilde{b}'(A_t + a_t\tilde{b}\tilde{b}')^{-1}\tilde{b} = \tilde{b}'A_t^{-1}\tilde{b}/b_t$, by Slutsky’s theorem,

$$N^{-1}D_{t,N} \rightarrow_p a_t \left(1 - a_t\tilde{b}'(A_t^{-1} - \frac{a_tA_t^{-1}\tilde{b}\tilde{b}'A_t^{-1}}{(1 + a_t\tilde{b}'A_t^{-1}\tilde{b})})\tilde{b}\right)$$

but the right hand side simplifies yielding

$$N^{-1}D_{t,N} \rightarrow_p a_t b_t^{-1}$$

By the same arguments, since $(A_t + a_t\tilde{b}\tilde{b}')^{-1}\tilde{b}a_t = a_t b_t^{-1}A_t^{-1}\tilde{b}$, then

$$e^{(N)}_iC_{t,N} = b_t^{-1}e^{(N)}_iH_t^{-1}(e + a_t(e\tilde{b}' - B)A_t^{-1}\tilde{b}) + o_p(1).$$

(ii) For (41)

$$C_{t,N} = H_t^{-1}e - H_t^{-1}B(\Omega_t^{-1} + B'H_t^{-1}B)^{-1}B'H_t^{-1}e = H_t^{-1}e - H_t^{-1}B(B'H_t^{-1}B)^{-1}B'H_t^{-1}e$$

$$+ H_t^{-1}B(B'H_t^{-1}B)^{-1}B'H_t^{-1}e - H_t^{-1}B(\Omega_t^{-1} + B'H_t^{-1}B)^{-1}B'H_t^{-1}e$$

$$= H_t^{-1}e - H_t^{-1}B(B'H_t^{-1}B)^{-1}B'H_t^{-1}e$$

$$+ H_t^{-1}B(B'H_t^{-1}B)^{-1}\Omega_t^{-1}(\Omega_t^{-1} + B'H_t^{-1}B)^{-1}B'H_t^{-1}e.$$

$$N^{-1}B'H_t^{-1}B \rightarrow_p A_t + a_t\tilde{b}\tilde{b}' ,$$

$$N^{-1}B'H_t^{-1}e \rightarrow_p a_t\tilde{b} .$$

Hence, using the identity

$$(A_t + a_t\tilde{b}\tilde{b}')^{-1} = \left(A_t^{-1} - \frac{a_t}{(1 + a_t\tilde{b}'A_t^{-1}\tilde{b})}A_t^{-1}\tilde{b}\tilde{b}'A_t^{-1}\right),$$

which yields $\tilde{b}'(A_t + a_t\tilde{b}\tilde{b}')^{-1}\tilde{b} = \tilde{b}'A_t^{-1}\tilde{b}/b_t$, by Slutsky’s theorem,

$$N^{-1}D_{t,N} \rightarrow_p a_t \left(1 - a_t\tilde{b}'(A_t^{-1} - \frac{a_tA_t^{-1}\tilde{b}\tilde{b}'A_t^{-1}}{(1 + a_t\tilde{b}'A_t^{-1}\tilde{b})})\tilde{b}\right)$$

but the right hand side simplifies yielding

$$N^{-1}D_{t,N} \rightarrow_p a_t b_t^{-1}$$

By the same arguments, since $(A_t + a_t\tilde{b}\tilde{b}')^{-1}\tilde{b}a_t = a_t b_t^{-1}A_t^{-1}\tilde{b}$, then

$$e^{(N)}_iC_{t,N} = b_t^{-1}e^{(N)}_iH_t^{-1}(e + a_t(e\tilde{b}' - B)A_t^{-1}\tilde{b}) + o_p(1).$$

(ii) For (41)

$$C_{t,N} = H_t^{-1}e - H_t^{-1}B(\Omega_t^{-1} + B'H_t^{-1}B)^{-1}B'H_t^{-1}e = H_t^{-1}e - H_t^{-1}B(B'H_t^{-1}B)^{-1}B'H_t^{-1}e$$

$$+ H_t^{-1}B(B'H_t^{-1}B)^{-1}B'H_t^{-1}e - H_t^{-1}B(\Omega_t^{-1} + B'H_t^{-1}B)^{-1}B'H_t^{-1}e$$

$$= H_t^{-1}e - H_t^{-1}B(B'H_t^{-1}B)^{-1}B'H_t^{-1}e$$

$$+ H_t^{-1}B(B'H_t^{-1}B)^{-1}\Omega_t^{-1}(\Omega_t^{-1} + B'H_t^{-1}B)^{-1}B'H_t^{-1}e.$$
In relation to (70), we seek the asymptotic distribution of

\[ N^\frac{1}{2} \left( Nw_{it}^{\text{gmv}} - N\tilde{w}_{it}^{\text{gmv}} \right) \]

where

\[
w_{it}^{\text{gmv}} = \frac{1}{N} \left( C_{it,N} \right) = \frac{1}{N} \left( \frac{e_i^{(N)}H_t^{-1}e - e_i^{(N)}H_t^{-1}B(\Omega_t^{-1} + B'H_t^{-1}B)^{-1}B'H_t^{-1}e}{N^{-1}D_{t,N}} \right) + O_p(N^{-2}),
\]

\[
\tilde{w}_{it}^{\text{gmv}} = \frac{1}{N} \left( \frac{b_t^{-1}e_i^{(N)}H_t^{-1}(e + a_t(\bar{e}_t^2 - B)A_t^{-1}H_t^{-1}1)}{a_t^{-1}b_t^{-1}} \right).
\]

By the continuous mapping theorem

\[ N^\frac{1}{2} \left( Nw_{it}^{\text{gmv}} - N\tilde{w}_{it}^{\text{gmv}} \right) \rightarrow_d \zeta_{it}^{\text{gmv}}, \]

\[ N^\frac{1}{2} \left( \frac{1}{N^{-1}D_{t,N}} - \frac{1}{a_t^{-1}b_t^{-1}} \right) \rightarrow_d \zeta_2^{\text{gmv}}, \]

where \( \zeta_{it}^{\text{gmv}}, \zeta_2^{\text{gmv}} \) are a \( k \times 1 \) and a scalar normally distributed random variable, respectively, with zero mean. Therefore by standard results

\[ N^\frac{1}{2} (Nw_{it}^{\text{gmv}} - N\tilde{w}_{it}^{\text{gmv}}) \rightarrow_d \xi_{1,t} \zeta_{it}^{\text{gmv}} + \xi_{2,t} \zeta_2^{\text{gmv}} = z_{it}^{\text{gmv}}, \]

which is a mixture of normal random variables, unless \( \xi_{1,t}, \xi_{2,t} \) are both non-random.

Concerning term (71) that involves \( \Omega_t \)

\[ e_i'H_t^{-1}B(B'H_t^{-1}B)^{-1}\Omega_t^{-1}(\Omega_t^{-1} + B'H_t^{-1}B)^{-1}B'H_t^{-1}e \]

\[ = a_t e_i'H_t^{-1}B(A_t + a_t\beta\beta)^{-1}\Omega_t^{-1}N^{-1}(A_t + a_t\beta\beta)^{-1}\beta(1 + o_p(1)). \]

(iii) To establish (43), since:

\[ w_{it-1}^{\text{gmv}} = \frac{1}{e_i\Sigma_{i,t-1}^{-1}} \left[ e_i'\Sigma_{i,t-1}^{-1}B(\nu_{t-1} + u_t) + e_i'\Sigma_{i,t-1}^{-1}(a_{t-1} + \varepsilon_t) \right], \]

34
then by Lemma A the first term on the right hand side satisfies

\[ \frac{1}{e'} \Sigma_{t-1}^{-1} e' \Sigma_{t-1} B (\nu_{t-1} + u_t) = O_p(N^{-1}). \]

The covariance matrix of the term involving \( \varepsilon_t \) is

\[ (e' \Sigma_{t-1}^{-1} e)^{-2} e' \Sigma_{t-1} H_{t-1} \Sigma_{t-1}^{-1} e = O_p(N^{-1}), \]

since by easy calculations

\[ \frac{e' \Sigma_{t-1}^{-1} H_{t-1} \Sigma_{t-1}^{-1} e}{N} \to_p 0, \]

yielding \( \varepsilon_t w_{qmu}^2 = O_p(N^{-\frac{1}{2}}). \) Therefore, collecting terms

\[ \rho_{qmu}^t = \frac{N^{-1} e' \Sigma_{t-1}^{-1} \alpha_{t-1}}{N^{-1} e' \Sigma_{t-1}^{-1} e} + O_p(N^{-\frac{1}{2}}), \]

which is asymptotically equivalent, by part (i) of this proof, to

\[ \frac{1}{N \alpha_{t-1}} \left( e' H_{t-1}^{-1} \alpha_{t-1} + a_{t-1} b' A_{t-1}^{-1} (B' - b' e') H_{t-1}^{-1} \alpha_{t-1} \right) \to_p \frac{c_{t-1}}{a_{t-1}}. \]

(44) follows easily since \( \sigma_{gmu}^2 \) and where the limit of \( \rho_{qmu}^t \), just established, coincide with the limit of its conditional mean \( \mu_{qmu}^t \).

**Proof of Theorem 2.** All the limits below are based on \( N \to \infty. \)

(i) By identity (69)

\[ w_{qmu}^t = \frac{1}{\gamma_t} \Sigma_{t-1}^{-1} (\mu_t - er_{0t}) \]

\[ = \frac{1}{\gamma_t} \left( H_{t-1}^{-1} (\mu_t - er_{0t}) - H_{t-1}^{-1} B (\Omega_{t-1}^{-1} + B' H_{t-1}^{-1} B)^{-1} B' H_{t-1}^{-1} (\mu_t - er_{0t}) \right), \]

for \( N < \infty. \) Since

\[ e_t^{(N)} \Sigma_t^{-1} (\mu_t - er_{0t}) = e_t^{(N)} \Sigma_t^{-1} \alpha_t - e_t^{(N)} \Sigma_t^{-1} er_{0t} + e_t^{(N)} \Sigma_t^{-1} B \nu_t, \]

we just need to determine the behavior of the first term on the right hand side. In fact, the second term can be written as \( -e_t^{(N)} C_{t,Nr0t} \) with \( C_{t,N} \) defined in the proof of Theorem 1 and the third term, \( e_t^{(N)} \Sigma_t^{-1} B \nu_t, \) goes to
By the continuous mapping theorem
\[ e_i^{(N)} \sum_t^{-1} \alpha_t = e_i^{(N)} H_t^{-1} \alpha_t - e_i^{(N)} H_t^{-1} B (A_t + a_t \bar{\beta} \bar{\beta}^{-1} \bar{\beta}) c_t + o_p(1), \]
since \((A_t + a_t \bar{\beta} \bar{\beta}^{-1} \bar{\beta}) c_t = c_t b_t A_t^{-1} \bar{\beta},\) straightforward manipulation yields
\[ e_i^{(N)} \sum_t^{-1} \alpha_t = 1 b_t e_i^{(N)} (\alpha_t + (a_t \mu_t \bar{\beta} - B c_t) A_t^{-1} \bar{\beta}) + o_p(1). \]
\[ \square \]

(ii) For (47)
\[
\begin{align*}
    w_t^{meu} &= H_t^{-1}(\mu_t - e r_0) - H_t^{-1} B (B' H_t^{-1} B)^{-1} B' H_t^{-1}(\mu_t - e r_0) + O_p(N^{-1}) \\
    &= H_t^{-1}(I_N - B (B' H_t^{-1} B)^{-1} B' H_t^{-1})(\mu_t - e r_0) \\
    &= H_t^{-1} B (B' H_t^{-1} B)^{-1} (\Omega_t^{-1} + B' H_t^{-1} B)^{-1} (\mu_t - e r_0).
\end{align*}
\]

In relation to (72), we seek the asymptotic distribution of
\[
N^\frac{1}{2} (u_t^{meu} - \bar{w}_t^{meu})
\]
where
\[
\begin{align*}
    w_t^{meu} &= \frac{1}{\gamma_t} \left( e_i^{(N)} H_t^{-1}(\mu_t - e r_0) - e_i^{(N)} H_t^{-1} B (\Omega_t^{-1} + B' H_t^{-1} B)^{-1} B' H_t^{-1}(\mu_t - e r_0) \right) \\
    &= \frac{1}{\gamma_t} \left( e_i^{(N)} H_t^{-1}(\mu_t - e r_0) - e_i^{(N)} H_t^{-1} B (B' H_t^{-1} B)^{-1} B' H_t^{-1}(\mu_t - e r_0) \right) + O_p(N^{-1}) \\
    &= \frac{1}{\gamma_t} \left( e_i^{(N)} H_t^{-1}(\alpha_t - e r_0) - e_i^{(N)} H_t^{-1} B (B' H_t^{-1} B)^{-1} (a_t - a_t r_0) \bar{\beta} \right) + 2O_p(N^{-1}),
\end{align*}
\]
\[
\begin{align*}
    \bar{w}_t^{meu} &= \frac{1}{\gamma_t} \left( e_i^{(N)} H_t^{-1}(\alpha_t - e r_0) - e_i^{(N)} H_t^{-1} B (A_t + a_t \bar{\beta} \bar{\beta}^{-1} \bar{\beta}) c_t \right).
\end{align*}
\]

By the continuous mapping theorem
\[
N^\frac{1}{2} \left( (B' H_t^{-1} B)^{-1} B' H_t^{-1}(\alpha_t - e r_0) - (A_t + a_t \bar{\beta} \bar{\beta}^{-1} \bar{\beta}) c_t \right) \to_d \zeta_t^{meu},
\]
where \(\zeta_t^{meu}\) is a \(k \times 1\) normally distributed random vector with zero mean yielding
\[
N^\frac{1}{2} (w_t^{meu} - \bar{w}_t^{meu}) \to_d \gamma_t^{-1} \xi_{1t}^{meu},
\]
which is a mixture of normal random variables, unless \(\xi_{1t}\) is non-random.


