The dissipation rate coefficient of turbulence is not universal and depends on the internal stagnation point structure

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The energy dissipation rate coefficient of statistically stationary homogeneous isotropic turbulence depends on the external force sustaining the turbulence irrespective of Reynolds number. This non-universality is established by proving that the Taylor length is proportional to the mean distance between stagnation points and thereby relating the energy dissipation rate coefficient to the stagnation point structure of the turbulence which is shown to depend on the structure of the large eddies. Confirmation of these relations is obtained at moderate Reynolds numbers by a series of direct numerical simulations where the large-scale forcing is systematically varied.

I. THEORY

G. I. Taylor stated in 1935 [1] that the rate $\epsilon$ of turbulent kinetic energy dissipation (per unit mass) is determined by the r.m.s. velocity $u'$ and some spatial linear dimension $L$ defining the scale of the system, i.e.

$$\epsilon = C_\epsilon \frac{u'^3}{L}$$

(1)

where $C_\epsilon$ is a constant for high Reynolds number turbulence produced by geometrically similar boundaries. We follow the convention of the past half century [2], and adopt for $L$ the integral length of the longitudinal velocity correlation function. Since the early 1950s when G. K. Batchelor plotted experimental data of $C_\epsilon$ in his textbook [2], the following questions have remained open and at the centre of all strands of turbulence research [3–
(i) whether \( C_\varepsilon \) is or is not independent of the Reynolds number in the high Reynolds number limit, (ii) whether \( C_\varepsilon \) is or is not independent of the boundary/initial conditions and/or forces which generate the turbulence, and (iii) what determines the value of \( C_\varepsilon \). These questions are both of fundamental physical importance and engineering turbulence modelling relevance: indeed, \( C_\varepsilon \) is directly related to one of the model parameters of the \( K-\varepsilon \) model, and to the eddy viscosity in large-eddy simulations.

In the present article, by further developing the viewpoint proposed in Ref. [8] and carrying out a series of direct numerical simulations (DNS), we show that \( C_\varepsilon \) is determined by the distribution of stagnation points of the turbulent velocity field \( \mathbf{u}(\mathbf{x}, t) \). This result suggests that \( C_\varepsilon \) cannot be universal even in the high Reynolds number limit because the stagnation point distribution is determined by the external forcing or boundary conditions irrespective of the Reynolds number. It may be worth mentioning that the fluctuating velocity field \( \mathbf{u}(\mathbf{x}, t) \) is Galilean invariant because it is obtained by removing the mean flow from the actual fluid velocity field. Also, stagnation points where \( \mathbf{u} = (u_1, u_2, u_3) = 0 \) can be expected to be numerous because they are intersections of the instantaneous line \( u_1(\mathbf{x}, t) = u_2(\mathbf{x}, t) = 0 \) with the instantaneous surface \( u_3(\mathbf{x}, t) = 0 \).

As predicted theoretically [9–11] and confirmed experimentally [8, 12], the mean distance between zero-crossings of a component of the turbulent velocity \( \mathbf{u} \) is proportional to the Taylor length,

\[
\lambda = \frac{u'}{\sigma}
\]

where \( \sigma \) is the r.m.s. value of the longitudinal velocity derivative. It is important that this property of zero-crossings can be generalised (see Appendix A) to the stagnation points of \( \mathbf{u} \) which are more fluid dynamically meaningful than the zero-crossings. More precisely, as we show in Appendix A under relatively weak assumptions, the number density \( n_s \) per unit volume of the stagnation points of \( \mathbf{u} \) is related to \( \lambda \) by

\[
\lambda = B n_s^{-1/3}
\]

where \( B \) is a constant (unless the Reynolds number is too small or small-scale intermittency effects are taken into account in which case \( B \) may be a very weak function of the Reynolds number; see Appendix A).

On the other hand, it has been shown [13, 14] that the number density \( n_s^{(c)} \) of stagnation points in the coarse-grained velocity field \( \mathbf{u}^{(c)} \) at the cut-off length scale \( \ell_c \) obeys the following
Here, the inner cut-off \( \eta \) is proportional to the Kolmogorov length \( \eta_K \), i.e.

\[
\eta = A \eta_K = A (\nu^3/\epsilon)^{1/4},
\]

(\( A \) and \( \nu \) being a dimensionless constant and the kinematic viscosity of the fluid, respectively) and \( L_* \) is a reference length such that \( \eta \lesssim L_* \lesssim L \). In our previous studies, we set \( L_* = L \) for simplicity, but the introduction of the reference length \( L_* \) is essential in this paper’s arguments. In (4), \( C_s \) and \( D_s \) are constants, and \( D_s \) is related to the exponent \( p \) of the energy spectrum \( E(k) \sim k^{-p} \) (in the range \( \eta \ll k^{-1} \ll L \)) of the velocity field \( u \) by

\[
D_s = \frac{3(3-p)}{2}.
\]

A simple derivation of the fractal dimension (6) is given in Ref. [15]. Note that the inner cut-off \( \eta \) of scaling (4) may not actually be proportional to the Kolmogorov length as shown experimentally in Ref. [8] for zero-crossings where \( A \) in (5) turned out to be an increasing function of \( \log R_\lambda \). However, for simplicity, we do not take into account this small correction here and assume \( A \) to be independent of Reynolds number. We return to this point in the last section.

From \( \epsilon = 15\nu u'^2/\lambda^2 \) in conjunction with (3)—(5) we obtain

\[
\epsilon = \left[ 15 u'^2 A^{-2D_s/3} B^{-2} C_s 2/3 L_s^{-2+2D_s/3} \nu^{1-D_s/2} \right]^{1/(1-D_s/6)}.
\]

Then, using the Reynolds number independence of \( A \) and \( B \) as \( \nu \to 0 \) (or, more generally, assuming the Reynolds number independence of \( BA^{D_s/3} \) as \( \nu \to 0 \)), it follows that the energy dissipation \( \epsilon \) is independent of \( \nu \) in that limit provided that \( D_s = 2 \) which corresponds to

\[
p = \frac{5}{3}
\]

in (6). The Taylor relation (1) then follows with the coefficient

\[
C_\epsilon = \frac{15^{3/2} C_s L}{A^2 B^3 L_*}.
\]
Note that (8) corresponds to the well-known $-5/3$ power law spectrum [16] which we have derived here using the generalised Rice theorem (Appendix A). Since we have made the same assumption about $\epsilon$ as was made by Kolmogorov in the high Reynolds number limit [16], it may not be too surprising that we reach the same conclusion on the energy spectrum. However, it is not trivial that the dissipation coefficient $C_\epsilon$ is related to a constant $C_s$ which characterises the large-scale spatial distribution of stagnation points and therefore one basic aspect of large-scale flow topology. It may be worth noting that this relation (9) can also be derived without any assumptions on the Reynolds number dependencies of $A$ and $B$ if use is made of $D_s = 2$ on account of $p = 5/3$ and (6). In this case, both sides of (9) are Reynolds number dependent in principle, as is undoubtedly the case when the Reynolds number is not large enough. In Sec. III we provide DNS support for (9) at Reynolds numbers where both sides of (9) are Reynolds number dependent.

Note that the reference length $L_*$ in (4) is arbitrary as long as it lies in the scaling range $\eta \lesssim L_* \lesssim L$. However, our conclusions do not depend on the choice of $L_*$. Although the reference length $L_*$ appears on the right-hand side of (9), $C_\epsilon$ does not depend on $L_*$ because $C_s$ also depends on the choice of $L_*$. If we choose another reference length, say $\widetilde{L}_*$, the coefficient in (4) is $\widetilde{C}_s = \widetilde{L}_*^2 n_s^{(c)}(\ell_c = \widetilde{L}_*)$ which equals $C_s(\widetilde{L}_*/L_*)^{3-D_s}$. Then, setting $D_s = 2$ for (9) to hold, both choices of $L_*$ lead to the same dissipation coefficient $C_\epsilon$.

An important consequence of (9) is that the coefficient $C_\epsilon$ in the Taylor relation (1) may not be universal. This is because $C_s$ is the number of stagnation points in a cube of side $L_*$ in the coarse-grained velocity field at the reference scale $L_*$, i.e. $L_*^3 n_s^{(c)}(L_*)$, and because it depends on the large-scale structures. Note that $L_*$ can be as large as the integral length. This conclusion may seem to be in conflict with a recent report [5] based on the currently highest Reynolds number DNS available, but this is not so because these DNS were carried out with a common large-scale structure. The non-universality of $C_\epsilon$ has already been reported from comparisons of DNS with different external forcings (albeit with limited Reynolds numbers) [4] and from analyses of experimental data (see Ref. [3, 8] and references therein).

The purpose of what follows is to demonstrate the non-universality of $C_\epsilon$ and of $(C_sL)/(B^3L_*)$, but also the proportionality between them, by a systematic series of DNS with different large-scale structures.
II. DIRECT NUMERICAL SIMULATIONS

Statistically homogeneous isotropic turbulence is simulated by numerically integrating the Navier-Stokes equations for an incompressible fluid in a periodic box (with period $2\pi$) using the fourth-order Runge-Kutta method. Spatial derivatives in the equations are estimated by a de-aliased Fourier spectral method. We report results based on the analysis of such turbulence simulated using $1024^3$ grid points.

The initial condition of each DNS is taken from a lower resolution ($512^3$) velocity field in statistically stationary state. This lower resolution field has been obtained from a DNS with an initially random velocity field with prescribed spectrum

$$E(k) = \begin{cases} 
C_E k^q \exp \left[ -\frac{q}{2} \left( \frac{k}{k_0} \right)^2 \right] & \text{for } k \leq k_0, \\
C_E k^q \exp \left[ -\frac{q}{\alpha} \left( \frac{k}{k_0} \right)^{\alpha} + \frac{q}{\alpha} - \frac{q}{2} \right] & \text{for } k_0 \leq k.
\end{cases}$$

(10)

Here, the coefficient $C_E$ is chosen such that the total kinetic energy in the $512^3$ simulation is unity; the exponent $q$ in the small wavenumber range is either 2 or 4; the wavenumber $k_0$ where the energy spectrum peaks is either 5, 10 or 15; and the other exponent $\alpha$ in the exponential function is chosen ($\alpha = 1$ for $q = 2$, and $1/2$ for 4) such that the energy spectrum of the $1024^3$ simulation is a smooth function at the border wavenumber ($k_f$, see below) between forced and unforced wavenumber ranges. We have carried out 24 different runs for 24 different combinations of the large-scale structure ($q, k_0$) and the kinematic viscosity $\nu$. Numerical parameters adopted in our DNS are given in the first three columns of Table I.

The external large-scale forcing is implemented numerically by keeping the magnitudes of the Fourier components of velocity fixed in the low wavenumber range $k < k_f = 2.4k_0$. The value 2.4 has been chosen for the energy spectrum to be as smooth as possible at $k_f$. Note, however, that the phases of these Fourier components evolve temporally, and that the large-scale velocity field is therefore not steady.

After several large-eddy turnover times $T = L/u'$, turbulence reaches a statistically stationary state. The temporal averages of the Reynolds number

$$R_\lambda = \frac{\lambda u'}{\nu}$$

(11)
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**TABLE I:** Numerical parameters of the DNS, and statistics (time-averaged values in the statistically stationary state) of simulated turbulence. Note that the index \( k_{\text{max}} \eta_K \), where \( k_{\text{max}} = (\sqrt{2}/3) \times 10^{24} \approx 482 \) is the largest wavenumber of the DNS de-aliased by the phase shift method, is smaller than 1 in some of our simulations. This suggests that the smallest scale motions might not be fully well-resolved in those simulations with small \( \nu \). However, the dependence of \( C_\epsilon \) on \( R_\lambda \) appears smooth in Fig. 3 suggesting that our values of \( k_{\text{max}} \eta_K \) are suitable for this paper’s purposes which are concerned with \( C_\epsilon \).
FIG. 1: Energy spectra at the final time of each simulation for the three different peak wavenumbers: (a) $k_0 = 5$, (b) 10 and (c) 15. In each figure, we plot eight curves for all combinations of the four different kinematic viscosities and the two different low-wavenumber exponents ($q = 2$ or 4) of the energy spectrum ($E(k) \sim k^q$ for $k \ll k_0$). The dotted lines indicate power laws with exponents $-5/3$, 2 and 4.
and the Taylor length $\lambda = \sqrt{15\nu u'^2/\epsilon}$ in this state are shown in Table I together with the temporal averages of the integral length $L$, the Kolmogorov length $\eta_k$, the r.m.s. velocity $u'$, the energy dissipation rate $\epsilon$ and the large-eddy turnover time $T$. Here, we estimate the integral length $L$ by integrating the energy spectrum $E(k)$ as follows:

$$L = \frac{3\pi}{4} \int_0^\infty E(k) k^{-1} \, dk / \int_0^\infty E(k) \, dk.$$  \hspace{1cm} (12)

As seen in Fig. 1, $E(k)$ exhibits two power laws in the statistically stationary state: $E(k) \sim k^q$ for $k \ll k_0$ and $E(k) \sim k^{-\frac{5}{3}}$ for $k_0 \ll k (\ll 2\pi/\eta_k)$. We plot 8 curves, in each figure of Fig. 1, which correspond to different combinations of the exponents $q$ and the kinematic viscosities $\nu$. For the same $q$, the behaviour of $E(k)$ in the energy containing range is identical for different values of $\nu$, whereas for the same $\nu$, the behaviour in the dissipation range seems almost identical for different values of $q$ in these logarithmic plots.

III. VERIFICATION OF (9)

We plot in Fig. 2 the temporal evolution of the dissipation coefficient $C_\epsilon$ for each simulation where $\nu = 6 \times 10^{-4}$. (Other $\nu$ cases have similar behaviours.) Probably because we fix the energy spectrum at the lower wavenumbers, there are hardly any temporal fluctuations on this coefficient in the statistically stationary state. It therefore makes sense, as we do in Fig. 3(a), to plot the value of $C_\epsilon$ at the final instant of each simulation as a function

FIG. 2: Temporal evolution of the dissipation coefficient $C_\epsilon$. (a) $(q, \nu) = (2, 6.25 \times 10^{-4})$, (b) $(q, \nu) = (4, 6.25 \times 10^{-4})$. Solid curves, $k_0 = 5$; dotted curves $k_0 = 10$; dashed curves, $k_0 = 15$. 

FIG. 3: Temporal evolution of the dissipation coefficient $C_\epsilon$. (a) $(q, \nu) = (2, 6.25 \times 10^{-4})$, (b) $(q, \nu) = (4, 6.25 \times 10^{-4})$. Solid curves, $k_0 = 5$; dotted curves $k_0 = 10$; dashed curves, $k_0 = 15$. 

and the Taylor length $\lambda = \sqrt{15\nu u'^2/\epsilon}$ in this state are shown in Table I together with the...
of Reynolds number $R_\lambda$. It is clearly observed that the coefficient $C_\epsilon$ depends not only on the Reynolds number $R_\lambda$ but also on the exponent $q$ of the energy spectrum $E(k) \sim k^q$ in the low wavenumber range $k \ll k_0$. The fact that, for different values of $q$, there are distinct $C_\epsilon(R_\lambda)$ curves which appear to asymptote towards different constants as the Reynolds number increases supports the view that $C_\epsilon$ is not universal and depends on the large-scale structure even asymptotically ($R_\lambda \to \infty$).

In order to verify (9), we need to estimate $C_s/B^3$. However, since our simulations possess only limited inertial ranges, it is hard to estimate the coefficient $C_s$ in (4) by fitting data for $n_s^{(c)}$ with the scaling form (4). Instead, we estimate $C_s$ by the relation

$$C_s = L_s^3 n_s^{(c)}(\ell_c = L_s) .$$

(13)

The number density of the stagnation points in the coarse-grained field at the length scale $L_s$ may be counted numerically, but the number of such points is not large enough if we choose $L_s$ too close to $L$. We therefore use an alternative estimation of $n_s^{(c)}$ which proceeds by introducing the coarse-grained field’s Taylor scale

$$\lambda^{(c)} = \sqrt{\frac{5}{2\pi} \int_0^{L_*} E(k) \, dk \, \sqrt{\int_0^{L_*} k^2 E(k) \, dk}} .$$

(14)

The number density $n_s^{(c)}$ is then obtained by application of the generalised Rice theorem to the coarse grained field which gives

$$n_s^{(c)} = \left( \frac{B}{\lambda^{(c)}} \right)^3 .$$

(15)

From (13) and (15) we obtain

$$\frac{C_s}{B^3} = \left( \frac{L_s}{\lambda^{(c)}} \right)^3 ,$$

(16)

which we use to estimate $C_s/B^3$ from our simulations.

Because of the assumptions under which the generalised Rice theorem is derived (see Appendix A), $B$ in (15) and (16) may be considered constant if $L/L_s$ and $R_\lambda$ are large enough. In our DNS, however, $R_\lambda$ is not so large and $B$ can therefore be expected to have a residual dependence on it. This is confirmed in Fig. 3(b) where we plot $C_s/B^3$ as a function of $R_\lambda$. In this plot, $C_s/B^3$ is estimated from (16) by choosing $L_s = 0.2L$ so as to make sure that $L_s$ is within the inertial range, and that $L/L_s$ is as large as possible in our DNS.
One of the interesting observations in Fig. 3(a, b) is that both $C_\epsilon$ (Fig. 3a) and $C_s/B^3$ (Fig. 3b) are clearly dependent on $R_\lambda$ and $q$. We use (9) to define the normalised dissipation coefficient

$$\tilde{C}_\epsilon = \frac{C_\epsilon}{(15^{3/2} C_s L_*/B^3)} \left( = \frac{1}{A^2} \right)$$

which we plot in Fig. 3(c) as a function of $R_\lambda$. This plot clearly shows that $C_\epsilon$ and $C_s/B^3$ have the same dependence on the large-scale structure (i.e. $q$) even at low $R_\lambda$, and it strongly supports the validity of (9) because $A$ depends only on $R_\lambda$. The plot seems also to suggest that $\tilde{C}_\epsilon$ tends to a constant with increasing $R_\lambda$, thus implying that the dependence of $C_\epsilon$ on the large-scale structures and the independence of $\tilde{C}_\epsilon$ on these structures survive even in the limit $R_\lambda \to \infty$.

However, it is not conclusive from Fig. 3(a) whether, for a fixed $q$, $C_\epsilon$ does or does not become a constant in the limit $R_\lambda \to \infty$. Also, we are unable to determine $A$ independently and directly from (4) with the kind of accuracy which would allow to fully validate (9) quantitatively. Nevertheless, Fig. 3(c) suggests that $A \approx 10$ at high $R_\lambda$, which is consistent with the experimental data [8] on zero-crossings, and with the well-known fact that the inertial range of the energy spectrum is bounded by about one tenth of the Kolmogorov wavenumber. Figure 3 also suggests that much, though not all, of the Reynolds number dependence of $C_\epsilon$ is accountable to the Reynolds number dependence of $A$, in agreement with the conclusions in Ref. [8].

Incidentally, the coefficient $B$ can be expected to have a residual dependence on $L_*$ if $L_*$ is not small enough. We have checked that the coincidence observed in Fig. 3(c) is preserved even if we adopt a different $L_*$, although the absolute value of $\tilde{C}_\epsilon$ depends on its choice particularly if $L_*$ is not small enough because of the residual dependence of $B$ on $L_*$.  

As a final point, we use (16) to show that (9) can yield the relation (22) between $C_\epsilon$ and $q$ which is similar to the one recently derived and used in Ref. [7]. If we assume that $E(k)$ has the high $R_\lambda$ model functional form

$$E(k) = \begin{cases} 
C_E k^q & \text{for } k \leq k_0, \\
C_E k_0^{q+\frac{5}{3}} k^{-\frac{4}{3}} & \text{for } k_0 \leq k, 
\end{cases}$$

(18)

the integral length (12) becomes

$$L = \frac{3\pi}{4} \frac{\frac{1}{9} + \frac{3}{5}}{\frac{1}{q+1} + \frac{3}{2}} k_0^{-1}.$$ 

(19)
FIG. 3: Plots as functions of $R_\lambda$ of: (a) the energy dissipation coefficient $C_\epsilon$; (b) the stagnation point coefficient $C_s/B^3$; (c) and the normalised dissipation coefficient. Open symbols, $q = 2$; solid symbols, $q = 4$. Squares, $k_0 = 5$; circles, $k_0 = 10$; triangles, $k_0 = 15$. 
Defining \( k_\ast = 2\pi/L_\ast \) and assuming it to be in the inertial range (i.e. \( k_\ast \gg k_0 \)), the Taylor scale \( \lambda^{(c)} \) defined by (14) takes the form

\[
\lambda^{(c)} = \sqrt{\frac{10(3q + 5)}{3(q + 1)}} k_0^{-\frac{1}{3}} k_\ast^{-\frac{2}{3}}.
\]

(20)

Therefore, from (16), we obtain

\[
C_s = (2\pi B)^3 \left[ \frac{3(q + 1)}{10(3q + 5)} \right]^{\frac{2}{3}} \frac{k_0}{k_\ast}.
\]

(21)

Then, (9) leads to

\[
C_\epsilon = \frac{3\pi^3}{A^2} \left[ \frac{9(q + 1)}{2(3q + 5)} \right]^{\frac{2}{3}} \frac{1}{\frac{q}{q+1} + \frac{3}{2}}.
\]

(22)

which implies that \( C_\epsilon \) depends on \( q \), the exponent of \( E(k) \) in the low wavenumber range, even in the high Reynolds number limit (i.e. \( L/\eta \to \infty \)). Note that \( C_\epsilon \) in (22) depends only on \( q \) (but is independent of \( k_0 \)), and is a decreasing function of \( q \). These points are consistent with Fig. 3(a). Furthermore, the ratio \( C_\epsilon(q = 2)/C_\epsilon(q = 4) \) predicted by (22) is about 1.071, which is almost identical to the ratio 0.525/0.490 \( \approx 1.071 \) at the highest Reynolds number of our DNS plotted in Fig. 3.

IV. CONCLUSION

By considering the statistics of the spatial distribution of stagnation points, we can suggest answers to two (ii and iii) of the three open questions mentioned in the first section. That is, the dissipation coefficient \( C_\epsilon \) is not universal, and it depends on the internal stagnation point structure of the turbulence, which itself depends on the structure of the large eddies.

However, we have not answered the question (i) whether \( C_\epsilon \) is asymptotically independent of the Reynolds number. Instead we have shown that if we assume it is, we can then use the generalised Rice theorem and (4) to derive the Kolmogorov wavenumber form of the energy spectrum and relation (9) between \( C_\epsilon \) and \( C_s/B^3 \). Incidentally, this relation (9) can also be obtained without an assumption on the Reynolds number dependence of \( C_\epsilon \): assuming a Kolmogorov shaped energy spectrum, i.e. \( p = 5/3 \), and using (6) to derive \( D_s = 2 \) also leads to (9). As observed in Fig. 3(a), for a common large-scale structure, \( C_\epsilon \) is a monotonically decreasing function of \( R_\lambda \), but the dependence seems to be weaker for higher
$R_\lambda$ in this linear-linear plot. In order to determine the subtle asymptotic $R_\lambda$ dependence or independence of $C_\epsilon$, we must carefully consider small-scale intermittency effects and the resulting $R_\lambda$ dependence of the constant $B$ in the generalised Rice theorem (3) (see Appendix A) as well as the $R_\lambda$ dependence of $A$, the ratio of the inner cut-off length to the Kolmogorov length (recall the statements below (6)). However, as experimentally suggested [8] for zero-crossings, these dependencies on $R_\lambda$ may be logarithmic and the verification of such weak dependencies is beyond the scope of the present paper which is based on DNS carried out over a narrow $R_\lambda$ range.

Another important work which must be left for future much larger DNS is the extension of this paper’s arguments to decaying turbulence so as to shed some light on the so-called permanence of large eddies. This seems a natural direction to follow because the spatial distribution of stagnation points in the turbulence is determined by the large-scale eddies and because the permanence of large eddies is therefore likely to be related to the permanence of a statistical property of stagnation points.

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The generalised Rice theorem states that, for large enough Reynolds number, the average distance between stagnation points of an incompressible statistically isotropic and homogeneous turbulent velocity field is proportional to the field’s Taylor scale provided that the velocity field and its spatial derivatives are statistically independent and provided that the shapes of the probability density functions (PDF) of the velocity components on the one hand and the velocity derivatives on the other are independent of Reynolds number and can be scaled, respectively, with \( u' \) and \( \sigma = \langle u_x^2 \rangle^{1/2} \). Here \( u_x = \partial u / \partial x \). It is also required that the PDF of velocity derivatives decays fast enough at infinity. The generalisation is twofold: the assumptions have been relaxed, and the statement is about stagnation points...
rather than zero-crossings. We now prove this statement.

The number \( N_s \) of stagnation points \( \mathbf{u} = (u, v, w) = 0 \) in a given volume can be expressed as an integral over that volume:

\[
N_s = \int dV \left| \nabla H[u(x)] \right| \left| \nabla H[v(x)] \right| \left| \nabla H[w(x)] \right|
\]

(A1)

where \( H \) is the Heaviside function. This formula leads to

\[
N_s = \int dV \delta[(u(x)] \delta[(v(x)] \delta[(w(x)] \left| \nabla u \right| \left| \nabla v \right| \left| \nabla w \right|
\]

(A2)

where \( \delta \) is the delta function. Statistical homogeneity allows us to replace this volume integral by an average weighted by the joint PDF \( P(\mathbf{u}, \nabla \mathbf{u}) \). We therefore obtain the following expression for the number density \( n_s \) per unit volume

\[
n_s = \int du dw dw = \int du dw dw \delta(u) \delta(v) \delta(w)
\]

(A3)

\[
P(\mathbf{u}, \nabla \mathbf{u}) \sqrt{u_x^2 + u_y^2 + u_z^2} \sqrt{v_x^2 + v_y^2 + v_z^2} \sqrt{w_x^2 + w_y^2 + w_z^2}
\]

(A4)

Following Rice [9–11] we assume statistical independence between \( \mathbf{u} \) and \( \nabla \mathbf{u} \), specifically \( P(\mathbf{u} = 0, \nabla \mathbf{u}) = P_l(\mathbf{u} = 0) P_s(\nabla \mathbf{u}) \) (where the suffixes \( l \) and \( s \) stand for large scale and small scale, respectively), which leads to

\[
n_s = P_l(\mathbf{u} = 0) \int du dw dw = \int du dw dw
\]

(A5)

\[
\sqrt{u_x^2 + u_y^2 + u_z^2} \sqrt{v_x^2 + v_y^2 + v_z^2} \sqrt{w_x^2 + w_y^2 + w_z^2}
\]

At this stage we assume statistical isotropy but we need to do this within the constraints of incompressibility. As demonstrated by Taylor [1], a statistically isotropic velocity gradient tensor field resulting from an incompressible velocity field obeys the following relations between averages of velocity gradient products: \( \langle u_x v_y \rangle = \langle v_y w_z \rangle = \langle w_z u_x \rangle = \langle u_y v_x \rangle = \langle u_z w_x \rangle = \langle w_y v_z \rangle = -\frac{1}{2} \langle u_x^2 \rangle = -\frac{1}{2} \langle v_y^2 \rangle = -\frac{1}{2} \langle w_z^2 \rangle \), \( 2 \langle u_x^2 \rangle = \langle u_y^2 \rangle = \langle u_z^2 \rangle \), \( 2 \langle v_y^2 \rangle = \langle v_x^2 \rangle = \langle v_z^2 \rangle \), \( 2 \langle w_z^2 \rangle = \langle w_x^2 \rangle = \langle w_y^2 \rangle \) and the averages of all other products of two velocity gradients vanish. These relations are satisfied if we make the assumption that the non-dimensional form of the PDF \( P_s \) can be written in terms of the single strain rate \( \sigma = \langle u_x^2 \rangle^{1/2} \). Specifically,

\[
P_s(\nabla \mathbf{u}) du dw dw dw = P_s(\nabla \mathbf{u} / \sigma) d \left( \frac{u_x}{\sigma} \right) d \left( \frac{u_y}{\sigma} \right) d \left( \frac{u_z}{\sigma} \right) d \left( \frac{v_x}{\sigma} \right) d \left( \frac{v_y}{\sigma} \right) d \left( \frac{v_z}{\sigma} \right) d \left( \frac{w_x}{\sigma} \right) d \left( \frac{w_y}{\sigma} \right) d \left( \frac{w_z}{\sigma} \right)
\]

(A6)
This assumption replaces and is more general than Rice’s assumption of Gaussianity of velocity derivatives. A similar assumption can be made for $P_l(u)$ which was also assumed by Rice to be Gaussian. Here we only assume $P_l(u) \, du = P_l(u/u') \, du/u'^3$ in agreement with our assumption of isotropy which demands that $u'^2 = \langle u^2 \rangle = \langle v^2 \rangle = \langle w^2 \rangle$. Clearly, appropriate Gaussian forms for the large-scale and small-scale distributions are particular cases which satisfy our scaling assumptions on $P_s$ and $P_l$.

It follows that $P_l(u/u' = 0) \sim u'^{-3}$ and therefore that

$$n_s \sim u'^{-3} \sigma^3. \quad (A7)$$

Hence,

$$n_s^{-1/3} = B \frac{u'}{\sigma} = B \lambda \quad (A8)$$

where $B$ is a dimensionless constant determined by the actual value of $P_l(u/u' = 0)$ and by the details of the probability distribution $P_s$. If we finally assume this value and this distribution to be independent of Reynolds number, then the generalised Rice theorem is established as it then follows that $B$ is also independent of Reynolds number. (The benign assumption that the PDF of velocity derivatives decays fast enough at infinity ensures that the integral in (A5), and therefore $B$, are finite.)

Whilst it is reasonable to expect that $P_l(u = 0)$ is independent of Reynolds number, it is well known that the PDF of turbulence velocity derivatives become increasingly non-Gaussian with increasing Reynolds number because of small-scale intermittency. There may therefore be a resulting dependence of $B$ on Reynolds number, but the results on zero-crossings presented in Ref. [8] suggest that this dependence might be as weak as logarithmic. If the Reynolds number is too small, then the PDF of turbulence velocity derivatives can have a residual Reynolds number dependence caused by insufficient development of inertial range dynamics. Our assumption is that the Reynolds number is large enough for this not to happen and that small-scale intermittency effects are negligible in this limit, so that $B$ is asymptotically independent of the Reynolds number as a result.

In pages 26 to 29 of a recent book by Joseph et al. [17] a relation is found between dissipation and number of stagnation points in a Taylor vortex array. This relation corresponds to a Rice theorem and a relation such as (9) for the particular flow considered by these authors.