The Minimal Entropy Martingale Measure for General Barndorff-Nielsen/Shephard Models

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Abstract

We determine the minimal entropy martingale measure for a general class of stochastic volatility models where both price and volatility process contain jump terms which are correlated. This generalizes previous studies which have treated either the geometric Lévy case or continuous price processes with an orthogonal volatility process. We proceed by linking the entropy measure to a certain semi-linear Integro-PDE for which we prove the existence of a classical solution.

1 Introduction

The main contribution of this paper is the calculation of the minimal entropy martingale measure (MEMM) for a general class of stochastic volatility models encompassing the simpler cases where either the dynamics of the risky asset is modelled as a geometric Lévy process or the price process is continuous with an orthogonal pure jump volatility process. These cases, as will be discussed below, have been studied separately and with different methods. Our approach presents a unifying framework which moreover covers models like the Barndorff-Nielsen and Shephard (BN-S) model where both price and volatility process contain jump terms which are correlated. It turns out that due to the correlation this general case is much more difficult and can be considered as a non-trivial mixture of the two cases studied previously.

Asset process models driven by non-normal Lévy processes date back to the work of Mandelbrot (1967). More recently, rather complex models like the stochastic volatility model of Barndorff-Nielsen and Shephard (2001) have been developed. This model is constructed via a jump-diffusion price process together with a mean reverting, stationary volatility process.

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of Ornstein-Uhlenbeck type driven by a subordinator, i.e. an increasing Lévy process. Moreover, the negative correlation between price and volatility process in this model allows to deal with the so called leverage problem, i.e. for equities a fall in price level typically is associated with an increase in volatility. One main reason for the use of Lévy-driven asset models is the flexibility when fitting a model to observed asset prices. However, the corresponding financial market then is typically incomplete which results in the existence of multiple equivalent martingale measures. A standard approach is to identify an optimal martingale measure on the basis of the utility function of the investor, see Kallsen (2001). In this paper, we consider the exponential utility function which corresponds via an asymptotic utility indifference approach to taking the MEMM as pricing measure (Becherer (2004), Delbaen et al. (2002)).

In case the price process is an exponential Lévy process, the MEMM has been calculated by several authors in varying degrees of generality (e.g. Chan (1999), Miyahara (2001), Fujiwara and Miyahara (2003) and Esche and Schweizer (2005)). Grandits and Rheinländer (2002), and Benth and Meyer-Brandis (2004) determine the MEMM in stochastic volatility models where the price process is driven by a Brownian motion \( B \), whereas the volatility process may contain jump terms and is orthogonal to \( B \). Still assuming a continuous price process, Becherer (2004) considers a model with interacting Itô- and point processes.

With respect to the BN-S model with leverage effect, Nicolato and Venardos (2003) analyze the class of all equivalent martingale measures, with a focus on the subclass of structure preserving martingale measures (i.e., the price process is also of BN-S-type under those martingale measures). While in case of exponential Lévy processes, the asset process under the MEMM is again an exponential Lévy process (see in particular Esche and Schweizer (2005)), one major implication of the results in this paper is that the volatility process in the BN-S model in general has no longer independent increments when seen under the MEMM. Therefore, only considering the class of structure preserving martingale measures seems to be a too narrow approach, especially in the context of exponential utility maximization.

The paper is structured as follows. Section 2 introduces our setup and the martingale approach for determining the MEMM in case of a general Lévy process-driven asset model. In Section 3 we consider a general class of stochastic volatility models. We derive the structure of the MEMM by linking it to the solution of a certain semi-linear Integro-PDE to which a unique classical solution is shown to exist. We conclude this paper in section 4 by applying this result to the two extreme cases (price process is Lévy process respectively continuous with an orthogonal stochastic volatility process) as well as to the BN-S model. The latter case presents an additional technical difficulty since the volatility process is unbounded. This issue has been resolved in Steiger (2005).

The present approach has been influenced by the martingale duality approach in Rheinländer (2005) where the MEMM was linked to the solution of a certain equation in case of a filtration where all martingales are continuous. This has been applied in Hobson (2004) and Rheinländer (2005) to stochastic volatility models driven by Brownian motions. The presence of jumps, however, calls for more general techniques. Our method was inspired by Becherer’s (2001) approach to consider interacting systems of semi-linear PDEs.
2 Preliminaries and General Results

We start with some general assumptions, which are valid throughout the paper. Let \((\Omega, \mathcal{F}, \mathbb{F}, P)\) be a filtered probability space and \(T\) some fixed finite time horizon. We assume that \(\mathcal{F}_0\) is trivial and that \(\mathcal{F} = \mathcal{F}_T\). The filtration \(\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) fulfills the usual conditions and is generated by a Lévy process \(Y\) where \(Y^c\) (\(Y^d\)), \(\mu_Y\), and \(\nu_Y(dx, dt) = \nu(dx)dt\) denote its continuous (discontinuous) martingale part, the jump measure, and its compensator, respectively. For simplicity, we assume that \(\langle Y^c \rangle_t = t\). We refer to Jacod and Shiryaev (2003) (abbreviated JS) with respect to the notation used in this paper. In particular, \(G_{\text{loc}}(\mu_Y)\) is defined in JS, Definition II.1.27.

Remark 2.1 By JS, Theorem III.4.34, we have the following representation property: every \((P, \mathbb{F})\)-local martingale \(M\) can be written as

\[ M = M_0 + \int H \, dY^c + W \ast (\mu_Y - \nu_Y) \]

for some \(H \in L^2_{\text{loc}}(Y^c), W \in G_{\text{loc}}(\mu_Y)\).

We denote by \(S\) an \(\mathbb{F}\)-adapted, locally bounded semimartingale (modelling the price process of a risky asset), which has the following canonical decomposition:

\[ S = S_0 + M + A \]

where \(M\) is a locally bounded local martingale with \(M_0 = 0\) and \(A\) is a process of locally finite variation. By the representation property, we write \(M\) as

\[ M = M^c + M^d = \int \sigma^M \, dY^c + W^M(x) \ast (\mu_Y - \nu_Y) \]

where \(M^c\) and \(M^d\) are the continuous and the discontinuous part of the local martingale \(M\), respectively, \(\sigma^M\) is predictable and \(W^M \in G_{\text{loc}}(\mu_Y)\). Moreover, we assume that the asset price process \(S\) satisfies the following

Assumption 2.2 (Structure Condition) There exists a predictable process \(\lambda\) satisfying

\[ A = \int \lambda \, d\langle M \rangle, \]

with

\[ K_T := \int_0^T \lambda_s^2 \, d\langle M \rangle_s < \infty \quad P - \text{a.s.} \]

Definition 2.3 \(\mathcal{V}\) is the linear subspace of \(L^\infty(\Omega, \mathcal{F}, P)\), spanned by the elementary stochastic integrals of the form \(f = h(S_{T_2} - S_{T_1})\), where \(0 \leq T_1 \leq T_2 \leq T\) are stopping times such that the stopped process \(S_{T_2}\) is bounded and \(h\) is a bounded \(\mathcal{F}_{T_1}\)-measurable random variable. A martingale measure is a probability measure \(Q \ll P\) with \(E[\frac{dQ}{dP} f] = 0\) for all \(f \in \mathcal{V}\).
We denote by $\mathcal{M}$ the set of all martingale measures for $S$ and by $\mathcal{M}^e$ the subset of $\mathcal{M}$ consisting of probability measures which are equivalent to $P$. Here and in the sequel, we identify measures with their densities. Note that, as $S$ is locally bounded, a probability measure $Q$ absolutely continuous to $P$ is in $\mathcal{M}$ if and only if $S$ is a local $Q$-martingale.

**Definition 2.4** The relative entropy $I(Q, R)$ of the probability measure $Q$ with respect to the probability measure $R$ is defined as

$$I(Q, R) = \begin{cases} 
E_R \left[ \frac{dQ}{dR} \log \frac{dQ}{dR} \right] & \text{if } Q \ll R, \\
+\infty & \text{otherwise}
\end{cases}$$

It is well known that $I(Q, R) \geq 0$ and that $I(Q, R) = 0$ if and only if $Q = R$.

**Definition 2.5** The minimal entropy martingale measure $Q^E$, in the following also abbreviated MEMM, is the solution of

$$\min_{Q \in \mathcal{M}} I(Q, P).$$

Theorems 1, 2 and Remark 1 of Frittelli (2000) as well as the fact that $\mathcal{V} \subset L^\infty(P)$ yield the following

**Theorem 2.6** [Frittelli (2000)] If there exists $Q \in \mathcal{M}^e$ such that $I(Q, P) < \infty$, then the minimal entropy martingale measure exists, is unique and moreover is equivalent to $P$.

Let us restate the following criterion for a martingale measure to coincide with the MEMM.

**Theorem 2.7** [Grandits and Rheinländer (2002)] Assume there exists a $Q \in \mathcal{M}^e$ with $I(Q, P) < \infty$. Then $Q^*$ is the minimal entropy martingale measure if and only if there exists a constant $c$ and an $S$-integrable predictable process $\phi$

$$\frac{dQ^*}{dP} = \exp \left( c + \int_0^T \phi_t dS_t \right)$$

such that $E_Q[\int_0^T \phi_t dS_t] = 0$ for all $Q \in \mathcal{M}^e$ with finite relative entropy.

**Remark 2.8** Based on the above results, we will pursue the following strategy to determine the MEMM: Find first some candidate measure $Q^*$ which can be represented as in (2.1). To verify that $Q^*$ is indeed the entropy minimizer, we shall proceed in three steps:

1. Show that $Q^*$ is an equivalent martingale measure;
2. $I(Q^*, P) < \infty$;
3. $\int \phi \, dS$ is a true $Q$-martingale for all $Q \in \mathcal{M}^e$ with finite relative entropy.

This martingale approach yields a necessary equation for $\phi$ and $c$: 4
Theorem 2.9 Assume that the MEMM $Q^*$ exists. The strategy $\phi$ and the constant $c$ in (2.1) satisfy the equation
\[
c + \int_0^T \left[ \frac{1}{2}(\sigma_t^L - \lambda_t)^2 + \phi_t \lambda_t (\sigma_t^M)^2 + \phi_t \lambda_t \int_\mathbb{R} (W_t^M(x))^2 \nu(dx) \right] dt
\]
\[
= \int_0^T \left( \sigma_t^L - (\phi_t + \lambda_t) \sigma_t^M \right) dY_t^c
\]
\[
+ \left( \left( W_t^L(x) - (\phi + \lambda) W_t^M(x) \right) * (\mu_Y - \nu_Y) \right)_T
\]
\[
+ \left( \left( \log(1 - \lambda W_t^M(x) + W_t^L(x)) + \lambda W_t^M(x) - W_t^L(x) \right) * \mu_Y \right)_T
\]
\tag{2.2}
\end{equation}

with predictable processes $\sigma^L \in L^2_{loc}(Y^c)$ as well as $W^L \in G_{loc}(\mu_Y)$ such that
\[
\sigma_t^M \sigma_t^L + \int_\mathbb{R} W_t^M(x) W_t^L(x) \nu(dx) = 0 \quad \forall t \in [0, T]. \tag{2.3}
\]

**Proof:** By Girsanov’s theorem together with the Structure Condition, the density process $Z = (Z_t)$ of $Q^*$ is a stochastic exponential of the form
\[
Z = \mathcal{E}(- \int \lambda \, dM + L)
\]
where $L$ and $[M, L]$ are local $P$-martingales. Let us write the local martingale $L$ by the representation property in the following way:
\[
L = \int \sigma^L \, dY^c + W^L(x) * (\mu_Y - \nu_Y), \tag{2.4}
\]
for some $\sigma^L \in L^2_{loc}(Y^c)$, $W^L \in G_{loc}(\mu_Y)$. We therefore get
\[
[M, L] = \int_0^T \sigma_s^M \sigma_s^L ds + W^M(x) W^L(x) * \mu_Y.
\]

We further know from Dellacherie and Meyer (1980), VII.39, that the predictable bracket process
\[
\langle M, L \rangle = \int_0^T \sigma_s^M \sigma_s^L ds + W^M(x) W^L(x) * \nu_Y
\]
exists, since $M$ is locally bounded. However, $\langle M, L \rangle$ is equal to zero since $[M, L]$ is a local martingale. Therefore, we get condition (2.3). We now apply Itô’s formula to $\log Z$ to get, for $t \in [0, T]$, that
\[ \log Z_t = \int_0^t \frac{1}{Z_{s-}} dZ_s - \frac{1}{2} \int_0^t \frac{1}{Z_{s-}^2} d(Z^c)_s + \sum_{s \leq t} (\log Z_s - \log Z_{s-} - \frac{1}{Z_{s-}} \Delta Z_s) \]
\[ = - \int_0^t \lambda_s dM_s + L_t - \frac{1}{2} \int_0^t \lambda_s^2 d(M^c)_s + \int_0^t \lambda_s d(M^c, L^c)_s - \frac{1}{2} \langle L^c \rangle_t \]
\[ + \sum_{s \leq t} (\log \frac{Z_s}{Z_{s-}} + \Delta \int_0^s \lambda dM - \Delta L_s) \]
\[ = \int_0^t (\sigma_s^L - \lambda_s \sigma_s^M) dY_s^c - \frac{1}{2} \int_0^t (\lambda_s \sigma_s^M - \sigma_s^L)^2 ds \]
\[ + \left( \left( W^L(x) - \lambda W^M(x) \right)^* (\mu_Y - \nu_Y) \right)_t \]
\[ + \left( \left( \log(1 - \lambda W^M(x) + W^L(x)) + \lambda W^M(x) - W^L(x) \right)^* \mu_Y \right)_t. \]

Moreover, due to Theorem 2.7, at the time horizon we have
\[ \log Z_T = c + \int_0^T \phi_t dS_t \]
\[ = c + \int_0^T \phi_t \sigma_t^M dY_t^c + \left( \phi_t W^M(x)^* (\mu_Y - \nu_Y) \right)_T \]
\[ + \int_0^T \left( \phi_t \lambda_t (\sigma_t^M)^2 + \phi_t \lambda_t \int_{\mathbb{R}} (W^M_t(x))^2 \nu(dx) \right) dt. \]

We arrive at equation (2.2) upon combining the two equations above. \[ \blacksquare \]

**Corollary 2.10** Equation (2.2) in Theorem 2.9 is fulfilled once the following conditions are satisfied:

(i) \(|W^L(x) - (\phi + \lambda)W^M(x)| \* \mu_Y \in \mathcal{A}^+_t\)

(ii) It holds

\[ c + \int_0^T \left[ \frac{1}{2} (\sigma_t^L - \lambda_t \sigma_t^M)^2 + \phi_t \lambda_t (\sigma_t^M)^2 \right] dt \]
\[ + \int_0^T \int_{\mathbb{R}} \left( \phi_t W^L_t(x) - (\phi_t + \lambda_t) W^M_t(x) + \phi_t \lambda_t (W^M_t(x))^2 \right) \nu(dx) dt \]
\[ = \int_0^T \left( \sigma_t^L - (\phi_t + \lambda_t) \sigma_t^M \right) dY_t^c \]
\[ + \left( \left( \log(1 - \lambda W^M_t(x) + W^L_t(x)) - \phi W^M_t(x) \right)^* \mu_Y \right)_T. \] (2.5)
Proof: By JS, Proposition II.1.28, condition (i) implies that we can write
\[
(W^L(x) - (\phi + \lambda)W^M(x)) * (\mu_Y - \nu_Y) = \left( W^L(x) - (\phi + \lambda)W^M(x) \right) * \mu_Y - \left( W^L(x) - (\phi + \lambda)W^M(x) \right) * \nu_Y.
\]
Taking this into account, equation (2.2) reduces to the simpler equation (2.5).

Once we have, by the solution of (2.2) together with (2.3), found a candidate martingale measure, we still have to carry out the verification procedure as outlined above. We will need the following lemma, which is a generalization of the Novikov condition to discontinuous processes:

Lemma 2.11 [Lepingle and Mémin (1978)] Let $N$ be a locally bounded local $P$-martingale. Let $Q$ be a measure defined by
\[
\frac{dQ}{dP} \bigg|_{\mathcal{F}_t} = Z_t = \mathcal{E}(N)_t,
\]
where $\Delta N > -1$. If the process
\[
U_t = \frac{1}{2} (N^c)_t + \sum_{s \leq t} \{(1 + \Delta N_s) \log(1 + \Delta N_s) - \Delta N_s\}
\]
belongs to $\mathcal{A}_{loc}$, and therefore has a predictable compensator $B_t$, as well as
\[
E[\exp B_T] < \infty,
\]
then $Q$ is an equivalent probability measure.

Finally, to cope with item 3 of our approach as in Remark 2.8, we mention the following result:

Lemma 2.12 [Rheinländer (2005)] Let $Q$ be an equivalent martingale measure with finite relative entropy, and let $\int \psi dS$ be a local $Q$-martingale. Then $\int \psi dS$ is a true $Q$-martingale if, for some $\beta > 0$ small enough, $\exp \left\{ \beta \int_0^T \psi_t^2 d[S]_t \right\}$ is $P$-integrable.

3 A General Jump-Diffusion Model

From now on, we restrict the Lévy process $Y$ to the case where $\nu(\mathbb{R}) < \infty$, i.e. $Y^d$ is a compensated compound Poisson process. Let us consider a class of stochastic volatility models with asset prices of the following type:
\[
\frac{dS_t}{S_{t-}} = \eta^M(t, V_{t-}) dt + \sigma^M(t, V_{t-}) dY^c_t + d\left( W^M(\cdot, V_{t-}, x) * (\mu_Y - \nu_Y) \right)_t \quad (3.1)
\]
\[
dV_t = \eta^V(t, V_{t-}) dt + d\left( W^V(\cdot, V_{t-}, x) * \mu_Y \right)_t, \quad (3.2)
\]
where $V$ is defined on some interval $E \subset \mathbb{R}$. In the notation we will often suppress the dependence on $V$ of the various processes. Our basic assumptions are as follows:

**Assumption 3.1**

1. The coefficient $\eta^V$ is differentiable in $y$ with bounded continuous partial derivative and is locally Lipschitz-continuous in $t$. $W^V$ is differentiable with bounded derivative in $y$ and continuous in $t$.

2. The coefficients $\eta^M$, $\sigma^M$ and $W^M$ are locally Lipschitz-continuous in $t$ and differentiable in $y$ with bounded derivative. Furthermore, $\eta^M$ is positive, $\sigma^M$ is positive and uniformly bounded away from zero on $[0, T] \times E$, and $W^M_t(y, \cdot) : \text{supp}(\nu) \to (-1, \infty)$ is uniformly bounded.

3. The functions $W^M$ and $W^V$ are in $G_{\text{loc}}^\mu$.

4. $b := \eta^M + (\sigma^M)^2 + \int (W^M(x))^2 \nu(dx)$ is uniformly bounded on $[0, T] \times E$.

**Remark 3.2** By Protter (2004), Theorem V.38 and the remark following it, Assumptions 3.1.1 - 3.1.3 ensure that there exists a unique solution $(S, V)$ to equations (3.1) and (3.2) which does not explode in $[0, T]$.

Let us turn to our basic equation (2.5). The functions $\sigma^M$ and $W^M(x)$ of section 2 correspond now, with a slight abuse of notation, to $S_- \sigma^M$ and $S_- W^M(x)$, respectively. The jump times of $Y$ may, because of the finite Lévy measure $\nu$, be counted in increasing order $0 =: \tau_0 < \tau_1 < \cdots$ such that we can write

$$
\begin{align*}
\left( \left[ \log \left( 1 - \lambda S_- W^M(x) + W^L(x) \right) - \phi S_- W^M(x) \right] * \mu_Y \right)_T \\
= \sum_{i=1}^{\infty} \left[ \log \left( 1 - \hat{\lambda}_i \tau_i W^M_{t_i}(x) + W^L_{t_i}(x) \right) - \hat{\phi}_i \tau_i W^M_{t_i}(x) \right] 1_{\tau_i \leq T}
\end{align*}
$$

with $\hat{\lambda} := \lambda S_- \text{ and } \hat{\phi} := \phi S_-$. We denote

$$
\Delta^u_{t,y}(x) := u(t, y + W^V(t, y, x)) - u(t, y),
$$

and work with the Ansatz that there exists a function $u$ such that

$$
\log \left( 1 - \hat{\lambda} W^M(x) + W^L(x) \right) - \hat{\phi} W^M(x) = \Delta^u_{t, V_-}(x), \quad (3.3)
$$

i.e. the jumps of the RHS of (2.5) correspond to the jumps of some function $u$ along the paths of process $V$. In addition, we set

$$
u(T, \cdot) = 0 \quad \text{on } E \quad (3.4)$$
Let us introduce a solution to this problem might be to require that

and we assume that \( u \) is a sufficiently smooth function. Taking into account that there are no jumps of \( Y \) in \([\tau_i, \tau_{i+1})\), we get

\[
\begin{align*}
    u(\tau_{i+1}, V_{\tau_{i+1}}) &= u(\tau_i, V_{\tau_i} + W_{\tau_i}^V(\Delta Y_{\tau_i})) \\
    &= u(\tau_{i+1}, V_{\tau_{i+1}}) - u(\tau_i, V_{\tau_i}) \\
    &= \int_{(\tau_i, \tau_{i+1})} du(t, V_t) \\
    &= \int_{(\tau_i, \tau_{i+1})} \left[ \frac{\partial}{\partial t} u(t, V_t) + \eta_t^V \frac{\partial}{\partial V} u(t, V_{\tau_i}) \right] dt.
\end{align*}
\]

We may therefore rewrite equation (2.5) as

\[
\begin{align*}
c + u(0, V_0) &= -\int_0^T \left[ \frac{1}{2} (\sigma^L_t - \bar{\lambda} \sigma^M_t)^2 + \phi_t \dot{\lambda}_t (\sigma^M_t)^2 + \frac{\partial}{\partial t} u(t, V_{\tau_i}) + \eta_t^V \frac{\partial}{\partial V} u(t, V_{\tau_i}) \\
   &\quad + \int \left( W^L_t(x) - (\bar{\phi} + \bar{\lambda}) W^M_t(x) + \phi_t \dot{\lambda}_t (W^M_t(x))^2 \right) \nu(dx) \right] dt \\
   &\quad + \int_0^T \left[ \sigma^L_t - (\bar{\phi} + \bar{\lambda}) \sigma^M_t \right] dY^c_t.
\end{align*}
\]

A solution to this problem might be to require that

\[
\begin{align*}
    \frac{1}{2} (\sigma^L_t - \bar{\lambda} \sigma^M_t)^2 + \phi_t \dot{\lambda}_t (\sigma^M_t)^2 + \frac{\partial}{\partial t} u(t, V_{\tau_i}) + \eta_t^V \frac{\partial}{\partial V} u(t, V_{\tau_i}) \\
    + \int \left( W^L_t(x) - (\bar{\phi} + \bar{\lambda}) W^M_t(x) + \phi_t \dot{\lambda}_t (W^M_t(x))^2 \right) \nu(dx) &= 0,
\end{align*}
\]

(3.6)

together with (3.4) and

\[
    c = -u(0, V_0), \quad \sigma^L = (\bar{\phi} + \bar{\lambda}) \sigma^M.
\]

Let us introduce \( u_t := u(t, \cdot) : E \to \mathbb{R} \) and

\[
\begin{align*}
g^u(t, u_t) := \frac{1}{2} \left( \sigma^L_t(y) - \bar{\lambda}_t(y) \sigma^M_t(y) \right)^2 + \phi_t(y) \dot{\lambda}_t(y) (\sigma^M_t(y))^2 \\
   &\quad + \int \left( W^L_t(y, x) - (\bar{\phi}_t(y) + \dot{\lambda}_t(y)) W^M_t(y, x) + \phi_t(y) \dot{\lambda}_t(y) (W^M_t(y, x))^2 \right) \nu(dx).
\end{align*}
\]

(3.8)

Provided that \( \phi_t, \sigma^L_t \) and \( W^L_t(x) \) are functions of \( u_t \), (3.6) is an Integro-PDE for \( u \) of the form

\[
\begin{align*}
    \frac{\partial}{\partial t} u(t, y) + \eta_t^V \frac{\partial}{\partial y} u(t, y) + g^u(t, u_t) &= 0 \quad \text{for all } y \in E.
\end{align*}
\]

(3.9)

\[
\begin{align*}
u(T, y) &= 0 \quad \text{for all } y \in E.
\end{align*}
\]

(3.10)
By equation (3.7) together with condition (2.3), we get
\[ \hat{\phi} = -\hat{\lambda} - \frac{\int W^M(x)W^L(x)\nu(dx)}{(\sigma^M)^2}, \] (3.11)
which, by equation (3.3), leads to
\[
\exp \left\{ \Delta^u(x) - \left[ \hat{\lambda} + \frac{\int W^M(z)W^L(z)\nu(dz)}{(\sigma^M)^2} \right] W^M(x) \right\} = 1 - \hat{\lambda}W^M(x) + W^L(x). \] (3.12)
To make this intuitive approach rigorous, we shall proceed as follows: we show in Corollary 3.4 below that each \( u \in C_b([0, T] \times E) \) gives via \( \Delta^u \) uniquely a bounded function \( W_t^L \) solving (3.12). We then define \( \hat{\phi} \) as in (3.11), \( \sigma^L \) as in (3.7) and \( g^u \) as in (3.8). In Theorem 3.8 below it is then shown that there exists a classical solution to the Integro-PDE (3.9), (3.10). Finally, we provide the verification results in Theorem 3.9.

For the discussion of equation (3.12) we first provide a preparatory result.

**Lemma 3.3** Let \( \beta > 0, f \in l^\infty(\text{supp(\nu)}) \), the set of bounded functions from \( \text{supp(\nu)} \) into \( \mathbb{R} \), and \( k \) be a function on \( \text{supp(\nu)} \), which is bounded from above. Then, the function \( \varphi : \text{supp(\nu)} \to \mathbb{R} \), given as
\[ \varphi(x) = \exp \left\{ k(x) - \beta f(x) \int f(z)\varphi(z)\nu(dz) \right\}, \]
is well-defined and bounded.

**Proof:** See Appendix. \[ \blacksquare \]

**Corollary 3.4** Under Assumption 3.1 as well as \( u \in C_b([0, T] \times E) \), \( u_t \) uniquely defines a function \( W_t^L = \cdot W_t^L(u_t) \in l^\infty(\text{supp(\nu)}) \) which fulfills equation (3.12). \( W_t^L \) and therefore also \( \hat{\phi} \) and \( \sigma^L \) are uniformly bounded for all \((t, y) \in [0, T] \times E \).

**Proof:** Introducing
\[ \varphi(x) := (W_t^L(u_t))(x) - \hat{\lambda}W_t^M(x) + 1, \] (3.13)
we may write equation (3.12) in the form
\[ \varphi(x) = \exp \left\{ k(x) + \beta f(x) \int f(z)\varphi(z)\nu(dz) \right\}, \]
with
\[
\begin{align*}
f(x) &:= W_t^M(x), \\
k(x) &:= \Delta^u(x) - W_t^M(x) \left[ \hat{\lambda} \left( 1 + \frac{\int W_t^M(z)^2\nu(dz)}{(\sigma_t^M)^2} \right) - \frac{\int W_t^M(z)\nu(dz)}{(\sigma_t^M)^2} \right], \\
\beta &:= \frac{1}{(\sigma_t^M)^2}.
\end{align*}
\]
Since $u_t \in C_b(E)$, we have $k \in l^\infty(\text{supp}(\nu))$ by Assumption 3.1 and we may apply Lemma 3.3. By the definition of $\varphi$ in (3.13), one directly gets that also $W^L$ fulfills equation (3.12). $W^L$ is uniformly bounded since $\Delta_t^v$ is uniformly bounded, which is a direct consequence of $u \in C_b([0,T] \times E)$. 

The function $W^L_{t,y} : C_b(E) \to l^\infty(\text{supp}(\nu))$ is not uniformly bounded. However, we can ensure boundedness by restricting the space $C_b(E)$ to the set

$$C^Q_b(E) := \{v \in C_b(E), \|v\|_\infty \leq Q\},$$

with a constant $Q > 0$. In fact, we even get the following statement:

**Lemma 3.5** For $(t,y) \in [0,T] \times E$ fixed,

$$W^L_{t,y} : C^Q_b(E) \to l^\infty(\text{supp}(\nu))$$

is Lipschitz-continuous, uniformly with respect to $t \in [0,T]$.

**Proof:** See Appendix.

We turn now to the existence of a solution for the Integro-PDE (3.9)-(3.10). The following two theorems provide some general existence results:

**Theorem 3.6** Let $E \subset \mathbb{R}$ be some interval. For $(t,z) \in [0,T] \times E$, consider

$$Z^{t,z} = z + \int_t^T b(u, Z_{u}^{t,z})du,$$

for a continuous process $b : [0,T] \times E \to \mathbb{R}$, such that $Z^{t,z}$ stays in $E$. Let us consider the partial differential equation with boundary condition:

$$
\begin{align*}
\frac{\partial}{\partial t} u(t,z) + b(t,z) \frac{\partial}{\partial z} u(t,z) + g^*(t, u_t) &= 0, \\
u(T,z) &= h(z) \quad \forall \ z \in E,
\end{align*}
$$

for which we shall assume:

**a-1** $b$ is locally Lipschitz-continuous.

**a-2** $g : [0,T] \times C_b(E) \to C_b(E)$ is a Lipschitz-continuous function in $v \in C_b(E)$, uniformly in $t$, i.e. there exists a constant $L < \infty$ such that

$$\|g(t,v_1) - g(t,v_2)\|_\infty \leq L \|v_1 - v_2\|_\infty \quad \forall t \in [0,T], \ v_1, v_2 \in C_b(E).$$

**a-3** $h : E \to \mathbb{R} \in C_b(E)$.
Then, there exists a unique solution \( \hat{u} \in C_b([0, T] \times E) \) which solves the boundary problem (3.15)-(3.16) in the sense of distributions. It can be written as

\[
\hat{u}(t, z) = h(Z^z_t) + \int_t^T g^z_s (s, \hat{u}_s)ds.
\]

**Proof:** See Appendix. \( \blacksquare \)

Existence of a strong solution can be ensured in the following special case:

**Theorem 3.7** Let us assume that all conditions of Theorem 3.6 are fulfilled. Let us further assume that \( E \subset \mathbb{R} \) is compact and that the following hold true.

**b-1** \( b \) has a uniformly bounded, continuous derivative \( \frac{\partial}{\partial z} b \).

**b-2** For any \( v \in C^1_b(E) \), \( g^z(t, v) \) is differentiable in \( z \) with \( \frac{\partial}{\partial z} g^z(t, v) = \hat{g}^z(t, \frac{\partial}{\partial z} v) \) for some suitable continuous function \( \hat{g} \), fulfilling

- there exist some constants \( L, K \) such that we may write
  \[
  \|\hat{g}(s, v_s)\|_\infty \leq L\|v_s\|_\infty + K, \tag{3.17}
  \]
- for any \( R > 0 \), \( \hat{g} \) is uniformly continuous on \( [0, T] \times M \times E \) with \( M = C^R_b(E) \).

**b-3** \( h \in C^1_b(E) \).

Then the weak solution \( \hat{u} \in C_b([0, T] \times E) \) is differentiable in the space variable and therefore, it is also the strong solution to the boundary problem (3.15)-(3.16).

**Proof:** See Appendix. \( \blacksquare \)

Let us apply this result to \( g^\phi(t, u_t) \) having the form (3.8). In this case, \( g^\phi(t, u_t) \) does not have to be Lipschitz-continuous. However, using a truncation argument we get the following result:

**Theorem 3.8** Let Assumption 3.1 be in place and let \( g^\phi(t, u_t) \) be of the form (3.8). Let \( E \) be a compact interval such that \( \sigma^M \) is uniformly bounded on \([0, T] \times E \). Then there is a classical solution \( \hat{u} \in C^{1,1}_b([0, T] \times E) \) to the Integro-PDE

\[
\frac{\partial}{\partial t} u(t, y) + \eta^v_t \frac{\partial}{\partial y} u(t, y) + g^\phi(t, u_t) = 0 \tag{3.18}
\]

with boundary condition

\[
u(T, y) = 0. \tag{3.19}\]

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\( \hat{u} \) satisfies

\[
\hat{u}(t, y) = \int_t^T g^{\hat{t}, y}(s, \hat{u}_s)ds
\]

with

\[
d\hat{V}^t_y = \eta^V(s, \hat{V}^t_y)ds
\]

and \( \hat{V}^t_y = y \).

**Proof:** Let us rewrite (3.8) using (3.7) and (3.11) as

\[
g(t, v) = \frac{1}{2} \left[ \left( \int \frac{W^M(x)W^L(x)\nu(dx)}{\sigma^M} \right)^2 - \lambda^2 (\sigma^M)^2 \right]
\]

\[
+ \frac{1}{(\sigma^M)^2} \int W^M(x)\nu(dx) - \frac{\lambda}{(\sigma^M)^2} \int W^M(x)^2\nu(dx)
\]

\[
+ \int W^L(x)\nu(dx) - \frac{\lambda}{(\sigma^M)^2} \int (W^M(x))^2\nu(dx),
\]

which is in general not Lipschitz-continuous. We circumvent this problem by introducing a truncating, auxiliary function \( \tilde{g} \). We will show that the weak solution \( \hat{u} \in C_b([0, T] \times \mathbb{R}) \) to the Integro-PDE

\[
\frac{\partial}{\partial t} u(t, y) + \eta^V \frac{\partial}{\partial y} u(t, y) + \tilde{g}^y(t, u_t) = 0,
\]

\[
u(T, y) = 0,
\]

fulfills the equation

\[
\tilde{g}^y(t, \hat{u}_t) = g^y(t, \hat{u}_t) \quad \forall (t, y) \in [0, T] \times E.
\]

We then conclude that \( \hat{u} \) is a weak solution to the partial differential equation (3.18) with boundary condition (3.19). In a final step, we will show that the solution is also a classical solution.

**Step 1: Definition of the auxiliary function \( \tilde{g} : [0, T] \times C_b(E) \to C_b(E) \)**

We introduce the function

\[
\tilde{g}(t, v) := g(t, \kappa(v, t)),
\]

defined on \([0, T] \times C_b(E)\), with the function \( \kappa \) truncating \( v \in C_b(E) \) in the following way. Let \( C \) be some positive constant, then

\[
\kappa(v, t)(x) := \max \left( \min(C(T-t), v(x)), -C(T-t) \right).
\]

**Step 2: Condition a-2 of Theorem 3.6 is fulfilled**
We have to prove that \( \tilde{g} \) is a Lipschitz-continuous function on \( \mathcal{C}_0(E) \), uniformly in \( t \), which follows if we can show that there exists a constant \( L \) independent of \((t,y) \in [0,T] \times E\) such that

\[
|g^y(t, v_1) - g^y(t, v_2)| \leq L\|v_1 - v_2\|_{\infty}
\]

for all \( v_1, v_2 \in \mathcal{C}_0^Q(E) \), whereas \( Q = CT \). In the following, we fix a pair \((t,y) \in [0,T] \times E\) and drop the indices \((t,y) \) in the notation. We may write

\[
\begin{align*}
|g(\cdot, v_1) - g(\cdot, v_2)| & \leq \frac{1}{2(\sigma M)^2} \left[ \left( \int W^M(x) \left( W^L(v_1) \right)(x) \nu(dx) \right)^2 - \left( \int W^M(x) \left( W^L(v_2) \right)(x) \nu(dx) \right)^2 \right] \\
& \quad + \left| \int W^M(x) \nu(dx) - \tilde{\lambda} \int \left( W^M(x) \right)^2 \nu(dx) \right| \\
& \quad \quad \times \left| \int \left( W^L(v_1) - W^L(v_2) \right)(x) W^M(x) \nu(dx) \right| \\
& \quad + \int \left( W^L(v_1) - W^L(v_2) \right)(x) W^M(x) \nu(dx) \right].
\end{align*}
\]

By Assumption 3.1, \( 1/(\sigma \sigma^M)^2 \) as well as \( \int W^M(x) \nu(dx) - \tilde{\lambda} \int \left( W^M(x) \right)^2 \nu(dx) \) are uniformly bounded on \([0,T] \times E\). Moreover, \( W^L(v) \) is uniformly bounded in \( v \in \mathcal{C}_0^Q(E) \) by some constant \( K \), and we may write, using the elementary inequality \( a^2 - b^2 \leq 2 \max(|a|, |b|) |a - b| \),

\[
\begin{align*}
|g(\cdot, v_1) - g(\cdot, v_2)| & \leq \left( \frac{1}{(\sigma M)^2} \left| \int W^M(x) \nu(dx) \right| \right) \left[ (K + 1) \left| \int W^M(x) \nu(dx) \right| + \tilde{\lambda} \int \left( W^M(x) \right)^2 \nu(dx) \right] + \nu(\mathbb{R}) \\
& \quad \times \left| W^L(v_1) - W^L(v_2) \right|_{\infty} \quad (3.25)
\end{align*}
\]

Due to Lemma 3.5 (Lipschitz-continuity of \( W^L \)), we conclude that \( \tilde{g} : [0, T] \times \mathcal{C}_b(E) \rightarrow \mathcal{C}_b(E) \) is Lipschitz-continuous on \( \mathcal{C}_b(E) \), uniformly in \( t \).

Now Theorem 3.6 can be applied to the problem (3.23)- (3.24) which gives us a unique bounded, weak solution \( \tilde{u} \in \mathcal{C}_b([0, T] \times E) \).

**Step 3:** There exists a constant \( C \) such that for all \((t, y) \in [0, T] \times E\),

\[
|\tilde{u}(t, y)| \leq (T - t)C. \quad (3.26)
\]

Let us fix \( t \in [0, T] \), \( y \in E \) as well as a positive constant \( C \) (to be specified below) and define (with \( \tilde{V} \) from (3.21)) the deterministic time \( \tau_y \) as

\[
\tau_y := \inf\{ s \in [t, T] \mid \tilde{u}(s, \tilde{V}^t_y) < (T - s)C \} \wedge T.
\]
Then \(\tilde{u}(s, V_t^{t,y}) \geq (T-s)C\) for all \(s \in [t, \tau_y]\) and \(\tilde{u}(\tau_y, V_{\tau_y}^{t,y}) \leq (T-\tau_y)C\). Since \(\tilde{u}(s, V_t^{t,y}) \geq (T-s)C\) for all \(s \in [t, \tau_y]\), we get (with the truncation function \(\kappa\) from step 1) \(\Delta_{s,V_t^{t,y}}^\kappa < 0\). It follows that, for \(s \in [t, \tau_y]\), the process

\[
W_{s,V_t^{t,y}}^L(\kappa(\tilde{u}_s), x) = \exp\left\{\Delta_{s,V_t^{t,y}}^\kappa(x) - \left[\bar{\lambda} + \frac{\int W_M(z)W_L(z)\nu(dz)}{(\sigma M)^2}\right]W_M(x)\right\} - 1 + \tilde{\lambda}W_M(x)
\]

is bounded by some constant independent of level \(\bar{C}\). By our assumptions, we then can conclude from (3.22) that there exists a constant \(C_1\), independent of \(\tau_y\) and hence also of \(C\), such that \(|\tilde{g}_{s}^{V_t^{t,y}}(s, \tilde{u}_s)| < C_1\) for all \(s \in [t, \tau_y]\). It results that

\[
\tilde{u}(t, y) = \int_t^T \tilde{g}_{s}^{V_t^{t,y}}(s, \tilde{u}_s)ds
\]

\[
= \int_t^{\tau_y} \tilde{g}_{s}^{V_t^{t,y}}(s, \tilde{u}_s)ds + \int_{\tau_y}^T \tilde{g}_{s}^{V_t^{t,y}}(s, \tilde{u}_s)ds
\]

\[
\leq (\tau_y - t)C_1 + (T-\tau_y)C.
\]

The lower bound can be shown directly. We know that \(W_{s}^L\) is bounded from below by \(-1 + \tilde{\lambda}W_M(x)\). As a direct consequence of this together with \(\sigma^M\) being bounded from above (this is the only place where we need this additional assumption), we get that \(\tilde{g}(s, \tilde{u}_s)\) is bounded from below. Therefore there exists a constant \(C_2 > 0\) such that

\[
\tilde{u}_t \geq -(T-t)C_2.
\]

If we now choose \(C \geq C_1 \lor C_2\), we directly get (3.26).

**Step 4: \(\tilde{u}\) is continuously differentiable in the space variable**

We use here an auxiliary function \(\tilde{g}\) slightly different from \(\tilde{g}\). A truncation function \(\tilde{\kappa}\) is now introduced in such a way that we do not bound \(u\) but the difference \(\Delta^u\), i.e. we consider \(\tilde{\kappa}(\Delta^u_t, t)\) instead of \(\Delta^\kappa_{(u,t)}\). In terms of the function \(\tilde{g}\), it means that we work with the function

\[
\tilde{W}^L : C_b(E) \to l^\infty(\text{supp}(\nu))
\]

defined as

\[
\tilde{W}^L(x) := \exp\left\{\tilde{\kappa}(\Delta^u, t)(x) - \left[\bar{\lambda} + \frac{\int W_M(z)\tilde{W}^L(z)\nu(dz)}{(\sigma M)^2}\right]W_M(x)\right\} - 1 + \tilde{\lambda}W_M(x).
\]
In addition, to ensure that \( \hat{g}(t, u_t) \) is differentiable, we assume that \( \hat{\kappa} \) has the following form, with \( w \in L^\infty(\text{supp}(\nu)) \):

\[
\hat{\kappa}(w, t)(x) = \begin{cases} 
  v(w) & \text{if } |w(x)| \leq (T-t)C \\
  \varphi(w, t)(x) & \text{if } (T-t)C < |w(x)| < K + (T-t)C \\
  \text{sign}(w(x))(K + C(T-t)) & \text{if } |w(x)| \geq K + (T-t)C
\end{cases}
\]

for some fixed constants \( C, K \) and a suitable \( \varphi(w, t) \in L^\infty(\text{supp}(\nu)) \) with \( |\varphi(w, t)(x)| \leq K + (T-t)C \), such that \( \hat{\kappa} : L^\infty(\text{supp}(\nu)) \times [0, T] \rightarrow L^\infty(\text{supp}(\nu)) \) is differentiable in \( w \) with uniformly bounded partial derivative.

Reasoning as in Step 2, we get that \( \hat{g} \) is Lipschitz-continuous and, therefore, we may apply Theorem 3.6 which provides a solution \( \hat{u} \). Let us consider \( \hat{u} \) from above, which is bounded due to Step 3, i.e. there exists a pair \((C, K)\) such that \( \hat{\kappa}(\Delta^2, t) = \Delta^2 \) and therefore \( \hat{g}(t, \hat{u}_t) = g(t, \hat{u}_t) \). By uniqueness of solution, we conclude that \( \hat{u} = \hat{u} \).

Let us now assume that \( u_t \in C^1_b(E) \). By direct calculation,

\[
\frac{\partial}{\partial y} \hat{g}^\mu(t, u_t) = \int \left( \frac{\partial}{\partial y} \hat{\kappa}(\Delta^2, t)(x) \right) \tilde{W}_t^L(x) \nu(dx) + k(t, y, \tilde{W}_t^L(\Delta^2, t(y)))
\]

with \( \tilde{W}_t^L(x) := \tilde{W}_t^L(x) + 1 - \lambda W^M(x) \) and a uniformly bounded \( k(t, y, \tilde{W}_t^L(\Delta^2, t(y))) \). Let us now write

\[
\frac{\partial}{\partial y} \hat{g}^\mu(t, u_t) = \left. \frac{\partial}{\partial w} \hat{\kappa}(w, t)(x) \right|_{w=\Delta^2, y} \left( \frac{\partial}{\partial y} \Delta^2, t(y) \right) \tilde{W}_t^L(x) \nu(dx) + k(t, y, \tilde{W}_t^L(\Delta^2, t(y)))
\]

\[
= \int \frac{\partial}{\partial w} \hat{\kappa}(w, t)(x) \left|_{w=f_0 \eta_{t,y}} \right. \frac{\partial}{\partial y} \Delta^2, t(y) \tilde{W}_t^L(x) \nu(dx) + k(t, y, \tilde{W}_t^L(\Delta^2, t(y)))
\]

\[
= \int \frac{\partial}{\partial w} \hat{\kappa}(w, t)(x) \left|_{w=f_0 \eta_{t,y}} \right. \left( \frac{\partial}{\partial y} \Delta^2, t(y) \tilde{W}_t^L(x) + \frac{\partial}{\partial y} u(t, y) \right)
\]

\[
\times \tilde{W}_t^L(x) \nu(dx) + k(t, y, \tilde{W}_t^L(\Delta^2, t(y)))
\]

\[
= \hat{g}^\mu(t, \frac{\partial}{\partial y} u_t).
\]

Let us set \( v_t(y) = \frac{\partial}{\partial y} u_t(y) \), which belongs to \( C_b(E) \). We already know that \( \tilde{W}_t^L \) is uniformly continuous and bounded in \((t, y, v_t) \in [0, T] \times E \times C_b(E) \) and therefore \( k \) is uniformly continuous and bounded on this set. On the other hand, taking into account the definition of \( \hat{\kappa} \), one directly gets that condition (3.17) is fulfilled and that

\[
\frac{\partial}{\partial w} \hat{\kappa}(w, t)(x) \left|_{w=f_0 \eta_{t,y}} \right. \left( \frac{\partial}{\partial y} \Delta^2, t(y) \right) \tilde{W}_t^L(x) + \frac{\partial}{\partial y} u(t, y) \nu(dx)
\]

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is uniformly continuous in \((t, y, v_t) \in [0, T] \times E \times M\). Therefore, all conditions of Theorem 3.7 are fulfilled and therefore the solution \(\hat{u}\) to the PDE (3.18) with boundary condition (3.19) is continuously differentiable in the space variable. ■

Having proven existence of a solution to the partial differential equation (3.9) with boundary condition (3.10), we are in the position to determine the triplet \((\hat{\phi}, W^L, \sigma^L)\) which solves equation (2.5). Since \(\hat{u}\) is uniformly bounded, we directly see that this also holds for \(\hat{\phi}\). The extra assumption that \(\sigma^M\) is bounded from above is not fulfilled in some examples. We shall indicate later, using the result of Theorem 3.8, how to proceed in the standard BN-S-model without this assumption and still get a uniformly bounded \(\hat{\phi}\).

**Theorem 3.9** Let Assumption 3.1 be fulfilled. We further assume that \(\sigma^M\) is uniformly bounded from above on \([0, T] \times E\). Let us assume that the triplet \((\hat{\phi}, W^L, \sigma^L)\) solves equation (2.5) as well as (2.3), with \((\hat{\phi}, W^L, \sigma^L)\) uniformly bounded. Then the process \(Z = (Z_t)\) defined by

\[
Z_t = \frac{dQ^*}{dP} \bigg|_{\mathcal{F}_t} = \mathcal{E} \left( - \left[ \int (\tilde{\lambda} \sigma^M - \sigma^L) dY^c + (\tilde{\lambda} W^M(x) - W^L(x)) \ast (\mu_Y - \nu_Y) \right] \right)_t
\]

is the density process of the MEMM.

**Proof:** To show that \(Q^*\) is the MEMM, we show according to our approach as outlined in Remark 2.8 that \(Q^*\) is an equivalent martingale measure, that \(I(Q^*, P) < \infty\) and that \(\int \tilde{\lambda} dS\) is a true \(Q\)-martingale for all \(Q \in \mathcal{M}^e\) with finite relative entropy.

1. \(Q^*\) is an equivalent martingale measure: Let us first show that it is an equivalent probability measure by checking the conditions of Lemma 2.11. We consider the local martingale \(N\) defined by

\[
N = - \int \lambda dM + L = \int (\sigma^L - \tilde{\lambda} \sigma^M) dY^c + \left( W^L(x) - \tilde{\lambda} W^M(x) \right) \ast (\mu_Y - \nu_Y).
\]

Since \(W^L, \tilde{\lambda}\) and \(W^M\) are bounded, \(N\) is locally bounded and due to

\[W^L(x) - \tilde{\lambda} W^M(x) > -1,\]

we have \(\Delta N > -1\). Moreover, we set

\[
U = \frac{1}{2} \int (\sigma^L - \tilde{\lambda} \sigma^M)^2 ds + \left\{ \left( 1 - \tilde{\lambda} W^M(x) + W^L(x) \right) \log \left( W^L(x) - \tilde{\lambda} W^M(x) + 1 \right) + \tilde{\lambda} W^M(x) - W^L(x) \right\} \ast \mu_Y.
\]
Since $\nu(\mathbb{R}) < \infty$ and $\tilde{\lambda}, \sigma^M, \sigma^L, W^M, W^L$ and $\tilde{\varphi}$ are all uniformly bounded, $U$ has locally integrable variation and its compensator $B$ is uniformly bounded as well. Hence, condition (2.7) is naturally fulfilled and therefore, $Q^*$ is an equivalent probability measure. It is a martingale measure since its density process can be written as

$$Z = \mathcal{E}\left(-\int \lambda \, dM + L\right)$$

where $L$ as well as $[M, L]$ are locally bounded local $P$-martingales.

2. $I(Q^*, P) < \infty$: The density $Z^* = \frac{dQ^*}{dP}$ may be written as

$$Z^* = \exp\left\{c + \int_0^T \frac{\tilde{\varphi}_t}{S_t} \, dS_t\right\}$$

where $c$ is the normalizing constant. We get

$$I(Q^*, P) = E_{Q^*}[c + \int_0^T \frac{\tilde{\varphi}_t}{S_t} \, dS_t]$$

$$= E_{Q^*}\left[c + \int_0^T \tilde{\varphi}_t \left(\eta_t^M \, dt + \sigma_t^M \, dY_t^c\right) + \left(\tilde{\varphi}W^M(x) \ast (\mu_Y - \nu_Y)\right)\right].$$

We therefore have to show that

$$E_{Q^*}\left[\int_0^T \frac{\tilde{\varphi}_t}{S_t} \, dS_t\right] = 0,$$

(3.28)

since then $I(Q^*, P) = c$ which is finite by the previous step. Introducing

$$\nu^*_Y = (W^L(x) + 1 - \tilde{\lambda}W^M(x)) \ast \nu_Y,$$

(3.29)

we get by Girsanov’s Theorem that $W^M(x) \ast (\mu_Y - \nu^*_Y)$ and $\int \sigma^M \, dY^c + \int (\tilde{\lambda}^M - \sigma^L) \sigma^M \, dt$ are local $Q^*$-martingales. In fact, they are true $Q^*$-martingales since their quadratic variations are $Q^*$-integrable; for this, we need in particular that $\sigma^M$ is bounded. (3.28) follows since the dynamics of $S$ can be written as

$$\frac{dS_t}{S_t} = \sigma_t^M \, dY_t^c + (\tilde{\lambda}_t^M - \sigma_t^L) \sigma_t^M \, dt + d\left(W^M(x) \ast (\mu_Y - \nu^*_Y)\right).$$

3. $\int \tilde{\varphi} \, dS$ is a true $Q$-martingale for all $Q \in \mathcal{M}^e$ with finite relative entropy: We will check the condition of Lemma 2.12. As preparation, let us observe that for any positive constant $\alpha$ we have

$$E\left[\exp\left\{\alpha (W^M(x))^2 \ast \mu_Y\right\}_T\right] < \infty,$$

(3.30)

$$E\left[\exp\left\{\alpha \int_0^T (\sigma_t^M)^2 \, dt\right\}\right] < \infty.$$
The first inequality follows from
\[ E \left[ \exp \left\{ \left( \alpha(W^M(x))^2 * \mu_Y \right)_T \right\} \right] = \int_0^T \int_\mathbb{R} \left( e^{\alpha(W^M_t)^2} - 1 \right) \nu(dx)dt < \infty, \]
see e.g. He et al. (1992), Lemma 14.39.1. Inequality (3.31) follows since \( \sigma^M \) is uniformly bounded.

We have that \( \int \frac{\hat{\phi}}{S_{t-}} dS \) is a local \( Q \)-martingale. It will be a true \( Q \)-martingale by Lemma 2.12 if we can show that, for some \( \beta > 0 \),
\[ E \left[ \exp \left\{ \beta \int_0^T \frac{\hat{\phi}_t^2}{S_{t-}^2} d[S]_t \right\} \right] < \infty. \]

We denote \( k = \sup_{t \in [0, T]} \| \hat{\phi} \|_\infty \). Let us take \( \beta = \frac{1}{k^2} \). By the Cauchy-Schwarz inequality and (3.30), (3.31) we get
\[ E \left[ \exp \left\{ \beta \int_0^T \frac{\hat{\phi}_t^2}{S_{t-}^2} d[S]_t \right\} \right] \leq E \left[\right. \left. \int_0^T (\sigma^M_t)^2 dt + \left( (W^M(x))^2 * \mu_Y \right)_T \right] < \infty. \]

\[ \textbf{4 Computing the MEMM in Special Cases} \]

\[ \textbf{4.1 The Deterministic Volatility Case} \]

The purpose of this section is to show how we can recover in our setup some well-known results. We consider an asset process
\[ dS_t \quad \frac{S_t}{S_{t-}} = \eta^M(t, V_{t-}) dt + \sigma^M(t, V_{t-}) dY^c_t + d\left( W^M(\cdot, V_{t-}, x) * (\mu_Y - \nu_Y) \right)_t \]
\[ dV_t = \eta^V(t, V_{t-}) dt, \]
fulfilling the Assumptions 3.1. In this special case it is easy to see that we can drop the assumption that \( \nu(R) < \infty \).

**Corollary 4.1** Let the function \( \hat{\phi} : [0, T] \to \mathbb{R} \) fulfill for any \( t \in [0, T] \)
\[ \eta_t^M + (\sigma_t^M)^2 \hat{\phi}_t + \int_\mathbb{R} W^M_t(x) \left( \exp \{ \hat{\phi}_t W^M_t(x) \} - 1 \right) \nu(dx) = 0. \quad (4.1) \]
Then the MEMM \( Q^* \) is given by
\[ \frac{dQ^*}{dP} = \exp \left\{ c + \int_0^T \frac{\hat{\phi}_t}{S_{t-}} dS_t \right\}. \]
(with normalizing constant \( c \)). Its density process can be written as
\[ Z_t = \frac{dQ^*}{dP} \bigg|_{\mathcal{F}_t} = \mathcal{E} \left( \int \hat{\phi} \sigma^M dY^c + \left( \exp \{ \hat{\phi} W^M(x) \} - 1 \right) (\mu_Y - \nu_Y) \right)_t. \]
Proof: In the deterministic case we have that $\Delta^u = 0$ since $W^V = 0$, hence we get $W^L$ immediately from (3.3) as

$$W^L(x) = \widehat{\lambda} W^M(x) - 1 + \exp\{\widehat{\phi} W^M(x)\}.$$  

Equation (4.1) follows then from equation (3.11) and the definition of $\widehat{\lambda}$. ■

Remark 4.2 Equation (4.1) corresponds to a condition well-known in the literature (e.g. condition (C) in Fujiwara and Miyahara (2003), or condition (4.4) in Theorem B in Esche and Schweizer (2005)).

4.2 The Orthogonal Volatility Case

Let us consider the asset process

$$\frac{dS_t}{S_{t-}} = \eta^M(t, V_{t-}) dt + \sigma^M(t, V_{t-}) dY^c_t,$$

$$dV_t = \eta^V(t, V_{t-}) dt + d\left(W^V(\cdot, V_{t-}, x) * \mu_Y\right)_t,$$

fulfilling the Assumptions 3.1 with

$$\widetilde{\lambda}_t = \frac{\eta^M_t}{(\sigma^M_t)^2}$$

and $E$ being compact such that $\sigma^M$ is uniformly bounded. Then we get the following result:

Corollary 4.3 The optimal strategy is

$$\widehat{\phi}_t = -\widetilde{\lambda}_t,$$  \hspace{1cm} (4.2)

and the density process of the MEMM is given via

$$W^L(t, V_{t-}, x) = \frac{v(t, V_{t-} + W^V_t(x))}{v(t, V_{t-})} - 1,$$

$$\sigma^L(t, V_{t-}) = 0,$$

where $v$ is the classical solution of the partial differential equation

$$\frac{\partial}{\partial t} v(t, y) + \eta^V_t \frac{\partial}{\partial y} v(t, y) - \frac{1}{2} \lambda_t^2 (\sigma^M_t)^2 v(t, y)$$

$$+ \int_{\mathbb{R}} \left( v(t, y + W^V_t(x)) - v(t, y) \right) \nu(dx) = 0,$$

$$v(T, y) = 1.$$  \hspace{1cm} (4.3)
Proof: (4.2) as well as \( \sigma^L = 0 \) is a direct consequence of \( W^M(x) = 0 \) and equation (3.11). Further, (3.12) leads to

\[
W^L(t, V_{t-}, x) = \exp\{u(t, V_{t-} + W^V_t(x)) - u(t, V_{t-})\} - 1. \tag{4.5}
\]

We know from Theorem 3.9 that

\[
\frac{\partial}{\partial t} u(t, y) + \eta^V \frac{\partial}{\partial y} u(t, y) - \frac{1}{2} \lambda_t (\sigma^M_i)^2 + \int W^L(t, y, x) \nu(dx) = 0
\]

\[
u(T, y) = 0.
\]

has a classical solution \( \tilde{u} \), from which we can determine the MEMM. By using the transformation \( v(t, y) = \exp u(t, y) \) we get the linear boundary problem (4.3), (4.4).

1. The optimal strategy in this specific case had already been identified by Grandits and Rheinländer (2002) by a conditioning argument. However, while the density of the MEMM at a fixed time \( T \) has a very simple form, the corresponding density process turns out to be of more complicated structure. Becherer (2001) determines the density process in a model where the volatility process switches between a finite number of states.

2. The transformation \( v(t, y) = \exp u(t, y) \) is very useful here since it linearizes the partial differential equation to (4.3). However, this does not apply to the general case when the jump process directly influences the asset process. As can be seen already in the deterministic volatility case, the exponential element cannot be linearized in this way.

3. Benth and Meyer-Brandis (2004) determined the MEMM for the specific case of a simplified BN-S model where no jumps occur in the price process. Their results generalize our results in the sense that they allow \( \gamma^d \) to be a finite variation process instead of only being a compound compensated Poisson process. Since in their approach the Ornstein-Uhlenbeck process \( \sigma^2 \) needs not to be bounded, one sees that our result might be further generalized. In the following section, we will treat the unbounded case in the BN-S framework, including the model of Benth and Meyer-Brandis as a special case but also allowing for jumps in the asset process.

4. It follows from (3.29) that the measure \( \nu^Q_Y \), where \( Q \) is the MEMM, is given as \( \nu^Q_Y = (W^L(x) + 1) \ast \nu_Y \). Since \( W^L \) is specified by (4.5), in general it is a stochastic process and in that case \( Y \) cannot be an additive process under \( Q \). We conclude that the MEMM is not in general contained in the class of structure preserving martingale measures as considered in Nicolato and Venardos (2003).
4.3 The Barndorff-Nielsen Shephard Model with Jumps

In Barndorff-Nielsen and Shephard (2001), the price process of a stock \( S = (S_t)_{t \in [0,T]} \) is defined by the exponential \( \exp \{ X_t \} \) with \( X = (X_t) \) satisfying

\[
\begin{align*}
    dX_t &= (\mu + \beta \sigma_t^2)dt + \sigma_t dY_t^c + d\left( \rho x \ast \tilde{\mu}_Y \right)_t, \\
    d\sigma_t^2 &= -\lambda \sigma_t^2 dt + d\left( x \ast \tilde{\mu}_Y \right)_t,
\end{align*}
\]

where the parameters \( \mu, \beta, \rho, \lambda \) are real constants with \( \lambda > 0 \) and \( \rho \leq 0 \) and \( \tilde{\mu}_Y \) has compensator \( \tilde{\nu}_Y := \lambda \nu_Y \). In addition, \( Y^d \) is assumed to be a subordinator, i.e. with positive increments only. It can be easily shown that the process \( S \) may then be written as

\[
\frac{dS_t}{S_t} = \left( \mu + \int (e^{\rho x} - 1)\tilde{\nu}(dx) + \sigma_t^2 (\beta + \frac{1}{2}) \right) dt + \sigma_t dY_t^c + d\left( (e^{\rho x} - 1) \ast (\tilde{\mu}_Y - \tilde{\nu}_Y) \right)_t.
\]

The process \( \sigma_t^2 \) is an Ornstein-Uhlenbeck process reverting towards zero and having positive jumps given by the subordinator. An explicit representation of it is given by

\[
\sigma_t^2 = \sigma_0^2 \exp \{ -\lambda t \} + \int_0^t \exp \{ -\lambda (t - u) \} dY_{\lambda u}.
\]

We apply the results of Section 3 and refer for one technical step (regarding the unboundedness of \( \sigma^M \)) to Steiger (2005). One has to pay attention to the fact that we work in this specific example with the Lévy process \( Y = Y^c + \tilde{Y}^d \), whereas \( \tilde{Y}^d = Y_{\lambda}^d \).

**Corollary 4.4** We assume \( \nu(\mathbb{R}) < \infty \) with \( \text{supp}(\nu) = \mathbb{R}_+ \). Let us assume \( \mu + \int (e^{\rho x} - 1)\tilde{\nu}(dx) > 0 \) and \( \beta + \frac{1}{2} > 0 \). We denote

\[
\tilde{\lambda}_t = \tilde{\lambda}_t(y) := \frac{\mu + \int (e^{\rho x} - 1)\tilde{\nu}(dx) + ye^{-\lambda t}(\beta + \frac{1}{2})}{ye^{-\lambda t} + \int (e^{\rho x} - 1)^2 \tilde{\nu}(dx)}.
\]

Let \( \sigma_0^2 > 0 \) be fixed and assume

\[
\int \left( e^{\frac{(\beta + \frac{1}{2})^2 x}{\lambda}} - 1 \right) \tilde{\nu}(dx) < \infty. \tag{4.6}
\]

The MEMM in case of the BN-S model is determined as follows:

Let us denote

\[
g^y(t, u_t) = \frac{1}{2} \left( \sigma_t^2 - \tilde{\lambda}_t e^{-\frac{1}{2} M \sqrt{y}} \right)^2 + \tilde{\phi}_t \tilde{\lambda}_t e^{-\lambda y}
\]

\[
+ \int \left[ W_t^L(y, x) - (\tilde{\phi}_t + \tilde{\lambda}_t) (e^{\rho x} - 1) + \tilde{\phi}_t \tilde{\lambda}_t (e^{\rho x} - 1)^2 \right] \tilde{\nu}(dx)
\]

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where $W_t^L(y, x)$ is the solution to
\[
\exp \left\{ \Delta_{t,y}^u(x) - \left[ \tilde{\lambda}_t + \frac{\int (e^{\rho x} - 1) W_t^L(y, x) \tilde{\nu}(dz)}{ye^{-\lambda t}} \right] (e^{\rho x} - 1) \right\} \\
= 1 - \tilde{\lambda}_t (e^{\rho x} - 1) + W_t^L(y, x),
\]
and
\[
\Delta_{t,y}^u(x) = u(t, y + e^{\lambda_t} x) - u(t, y), \\
\phi_t = -\frac{\int (e^{\rho x} - 1) W_t^L(y, x) \tilde{\nu}(dx)}{ye^{-\lambda t}} - \tilde{\lambda}_t, \\
\sigma_t^L = -\frac{\int (e^{\rho x} - 1) W_t^L(y, x) \tilde{\nu}(dx)}{ye^{-\frac{1}{2}\lambda t}}.
\]

Then the classical solution $\tilde{u}$ of the Integro-PDE
\[
\frac{\partial}{\partial t} u(t, y) + g^y(t, u_t) = 0, \\
u(T, y) = 0 \ \forall \ y \in E := [\sigma_0^2, \infty)
\]
determines the MEMM via $W^L$ and $\sigma^L$:
\[
Z_t = \frac{dQ^x}{dP} \bigg|_{\mathcal{F}_t} = \mathcal{E} \left( \int \left( -\tilde{\lambda}_s \sigma_s + \sigma_s^L \right) dY_s^c + (-\tilde{\lambda}(e^{\rho x} - 1) + W^L(x)) (\tilde{\mu}_Y - \tilde{\nu}_Y) \right)_t.
\]

**Proof:** The PDE (4.7) with boundary condition (4.8) follows from the results in Section 3 by making the transformation
\[
\tilde{\sigma}_t^2 = e^{\lambda t} \sigma_t^2
\]
such that we get the dynamics
\[
\frac{dS_t}{S_t} = \left( \mu + \lambda \int (e^{\rho x} - 1) \nu(dx) + e^{-\lambda \tilde{\sigma}_t^2} (\beta + \frac{1}{2}) \right) dt \\
+ e^{-\frac{1}{2} \lambda \tilde{\sigma}_t^2} dY_t^c + d \left( (e^{\rho x} - 1) * (\tilde{\mu}_Y - \tilde{\nu}_Y) \right)_t
\]
with
\[
d\tilde{\sigma}_t^2 = \left( e^{\lambda t} * \tilde{\mu}_Y \right)_t.
\]
As $\tilde{\sigma}$ is in general not bounded, we may not directly apply Theorem 3.8 for proving that there exists a classical solution $\tilde{u}$ to the problem (4.7)-(4.8). Resolving this issue has turned out to be surprisingly technical and has been carried out in Steiger (2005). It results that we still have that $\Delta^{\tilde{u}}$ is bounded from above on $[0, T] \times E$. Hence, using Lemma 3.3.1, we directly get that $W^L$ and therefore also $\sigma^L$ and $\tilde{\phi}$ are uniformly bounded.

Based on this result, we show now that the three conditions of Remark 2.8 are fulfilled.
1. $Q^*$ is an equivalent martingale measure: Here we proceed similarly as in the proof of Theorem 3.9, and concentrate only on the verification of condition (??). Let us consider
\[ U = \frac{1}{2} \int \tilde{\phi}_t^2 \sigma_t^{-2} ds + W^U(x) \ast \tilde{\lambda}_Y \]
with
\[ W^U(x) := \left( W^L(x) + 1 - \tilde{\lambda}(e^{\rho x} - 1) \right) \log \left( W^L(x) + 1 - \tilde{\lambda}(e^{\rho x} - 1) \right) + \tilde{\lambda}(e^{\rho x} - 1) - W^L(x). \]
Since $e^{\rho x} - 1$ and $W^L$ are uniformly bounded, $U$ has locally integrable variation, and we get
\[ E \left[ \exp \left\{ 2\lambda \int_0^T W^U_t(x) \nu(dx) dt \right\} \right] < \infty. \]
Hence, (2.7) is fulfilled by the Cauchy-Schwarz inequality, if we can show that
\[ E \left[ \exp \left\{ \int_0^T \tilde{\phi}_t^2 \sigma_t^{-2} dt \right\} \right] < \infty. \]
By definition, we have
\[ \tilde{\phi}_t = -\tilde{\lambda}_t - \frac{\int (e^{\rho x} - 1) W^L_t(x) \nu(dx)}{\sigma_t^{-2}}. \]
Since $\tilde{\lambda}$ is positive and $W^L$ is bounded, $\tilde{\phi}_t$ is negative for $\sigma_{t-}$ big enough. Let us introduce $\bar{\sigma}$ such that for all $t \in [0, T]$,
\[ \tilde{\phi}_t < 0 \text{ for all } \sigma_{t-} > \bar{\sigma}. \]
And on the other hand, since $W^L_t(x) \geq -1 + \tilde{\lambda}_t(e^{\rho x} - 1)$, $\tilde{\phi}_t$ is bounded from below with
\[ \tilde{\phi}_t \geq -\tilde{\lambda}_t - \frac{\int (e^{\rho x} - 1)(-1 + \tilde{\lambda}_t(e^{\rho x} - 1)) \nu(dx)}{\sigma_t^{-2}} = -\left( \beta + \frac{1}{2} \right) - \frac{\mu}{\sigma_t^{-2}} \]
because of
\[ \tilde{\lambda}_t = \frac{\mu + \int (e^{\rho x} - 1) \nu(dx) + \sigma_t^{-2} (\beta + \frac{1}{2})}{\sigma_t^{-2} + \int (e^{\rho x} - 1)^2 \nu(dx)}. \]
Let us now analyze
\[ E \left[ \exp \left\{ \int_0^T \tilde{\phi}_t \sigma_t^{-2} dt \right\} \right] = E \left[ \exp \left\{ \int_0^T 1_{\{\sigma_{t-} \leq \bar{\sigma}\}} \tilde{\phi}_t \sigma_t^{-2} dt \right\} \exp \left\{ \int_0^T 1_{\{\sigma_{t-} > \bar{\sigma}\}} \tilde{\phi}_t \sigma_t^{-2} dt \right\} \right]. \]
We have that
\[
\exp \left\{ \int_0^T 1_{\{\sigma_t \leq \pi\}} \phi_t^2 \sigma_t^2 \, dt \right\}
\]
is uniformly bounded. Moreover,
\[
E \left[ \exp \left\{ \int_0^T 1_{\{\sigma_t > \pi\}} \phi_t^2 \sigma_t^2 \, dt \right\} \right]
\]
is finite since
\[
0 \geq \phi_t \geq -\left( \beta + \frac{1}{2} \right) - \frac{\mu}{\sigma_t^2}
\]
on the set \( \{ \sigma_t > \pi\} \) and condition (4.6), which, according to Benth et al. (2003), Lemma 3.1, ensures that
\[
E \left[ \exp \left\{ (\beta + \frac{1}{2})^2 \int_0^T \sigma_t^2 \, dt \right\} \right] < \infty.
\]

2. \( I(Q^*, P) < \infty \): We have to show that for \( \hat{\nu}_Y^Q = \left( W_L(x) + 1 - \hat{\lambda}(e^{px} - 1) \right) \ast \hat{\nu}_Y \)
\[(e^{px} - 1) \ast \left( \hat{\mu}_Y - \hat{\nu}_Y^Q \right) \]
as well as
\[
\int \sigma dY^c + \int (\hat{\lambda}\sigma - \sigma^L) \sigma dt
\]
are true \( Q^* \)-martingales, i.e. their quadratic variations are \( Q^* \)-integrable. We only have to consider the second term because of the boundedness of \( W^L \). Let us consider
\[
E_{Q^*} \left[ \left\langle \int \sigma dY^c \right\rangle_T \right] = E_{Q^*} \left[ \int_0^T \sigma_t^2 \, dt \right].
\]
It is well-known that we may write
\[
\int_0^T \sigma_t^2 \, dt = \lambda^{-1}(1 - e^{-\lambda T}) \sigma_0^2 + \left( \lambda^{-1}(1 - e^{-\lambda(T^{-1})}x \ast \hat{\mu}_Y \right)_T.
\]
Hence, \( E_{Q^*} \left[ \int_0^T \sigma_t^2 \, dt \right] \) is finite if \( E_{Q^*} \left[ \left( x \ast \hat{\nu}_Y^Q \right)_T \right] \) is finite, which is, since \( W^L \) is bounded, equivalent to showing \( \int x \nu(dx) < \infty \). However, this is fulfilled by the condition (4.6).

3. \( \int \hat{\phi}_{S_t} \, dS \) is a true \( Q \)-martingale for all \( Q \in \mathcal{M}^e \) with finite relative entropy: By Lemma 2.12, \( \int \hat{\phi}_{S_t} \, dS \) is a true \( Q \)-martingale if we can show that for some \( \gamma > 0 \),
\[
E \left[ \exp \left\{ \gamma \int \hat{\phi}_{S_t}^2 \, d[S]_t \right\} \right]
\]
\[
= E \left[ \exp \left\{ \gamma \int_0^T \hat{\phi}_t^2 \sigma_t^2 \, dt + \left( \gamma \hat{\phi}_t^2 (e^{px} - 1) \ast \hat{\mu}_Y \right)_T \right\} \right]
\]
\[
< \infty.
\]
We have
\[ E \left[ \exp \left\{ 2 \gamma \phi_t^2 (e^{\rho x} - 1)^2 * \check{\nu}_Y \right\} \right] < \infty, \]
and, for \( \gamma < \frac{\beta+\frac{1}{2}}{2 \max \phi_t} \), we get that
\[ E \left[ \exp \left\{ 2 \gamma \int_0^T \phi_t^2 \sigma_t^2 \, dt \right\} \right] < \infty. \]

Therefore, an application of the Cauchy-Schwarz inequality yields
\[ E \left[ \exp \left\{ \gamma \int_0^T \frac{\phi_t^2}{S_t^2} \, d[S]_t \right\} \right] < \infty. \]

References


[23] Steiger, G. (2005), Technical Note, Preprint, ETH Zurich


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\section*{A Appendix}

\textbf{Proof of Lemma 3.3:} We consider the equation
\[ \Phi = \int f(x) \exp \{ k(x) - \beta f(x) \Phi \} \nu(dx), \]
and will show that there exists a unique value \( \Phi_k \in \mathbb{R} \) which fulfills this equation. For this purpose, let us define
\[ H(z) = z - \int f(x) \exp \{ k(x) - \beta f(x)z \} \nu(dx). \]

Since
\[ \lim_{z \to \infty} -f(x) \exp \{ -\beta f(x) z \} = \begin{cases} 0 & f(x) \geq 0 \\ \infty & f(x) < 0 \end{cases}, \]
we get \( \lim_{z \to -\infty} H(z) = \infty \) and, for symmetry reasons, \( \lim_{z \to -\infty} H(z) = -\infty \). Furthermore, \( H \) is continuously differentiable with
\[ \frac{\partial}{\partial z} H(z) = 1 + \int \beta f^2(x) \exp \{ k(x) - \beta f(x) z \} \nu(dx) > 0. \]
Therefore, there exists a unique \( \Phi_k \in \mathbb{R} \) such that \( H(\Phi_k) = 0 \). We can moreover show that
\[ |\Phi_k| \leq \max_{x \in \text{supp}(\nu)} \left\{ \exp k(x) \right\} \int |f(x)| \nu(dx). \]

Let us assume that \( \Phi_k \geq 0 \). Then we get
\[
\begin{align*}
\Phi_k &= \int f(x) \exp \{ k(x) - \beta f(x) \Phi_k \} \nu(dx) \\
&\leq \int_{\{f(x) > 0\}} f(x) \exp \{ k(x) - \beta f(x) \Phi_k \} \nu(dx) \\
&\leq \int_{\{f(x) > 0\}} f(x) \exp \{ k(x) \} \nu(dx) \\
&\leq \max_{x \in \text{supp}(\nu)} \left\{ \exp k(x) \right\} \int_{\{f(x) > 0\}} f(x) \nu(dx) \\
&\leq \max_{x \in \text{supp}(\nu)} \left\{ \exp k(x) \right\} \int |f(x)| \nu(dx).
\end{align*}
\]

The lower bound can be shown in exactly the same way.

Let us now define the bounded function
\[ \varphi_k(x) := \exp \left\{ k(x) - \beta f(x) \Phi_k \right\}. \]
As we have
\[ \int f(x)\varphi_k(x)\nu(dx) = \int f(x)\exp\left\{ k(x) - \beta f(x)\Phi_k \right\}\nu(dx) = \Phi_k, \]
we get
\[ \varphi_k(x) = \exp\left\{ k(x) - f(x)\int f(z)\varphi_k(z)\nu(dz) \right\}; \]
and therefore, we conclude that \( \varphi := \varphi_k \) is well-defined and bounded. \( \blacksquare \)

**Proof of Lemma 3.5:** Since \( W^L \) is bounded on \( C^Q_b(E) \), we only have to show local Lipschitz-continuity of \( W^L \), i.e. we have to show that for any \( c > 0 \), there exists a constant \( L_c \) such that
\[ \| W^L(v_1) - W^L(v_2) \|_{\infty} \leq L_c\| v_1 - v_2 \|_{\infty} \]
for all \( v_1, v_2 \in C^Q_b(E) \) with \( \| v_1 - v_2 \|_{\infty} \leq \frac{c}{2} \). For that purpose, consider \( v_0 + rh \) where \( v_0 \in C^Q_b(E) \) and \( h \in C^Q_b(E) \) with \( \| h \|_{\infty} = \frac{c}{2} \).

Let \( k \in l^\infty(\text{supp}(\nu)) \) and define
\[ \varphi_k(x) := \exp\left\{ k(x) - \frac{W^M(x)}{(\sigma^M)^2} \int W^M(z)\varphi_k(z)\nu(dz) \right\}, \]
\[ \Phi_k := \int W^M(x)\varphi_k(x)\nu(dx). \]
By equation (3.12), we may write
\[ \varphi_{k^*(r)} = W^L(v_0 + rh) - \tilde{\lambda}W^M + 1 \]
for
\[ k^*(r) = \Delta^v_0 + r\Delta^h - W^M \tilde{\eta} \]
and
\[ \tilde{\eta} := \frac{\lambda}{2} \left[ 1 + \frac{\int [W^M(z)]^2\nu(dz)}{(\sigma^M)^2} \right] - \frac{\int W^M(z)\nu(dz)}{(\sigma^M)^2}. \]
The goal is to show that there is a constant \( C_1 \) such that we have for all \( r \in [0, 1] \)
\[ \| \varphi_{k^*(r)} - \varphi_{k^*(0)} \|_{\infty} \leq C_1 \| rh \|_{\infty} = \frac{C_1rc}{2}. \] (A.1)
Let us therefore analyze
\[ | \left( \varphi_{k^*(r)} - \varphi_{k^*(0)} \right)(x) | = \exp\left\{ \Delta^v_0(x) - \tilde{\eta}W^M(x) \right\} \times \exp\left\{ r\Delta^h(x) - \Phi_{k^*(r)} W^M(x) \left( \frac{1}{(\sigma^M)^2} \right) \right\} - \exp\left\{ -\Phi_{k^*(0)} W^M(x) \left( \frac{1}{(\sigma^M)^2} \right) \right\} \]
\[ \times \left( \frac{1}{(\sigma^M)^2} \right) - 1 \right| \] (A.2)
Since \( v_0 \) is uniformly bounded by \( Q \), the first term on the RHS is uniformly bounded for all \( x \in \text{supp}(\nu) \). The second term (to be labelled \( f_x(r) \)) needs further investigation. For this purpose, let us state the following property of \( \Phi_k \):

**Claim:** If we have two functions \( k_1, k_2 \in l^\infty(\text{supp}(\nu)) \) with

\[
\begin{aligned}
&\quad \{ k_1(x) \leq k_2(x) \quad \forall \ x \in \text{supp}(\nu) \text{ s.t. } W^M(x) < 0 \\
&\quad k_1(x) \geq k_2(x) \quad \forall \ x \in \text{supp}(\nu) \text{ s.t. } W^M(x) > 0 ,
\end{aligned}
\]

then we get \( \Phi_{k_1} \geq \Phi_{k_2} \).

**Proof:** Let us assume that \( \Phi_{k_1} < \Phi_{k_2} \). Then we get for any \( x \in \text{supp}(\nu) \) that

\[
\frac{\varphi_{k_2}(x)}{\varphi_{k_1}(x)} = \exp \left\{ k_2(x) - k_1(x) - \frac{W^M(x)}{(\sigma M)^2} (\Phi_{k_2} - \Phi_{k_1}) \right\}
\]

\[
\left\{ \begin{array}{ll}
> 1 & \forall \ x \in \text{supp}(\nu) \text{ s.t. } W^M(x) < 0 \\
< 1 & \forall \ x \in \text{supp}(\nu) \text{ s.t. } W^M(x) > 0 
\end{array} \right.
\]

However, this leads to a contradiction since then

\[
\Phi_{k_2} - \Phi_{k_1} = \int W^M(x) (\varphi_{k_2}(x) - \varphi_{k_1}(x)) \nu(dx) \geq 0.
\]

Therefore, we must have \( \Phi_{k_1} \geq \Phi_{k_2} \). \hfill \Box

Let us now fix \( x_0 \in \text{supp}(\nu) \) and analyze the term

\[
f_{x_0}(r) := \exp \left\{ r \Delta^h(x_0) - \left( \Phi_{k^*(r)} - \Phi_{k^*(0)} \right) \frac{W^M(x_0)}{(\sigma M)^2} \right\} - 1.
\]

Obviously, we have \( f_{x_0}(0) = 0 \). Let us now assess the upper and lower bounds of \( f_{x_0} \) for \( r \in [0, 1] \). For this purpose, we introduce

\[
k^+(r, x) := \Delta^v_0(x) - W^M(x) \tilde{\eta} + r c \left( 1_{\{ W^M(x) > 0 \}} - 1_{\{ W^M(x) < 0 \}} \right)(x)
\]

\[
k^-(r, x) := \Delta^v_0(x) - W^M(x) \tilde{\eta} - r c \left( 1_{\{ W^M(x) > 0 \}} - 1_{\{ W^M(x) < 0 \}} \right)(x).
\]

We will use in the following the notation

\[
1^+(x) := 1_{\{ W^M(x) > 0 \}} - 1_{\{ W^M(x) < 0 \}}.
\]

It results from the claim that

\[
\Phi_{k^-(r)} - \Phi_{k^-(0)} \leq \Phi_{k^*(r)} - \Phi_{k^*(0)} \leq \Phi_{k^+(r)} - \Phi_{k^+(0)} . \tag{A.3}
\]

Let us now consider in detail the upper bound,

\[
\Phi_{k^+(r)} - \Phi_{k^+(0)} = \int_0^r \frac{\partial}{\partial r} \Phi_{k^+(s)} ds.
\]
Here the existence of the derivative can be guaranteed by an application of the Implicit Function Theorem for Banach spaces (see e.g. Zeidler (1986), p. 150) to the equation

\[
\Phi_{k^+(r)} = \int W^M(x) \exp \left\{ k^+(r, x) - \frac{W^M(x)}{(\sigma M)^2} \Phi_{k^+(r)} \right\} \nu(dx).
\]

We get

\[
\frac{\partial}{\partial r} \Phi_{k^+(r)} = \int W^M(x) \left[ c 1^+(x) - \frac{W^M(x)}{(\sigma M)^2} \frac{\partial}{\partial r} \Phi_{k^+(r)} \right] \times \exp \left\{ k^+(r, x) - \frac{W^M(x)}{(\sigma M)^2} \Phi_{k^+(r)} \right\} \nu(dx),
\]

such that we may write (recall that \( \varphi_{k^+(r)}(x) = \exp \left\{ k^+(r, x) - \frac{W^M(x)}{(\sigma M)^2} \Phi_{k^+(r)} \right\} \))

\[
\frac{\partial}{\partial r} \Phi_{k^+(r)} = c \left( \frac{\int W^M(x) 1^+(x) \varphi_{k^+(r)}(x) \nu(dx)}{1 + \int \frac{(W^M(x))^2}{(\sigma M)^2} \varphi_{k^+(r)}(x) \nu(dx)} \right).
\]

\[
< c \int |W^M(x)| \varphi_{k^+(r)}(x) \nu(dx).
\]

Since \( k^+(s) \in L^\infty(supp(\nu)) \), it follows from the definition of \( \varphi_{k^+(s)} \) and Lemma 3.3 that \( \varphi_{k^+(s)} \) is uniformly bounded by some constant \( K^* \) for any \( s \in [0, r] \). Therefore, we get

\[
\Phi_{k^+(r)} - \Phi_{k^+(0)} < crK^* \int |W^M(x)| \nu(dx).
\]

Applying the same steps to the lower bound, we get

\[
\Phi_{k^-(r)} - \Phi_{k^-(0)} > -crK^* \int |W^M(x)| \nu(dx).
\]

Taking into account the inequalities of (A.3), we get the following bounds:

\[
\exp\{rc\tilde{K}\} - 1 \geq \sup_{x \in supp(\nu)} f_x(r) \geq f_{x_0}(r) \geq \inf_{x \in supp(\nu)} f_x(r) \geq \exp\{-rc\tilde{K}\} - 1
\]

with

\[
\tilde{K} := 1 + K^* \frac{\max_{x \in supp(\nu)} |W^M(x)|}{(\sigma M)^2} \int |W^M(x)| \nu(dx).
\]

Therefore, we get for \( r \in [0, 1] \) that

\[
\sup_{x \in supp(\nu)} |f_x(r)| \leq (\exp\{c\tilde{K}\} - 1)r
\]

and hence, via (A.1), the Lipschitz-continuity of \( W^L \) is shown. □
Proof of Theorem 3.6: Let us fix some \( u \in C_b([0, T] \times E) \) and consider the PDE

\[
\frac{\partial}{\partial t} w(t, z) + b(t, z) \frac{\partial}{\partial z} w(t, z) + g^z(t, u(t)) = 0, \tag{A.4}
\]

\[ w(T, z) = h(z) \quad \forall \ z \in E. \tag{A.5} \]

It is straightforward to see that

\[
 w(t, z) = h(Z_{T-t}^z) + \int_t^T g^{Z_{s-t}^z}(s, u_s) ds.
\]

solves the boundary problem in the weak sense. Let us introduce the operator \( F : C_b([0, T] \times E) \rightarrow C_b([0, T] \times E) \) as

\[
(Fu)(t, z) = h(Z_{T-t}^z) + \int_t^T g^{Z_{s-t}^z}(s, u_s) ds.
\]

We have to prove that \( F \) is a contraction on the space \( C_b([0, T] \times E) \). Let us, for some \( \alpha \in \mathbb{R}_+ \), consider the norm

\[
\|u\|_{\alpha} := \sup_{(t, z) \in [0, T] \times E} e^{-\alpha(T-t)} |u(t, z)|,
\]

which is equivalent to the supremum-norm \( \|u\|_{\infty} \). Due to condition \( a-2 \), we obtain for \( u_1, u_2 \in C_b([0, T] \times E) \)

\[
e^{-\alpha(T-t)} |(Fu_1)(t, z) - (Fu_2)(t, z)|
\]

\[= \frac{1}{e^{\alpha(T-t)}} \left| \int_t^T \left( g^{Z_{s-t}^z}(s, u_{1,s}) - g^{Z_{s-t}^z}(s, u_{2,s}) \right) ds \right|
\]

\[\leq \frac{1}{e^{\alpha(T-t)}} \int_t^T \left| g^{Z_{s-t}^z}(s, u_{1,s}) - g^{Z_{s-t}^z}(s, u_{2,s}) \right| e^{-\alpha(T-s)} e^{\alpha(T-s)} ds
\]

\[\leq \frac{1}{e^{\alpha(T-t)}} L \|u_1 - u_2\|_{\alpha} \int_t^T e^{\alpha(T-s)} ds
\]

\[\leq \frac{L}{\alpha} \|u_1 - u_2\|_{\alpha}
\]

for all \( t \in [0, T] \) and \( z \in E \). Thus,

\[
\| (Fu_1)(t, z) - (Fu_2)(t, z) \|_{\alpha} \leq \frac{L}{\alpha} \|u_1 - u_2\|_{\alpha},
\]

and \( F \) is a contraction on the normed space \((C_b([0, T] \times E), \| \cdot \|_{\alpha})\) with \( \alpha > L \). Therefore, there exists a unique fixed point \( u \in C_b([0, T] \times E) \), which satisfies the PDE (3.15)-(3.16) in the weak sense.  

\[ \blacksquare \]
Proof of Theorem 3.7: Let us analyze the operator $\hat{G} : C_b([0, T] \times E) \to C_b([0, T] \times E)$, defined as
\[
(\hat{G}v)(t, z) = \frac{\partial}{\partial z}h(Z^t_z) + \int_t^T \left( \frac{\partial}{\partial z}Z^s_z \right) g(Z^s_z, v_s) ds.
\]

Let us first discuss $\frac{\partial}{\partial z}Z^t_z$, which is well-defined by Protter (2004), Theorem V.39. Differentiating (3.14), we get
\[
\frac{\partial}{\partial z}Z^t_z = 1 + \int_t^s \left( \frac{\partial}{\partial z}Z^u_u \right) \frac{\partial}{\partial Z^u_u}b(u, Z^u_u) du.
\]

By Gronwall’s Lemma, we directly can conclude that $\frac{\partial}{\partial z}Z^t_z$ is uniformly bounded, the bound denoted by $L_Z$. In analogy, let us denote $L_h := \|h'\|_\infty$.

Let us now discuss, for $v \in C_b([0, T] \times E)$,
\[
e^{-\beta(T-t)}||(\hat{G}v)(t, z)|| \leq e^{-\beta(T-t)}\left(\left|\frac{\partial}{\partial z}Z^t_z\right| \|h'(Z^t_z)\| + \int_t^T \left|\frac{\partial}{\partial z}Z^s_z\right| \|g(Z^s_z, v_s)\| ds\right)
\]
\[
\leq e^{-\beta(T-t)}L_Z\left(L_h + \int_t^T (L\|v_s\|_\infty + K)e^{-\beta(T-t)}e^{\beta(T-t)} ds\right)
\]
\[
\leq L_Z L \beta \|v\|_\beta + L_Z KT + L_Z L_h.
\]

Hence, for $\beta = 2L_Z L$ and $N := \{v \in C_b([0, T] \times E) \mid \|v\|_\beta \leq 2L_Z(KT + L_h)\}$, $\hat{G}$ maps $N$ into $N$. Using the Arzela-Ascoli Theorem, one can show that $\hat{G}$ is a compact operator on $N$. By Schauder’s Fixed Point Theorem, we conclude that $\hat{G} : N \to N$ has at least one fixed point $\hat{v}$. Let us assume $u \in C_b([0, T] \times E)$ being differentiable in the space variable. Hence,
\[
\frac{\partial}{\partial z}(Fu)(t, z) = \frac{\partial}{\partial z}h(Z^t_z) + \int_t^T \left( \frac{\partial}{\partial z}Z^s_z \right) \frac{\partial}{\partial Z^s_z}g(Z^s_z, u_s) ds
\]
\[
= (\hat{G} \frac{\partial}{\partial z}u)(t, z).
\]

Let us now consider the primitive with respect to $z \in E$ of $\hat{v}$, denoted as $\hat{u}$. We may write
\[
\frac{\partial}{\partial z}(F\hat{u})(t, z) = (\hat{G}\hat{v})(t, z) = \hat{v}(t, z) = \frac{\partial}{\partial z}\hat{u}(t, z).
\]

We therefore get that the function $\hat{u}$ may be written as
\[
\hat{u}(t, z) = (F\hat{u})(t, z) + C(t)
\]

with function $C : [0, T] \to \mathbb{R}$. On the other hand, we know by Theorem 3.6 that there exists a unique fixed point of operator $F$ in $C_b([0, T] \times E)$. Hence, choosing $C \equiv 0$, we get that $\hat{u}$ is uniquely defined. We therefore have shown that there exists a unique classical solution to the boundary problem (3.15)-(3.16).