FE MODEL VALIDATION
FOR STRUCTURAL DYNAMICS

by

Gan Chen

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Dynamics Section
Department of Mechanical Engineering
London, South Kensington

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Abstract

This thesis describes studies of finite element (FE) model validation methods for structural dynamics. There are usually discrepancies between predictions of the structural dynamic properties based on an initial FE model and those yielded by experimental data from tests on the actual structure. In order to make predictions from the model suitable for evaluating the dynamic properties of the structure, and thus to optimise its design, the model has to be validated. The process of model validation and the individual steps that make up this process are studied through theoretical analysis and case studies.

A general strategy for model validation is presented in the introduction to the thesis, describing the validation process as a sequence of distinct steps. A study of the strategy presented at the end of the thesis shows multiple approaches for model validation. Using experience obtained from validating models of example structures for constructing and validating the model of a new design of a similar structure could accelerate the process of FE model validation, and therefore benefit industry by reducing the time of the design-examine-redesign loop.

Through case studies, a new perspective on the performance of model updating is presented in this work showing that model updating can only minimise the discrepancies caused by parameter errors in the model. However, model updating procedures cannot reduce discrepancies caused by discretisation errors and/or configuration errors without losing the physical meaning of the updating parameters. In order to make sure that a model is suitable for being subjected to the updating step, the model must first be verified. Two verification methods, a convergence check by using two different formulae for constructing the mass matrix of each element and a configuration check by projecting experimental mode shapes onto the subspace spanned by FE model predicted eigenvectors, are proposed to deal with discretisation errors and configuration errors respectively in FE models.

In addition, this thesis presents research findings and ideas dealing with correlation and testing for the purpose of model validation. Special attention is given to the correlation performance for quasi-axisymmetric structures in order to get a "true" representation of the correlation by numerical parameters. Studies of test-planning and modal testing are also focused on cases with quasi-axisymmetric structures.
Acknowledgements

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Nomenclature

**Basic Terms, Dimensions and Subscripts**

\(x, y, z\)  
Translational degrees-of-freedom / coordinates

\(u, v, w\)  
Translational degrees-of-freedom

\(t\)  
Time

\(\omega\)  
Frequency of vibration, in rad/s

\(N\)  
Number of degrees-of-freedom (DOFs)

\(m\)  
Number of modes

\(i, j, k, r\)  
Integer indices

\(a, b, c, A, B, C\)  
Real constants

\(E\)  
Young’s Modules of materials

\(\rho\)  
Mass density of materials

**Matrices and vectors**

\([\, ]\)  
Two dimensional matrix

\(\{\, \}\)  
Column vector

\([\cdot\cdot\cdot]\)  
Diagonal matrix

\([\, ]^T, [\, ]^H\)  
Transpose of a matrix, complex conjugate transpose of a matrix

\(\{\, \}^T, \{\, \}^H\)  
Transpose of a vector, complex conjugate transpose of a vector

\([I]\)  
Identity matrix

\([\, ]^{-1}\)  
Inverse of a matrix

\([\, ]^+\)  
Generalized/pseudo inverse of a matrix

\([\mathcal{H}]\)  
Norm of a matrix / vector

\([U],[V]\)  
Matrices of left and right singular vectors

\([\Sigma], \sigma\)  
Singular value matrix, singular value
Structural dynamic properties

\[ [M] \] Mass matrix

\[ [K] \] Stiffness matrix

\[ m_r, k_r \] Generalized mass, stiffness of \( r^{th} \) mode

\[ \omega_r \] Natural frequency of \( r^{th} \) mode, in rad/s

\[ [\lambda] \] Eigenvalue matrix

\[ [\phi] \] Mode shape/eigenvector matrix

\{\phi,\} Mode shape/eigenvector of \( r^{th} \) mode

\[ \phi_{ij} \] \( i^{th} \) element of \( j^{th} \) mode shape/eigenvector

\[ [x], \{x\}, (x) \] Experimental properties

\[ [A], \{A\}, (A) \] Analytical properties

\[ H(\omega) \] Frequency Response Function (FRF)

For model updating study

\[ [S_s], [S_s] \] Eigenvalue and eigenvector sensitivity matrices

\[ J(p) \] Penalty function of updating equation

\{\delta p\} Updating parameter vector

\{R\} Residual vector of updating equation

\[ [W], [W_S], [W_R] \] Weighting factor matrices applied to updating equation

\[ C_{Rel} \] Release factor applied to solution of updating equation

For model verification study

\[ \lambda, \mu \] Eigenvalue (characteristic root of linear differential equations)

\[ U_\sigma \] Strain energy

\[ L \] Number of elements per wavelength

\[ \text{proj}_V u \] Projection of vector \( u \) onto space \( V \)

\[ [C] \] Projecting matrix

\{\bar{\phi}_x\}, [\bar{\phi}_x] \] Projection of reference eigenvector matrix

For correlation study

\[ m_{R_{i0}}, m_{T_{i0}}, m_{Z_{i0}}, \] Amplitudes for the \( m^{th} \) harmonic component of the \( i^{th} \) mode in the radial, tangential, and axial directions, respectively

\[ m_{\theta_{i0}} \] Spatial phase angle of the \( m^{th} \) harmonic component of the \( i^{th} \) mode
\(\theta_k\)  
Phase angle coordinate of \(k^{th}\) grid point

\(N_p\)  
Number of grid points around a circumference

\(N_c\)  
The highest order of the harmonic component

\(\{\tilde{\phi}\}\)  
Rotated eigenvector

\(\tilde{\theta}\)  
Shifting phase angle

\(\tilde{\theta}_m\)  
Shifting phase angle for \(m^{th}\) harmonic component

**For test study**

\([\phi_o], [\phi_i]\)  
eigenvector matrices on output DOFs and input DOFs

\(Q_r\)  
Scaling factor of \(r^{th}\) mode

\(N_t\)  
Experimental measurement frequency points

\(CMIF_i(\omega_k)\)  
\(i^{th}\) Complex Mode Indicator Function parameter at \(k^{th}\) frequency point
# Standard Abbreviations

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<tr>
<td>DOF(s)</td>
<td>Degree(s)-of-freedom</td>
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<tr>
<td>FE</td>
<td>Finite element</td>
</tr>
<tr>
<td>FRF</td>
<td>Frequency response function</td>
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<tr>
<td>MAC</td>
<td>Modal assurance criterion</td>
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<tr>
<td>NFD</td>
<td>Natural frequency difference</td>
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<tr>
<td>SVD</td>
<td>Singular value decomposition</td>
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<tr>
<td>CMP(s)</td>
<td>Correlated mode pair(s)</td>
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<tr>
<td>EI</td>
<td>Effective independence</td>
</tr>
<tr>
<td>CMIF</td>
<td>Complex mode indicator function</td>
</tr>
<tr>
<td>MSF</td>
<td>Modal scale factor</td>
</tr>
<tr>
<td>PBC</td>
<td>Perturbed boundary condition</td>
</tr>
<tr>
<td>nND</td>
<td>Nodal diametral mode of the $n^{th}$ family order</td>
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<tr>
<td>ana.</td>
<td>Analytical</td>
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<tr>
<td>exp.</td>
<td>Experimental</td>
</tr>
<tr>
<td>CAD</td>
<td>Computer aided design</td>
</tr>
<tr>
<td>CAE</td>
<td>Computer aided engineering</td>
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Chapter 1

Introduction

1.1 Finite Element (FE) model technique and model validation

Using Finite Element (FE) models to predict the dynamic properties of structures becomes more and more important in modern mechanical industries, such as the aerospace industry and the automobile industry. Whenever there is a new design or modification of an existing design, the structural dynamic properties of the product must be examined to fulfil some criteria proposed either by the industry itself and/or external agencies before the product can be launched on the market.

The traditional way for evaluating the structural dynamic properties of a product is to perform a series of dynamic tests on prototypes of the product and to demonstrate its capacity to withstand these tests. Until the experimental results show that the prototypes can be in compliance with the relevant criteria, the product has to be redesigned and another design-test loop is followed. In this design-test-redesign loop, much time and money is spent on producing prototypes and performing tests.

With the growing capabilities of computing techniques, and the strength of the competition between companies, FE model predictions are used more and more to take the place of practical dynamic test data. Furthermore, the FE modelling technique may also be used to predict the dynamic response of structures when working in an over-limit situation which is very difficult, if not impossible, to simulate by experiments. All of these depend on the accuracy of FE model predictions. So, the model must be validated.

The definition of FE model validation in this thesis refers to the process of creating an FE model which has acceptably similar dynamic behaviour to that of the actual structure under consideration and in demonstrating this similarity. A validated
model is generally based on an initial model but with updated parameter values. The validated model should have the ability to predict the dynamic behaviour of the structure that is represented in the experimental data used in the validation process. It should also be able to predict, with a certain accuracy, the dynamic properties of the structure under situations different to those in which the experiment was undertaken.

1.1.1 Discrepancies of model prediction from experimental data

In general, it cannot be guaranteed that the dynamic properties from the initial FE model of an industrial structure will have an accuracy that is acceptable for the predictions to be used directly to replace the experimental data – there are generally discrepancies between the model predictions and the experimental data from tests on the actual structure. Some of the discrepancies come from noise in the experimental data. However, most are the result of uncertainties and inaccuracies in the model. It is these discrepancies that make the FE model technique not used as widely as it would be if the model predictions were accurate.

There are three kinds of error in FE models that are considered to cause model predictions with discrepancies from experimental data. All three kinds of error can be introduced into an FE model during the construction of the model. The effects of these errors on the dynamic properties of FE models are summarised as follows.

(1) Discretisation errors. The predictions from a discrete FE model are different, more or less, from the dynamic properties of the continuous structure, even when the configuration and the parameters of the model are all correct. The source of these differences is referred to as "discretisation errors" of the FE model. For predicting dynamic properties from an FE model, both the mass and stiffness matrices of the model are involved in the calculation. Mathematically, constructing mass and stiffness matrices for each element of the model in a way that is different from constructing the equation of motion for a continuous system is a basic cause of discretisation errors.

(2) Configuration errors. The types of element used in an FE model, especially those elements that represent the “key” features of the structure, such as joints, significantly influence the dynamic properties of the model. In the modelling process, some physical features have to be simplified in order to make the model of a manageable size. For example, the casing of an airplane engine is generally modelled using shell elements, although in some locations the thickness of the casing is not suitable for being represented by shell elements. The flanges of the casing are mostly modelled by beam elements with offsets at the two ends of each beam element. In a model constructed in this way, the connection points between two sub-casings are not located on the flanges but on the grid points of the shell elements. The flexibility of the flange will not be involved in the “connecting stiffness” between the sub-casings. Considering low-frequency modes, the strain energy in the flanges can be neglected.
when comparing to the strain energy of the whole structure. Thus these simplified features might not affect significantly the dynamic properties of the model. However, they will definitely affect the dynamic properties of the model in the high-frequency range. Even more, if some parts or substructures representing the “key” features of the structure for some modes are missing from the model, then this cannot be expected to predict the dynamic properties of the structure for those modes, or at least cannot predict those modes accurately.

(3) Parameter errors. Because an FE model is a simplified model of a structure, there must be some parameters with uncertain values. Engineers constructing the model determine suitable parameter values based on their experience, or they assign parameters with approximate values. As the values of those parameters are not accurate, the dynamic properties predicted by the model are not accurate. If the parameter values are not too far away from the “true” values, the model can still keep the essential dynamic features of the structure. When subjecting the model to a model updating procedure, and if all the parameters with uncertainties are selected as updating parameters, it is expected that the results from the procedure will lead to parameters with the “correct” values.

In a model validation process, the initial model usually has all three kinds of error mentioned above. The effects of the different kinds of error are combined together to make the model validation process more complicated than cases where there is only one kind of error present in the model.

1.1.2 Strategy for model validation

A strategy for the process of model validation is shown as a block diagram in Figure 1-1. In this block diagram, there are series of distinct steps, including tests on a structure and numerical calculations based on the initial model.

Once a process of model validation on the initial FE model is in progress, some tests (usually, modal tests) have to be undertaken in order to obtain the actual dynamic properties of the structure which the model is expected to predict with a certain accuracy. In order to ensure that the results from the tests provide the information necessary for the model to be validated, and that the tests are undertaken efficiently, predictions from the initial model are used in the test planning procedure. Although these predictions may not be entirely accurate, the output from the test planning procedure with margins set for this inaccuracy can help the test engineer(s) to determine efficient test settings.

Model verification and correlation provide the information of the initial FE model about its consistency with the experimental data. Although model verification has hardly been mentioned in the literature of model validation, it is an important step in the process. It serves to debug the initial model and to check if it is suitable for being
subjected to the model updating procedure. After correlation and verification on the initial model, if the model is found not to be suitable for application of the updating procedure, the model has to be modified significantly, not only for its parameter values, but also for its configuration or even for its mesh size.

Model updating plays a central role in the process of model validation. By constructing and solving an updating equation, the step of model updating tries to find a group of values that modify the parameters in the initial model in order to reduce the discrepancies between the experimental data and the dynamic properties predicted by the updated model. The selection of the updating parameters will affect significantly the results from the model updating procedure. Although it cannot be guaranteed that all the parameters selected are those with incorrect values, the selection of parameters based on...
certain techniques helps the updating procedure to have convergent results.

When an initial model has been subjected to the model updating step, the updated model is one that can produce the dynamic property predictions with discrepancies from the experimental data minimised in the way defined in the updating equation or in the residual function. Because of the different choices of updating methods, updating parameters, and different choices for the experimental data, there can be many updated models. Thus, each updated model must be assessed to check if it is optimised in a pre-defined way and is capable of representing the dynamic properties of the structure. This procedure of assessing updated models is the final and crucial procedure in the process of model validation. Only after an updated model is assessed in some way and the assessment yields a satisfactory result, can the model be said to be a validated model.

1.2 Objectives of research

The overall objective of the research reported here is to develop a strategy for FE model validation in structural dynamics which includes provision of a balanced set of algorithms for all the procedures in the process of FE model validation. According to the above summary of all steps in the process of FE model validation, the specific objectives of the research in this thesis are as follows.

- To analyse the effects of the three kinds of error in an FE model on the dynamic properties predicted by the model and on the results from an updating procedure undertaken on the model;
- To determine the requirements for an FE model to be suitable for updating and to develop new methods for model verification;
- To study the procedures of correlation and modal test on quasi-axisymmetric structures in the model validation process.

1.3 Outline of Thesis

Chapter 2 prescribes a literature review of the topic of model validation. The focus is put on three aspects: (i) test planning, (ii) FE model verification and (iii) FE model updating. On test planning, several techniques are discussed for optimal sensor location selection and optimal excitation point(s) selection. On model verification, the methods for estimating and reducing discretisation errors are described. On FE model updating, the emphasis is put on the iterative methods using penalty functions, although some other methods are also described and discussed briefly.

Chapter 3 describes the structure and the features of the model updating procedure, and the case studies of updating several FE models of simple structures and complicated industrial structures. Through the analysis of case study results, the
requirements for a model to be suitable for being subjected to an updating procedure are discussed in this chapter.

Chapter 4 addresses the problems in the model verification procedure. In this chapter, the definition of model verification is discussed and followed by the proposal of two methods, which deal with discretisation errors and configuration errors respectively. The case studies of using these methods show that they are useful and efficient to verify FE models in the light of model validation.

Chapter 5 is concerned with the correlation procedure in the process of model validation as a whole. The particular problems encountered when performing a correlation calculation for quasi-axisymmetric structures are studied and a method to obtain a "true" numerical presentation of the correlation for this kind of structure is developed.

Chapter 6 presents a study of methods for test planning and modal tests for the purpose of model validation. What kind of information is needed from the tests for validating an FE model for structural dynamics and how can the information be best obtained? With the case studies of using a test planning technique and the CMIF on quasi-axisymmetric structures, these questions are explored.

After presenting studies on the major procedures in the model validation process, a case study for validating the FE model of an industrial structure is presented in Chapter 7. Most procedures in the model validation process are performed in this case study. The updated model for the structure is assessed in the end of the chapter.

Chapter 8 presents a discussion on the strategy of model validation. In order to speed up the design cycle for products in industry, the model validation process needs to be accelerated. Thus, a process of model validation could be started, if possible, before the prototypes of a new design are available to undertake tests. The possibility does exist provided that other sources of information about the dynamic properties of the structure can be made available.

Chapter 9 concludes the discussions in the previous chapters and summarises the contributions in the project of this thesis. For further studies in the area of model validation, some general suggestions are given in this chapter.
Chapter 2

Literature Review

2.1 Introduction

This chapter reviews the literature on FE model validation.

The process of FE model validation consists of a series of procedures (see 1.1 of the thesis) performed on both an FE model and experimental data. Although all these procedures are important to the final results of FE model validation, the emphasis of the literature review in this chapter is placed on the procedures of test planning, FE model verification and FE model updating. Test planning is the first step in FE model validation and it is essential that the correct data are available for the subsequent procedures of verification and updating. Model updating plays a central role in the process. Only when these two steps are performed correctly, can the requirements from and on the other procedures be fulfilled and proposed clearly and the whole process gets a solid base. Considering the topic on FE model verification, there are only a few papers published on this issue. It is worth reviewing these papers, which present the definition and contents on the procedure. Some of the other topics - for example, the correlation between FE model predictions and experimental data - are widely reviewed by many papers [33, 63, 19 and 97]. Literature reviews on these topics are not included in this chapter. However, they will be briefly reviewed in the following chapters where the topics are concerned.

Since FE model validation was first proposed in late 1970s or early 1980s, many researchers have invested a great deal of effort into this area that even a full list of the papers published is impossible to be given here. The author has sought to collect the most important papers published in recent years. From these papers, one can get the information about what is being studied and what significant progress has been achieved.
2.2 Test planning for model validation

In order to make sure that the data from a test, especially a modal test undertaken for the process of model validation, can give useful information about the structure and the test can be carried out effectively, the test settings must be well planned. Among the considerations in the settings, selecting suspension points, driving point(s) and response points are the most important. Test planning in the process of model validation is undertaken to help to decide the optimum consideration for the test settings. That is to determine how and where to suspend the test structure, how to select the point(s) in the test structure as excitation point(s), and – most importantly – how to choose the response DOFs that must be measured. In the process of FE model validation, the initial FE model may not be accurate enough to predict dynamic properties of the structure accurately. However, it will usually provide a useful guide to these properties. Test planning uses results predicted by such an initial FE model and, by performing a series of calculations with a certain amount of margin in the light of the expected inaccuracy of the model, gives some guidance for the test.

Most of papers dealing with test planning techniques focus on optimal sensor location selection and excitation DOFs selection.

2.2.1 Optimal sensor location selection

2.2.1.1 Test-Analysis Model (Guyan Reduction)

Kammer [38] used the concept of a Test-Analysis Model (TAM) to select accelerometer positions for modal tests on large structures.

From an initial FE model:

\[
(K - \omega^2 M)x = 0
\]  

(2.1)

A Test-Analysis Model is needed. This model has fewer DOFs than the initial model has but can predict proximately the dynamic properties of the initial model for the low frequency modes. The measurement DOFs can be selected which are the same DOFs in the Test-Analysis Model. The Guyan model reduction method was used to get the Test-Analysis Model in Kammer’s article [38].

By the Guyan Reduction method [26], the DOFs of the initial model are divided into two parts: \(\{x\} = \{x_m\} \cup \{x_s\}\). \(\{x_m\}\) and \(\{x_s\}\) denote the master degrees of freedom and the slave degrees of freedom, respectively. Equation (2.1) becomes:

\[
\begin{pmatrix}
K_{mm} & K_{ms} \\
K_{sm} & K_{ss}
\end{pmatrix} - \omega^2
\begin{pmatrix}
M_{mm} & M_{ms} \\
M_{sm} & M_{ss}
\end{pmatrix} \cdot \begin{pmatrix}
x_m \\
x_s
\end{pmatrix} = 0
\]  

(2.2)

Assuming that the inertial items on the slave DOFs can be ignored, from equation (2.2) it can be obtained that:
\begin{equation}
[K_{sm}][x_m] + [K_{ss}][x_s] = 0 \tag{2.3}
\end{equation}

\begin{equation}
[x_s] = -[K_{ss}]^{-1}[K_{sm}][x_m]
\end{equation}

\begin{equation}
[x] = \begin{bmatrix} x_m \\ x_s \end{bmatrix} = \begin{bmatrix} I \\ -[K_{ss}]^{-1}[K_{sm}] \end{bmatrix} [x_m] \tag{2.4}
\end{equation}

The reduced mass matrix and stiffness matrix can be calculated as:

\begin{equation}
[M_m] = \begin{bmatrix} I \\ -[K_{ss}]^{-1}[K_{sm}] \end{bmatrix}^T [M] \begin{bmatrix} I \\ -[K_{ss}]^{-1}[K_{sm}] \end{bmatrix}
\end{equation}

and

\begin{equation}
[K_m] = \begin{bmatrix} I \\ -[K_{ss}]^{-1}[K_{sm}] \end{bmatrix}^T [K] \begin{bmatrix} I \\ -[K_{ss}]^{-1}[K_{sm}] \end{bmatrix}
\end{equation}

The Guyan reduction method is based on the assumption that the inertial values on the slave DOFs can be omitted [73]. By repeating the removal of one DOF with the largest ratio of \((k_{ii}/m_{ii})\) and re-generating the reduced mass and stiffness matrices until the number of the DOFs in the reduced model equals to the required number of measurement DOFs, the retained DOFs in the final reduced model are the measurement DOFs and the sensor locations can be determined. The final reduced mass matrix and stiffness matrix can also be used in the orthogonality check between the experiment mode shapes and those predicted by the initial FE model.

Kammer used the concept of the Test-Analysis Model (TAM) [38] on a component-by-component basis, allowing the determination of the TAM at the much smaller component level.

Brian used the Guyan reduction method to select the sensor locations that have smaller values of \((k_{ii}/m_{ii})\) among all DOFs in the initial FE model and, in his opinion, the selected sensor location set can also enhance the modal test results [10].

Some other model reduction methods – the Improved Reduction System (IRS), modal reduction method, and the hybrid TAM method – are also used for reducing FEM to TAM [22]. These methods differ from the Guyan reduction method in the formulae used for the reduced mass and stiffness matrices. Once the matrices are calculated, the criterion to remove a slave DOF is the same as that described above. Because of the different formulae for the reduced mass and stiffness matrices, the removed slave DOFs will be different by the different methods.

2.2.1.2 Energy distribution method

The use of kinetic energy for test planning has been discussed in a few papers [61, 10, 73].

From an FE model with the dynamic equation \([2.1]\) the expression for the
kinetic energy at the \( j \)th DOF for \( m \) modes of interest can be written as:

\[
KE_j = \frac{m_j}{2} \sum_{i=1}^{m} \omega_i^2 \phi_{ij}^2
\]  

(2.6)

It is assumed that by placing the sensors at points of maximum kinetic energy, the sensors will have the maximum observability of the structural parameters. Those DOFs with the maximum kinetic energy for a mode (or modes) of interest would be chosen as sensor locations.

The energy distribution method uses the mode shape data directly. There is no complicated calculation needed. An alternative method to use mode shape data directly is a method called the Eigenvector Component Product (ECP) method [61]. A Product for a DOF is calculated by multiplying the eigenvector components on the DOF over the frequency range of interest. The DOFs with maximum values of this product should be retained as candidate sensor locations. This method will preclude all points at nodal points because of the zero products on these points.

There are some disadvantages for the kinetic energy method and the Eigenvector Component Product method. The selected sensor locations by these methods may be not good for correlation performance when the test data are used to compare the FE model predicted mode shapes. However, they can be used in combination with other methods in order to ensure that the test data have high observability.

2.2.1.3 Flexibility shape method

Flanagan et al developed a search algorithm to perform the sensor location selection based on the TAM concept [16]. They used the static flexibility shapes of the initial FE model to define an optimum reduced model.

The static flexibility shapes \( \phi \) of the FE model can be obtained by applying unit forces at each candidate DOF:

\[
K \cdot \phi = \phi
\]  

(2.7)

The mode shapes \( \phi \) predicted by the FE model in the frequency range of interest can be considered as linear combinations of the static flexibility shapes:

\[
\phi = [\phi] \cdot [C]
\]  

(2.8)

where \( [C] \) is the coefficient matrix for the linear combinations. When all DOFs in the FE model are retained as the master DOFs, this equation will be exactly correct. Otherwise, the coefficient matrix \( [C] \) can only be calculated by applying pseudo-inverse on the static flexibility shape matrix.

\[
[C] = ([\phi]^T \cdot [\phi])^{-1} \cdot [\phi]^T \cdot \phi
\]  

(2.9)

The error in the linear combinations of the static flexibility shapes with respect to the mode shapes predicted by the FE model can be expressed as:
The DOFs that can minimise the error are the optimal sensor locations for the test [16]. Flanigan et al and Stabb et al have developed several sub-optimum search algorithms to perform this method [16, 17, 78].

Effective Independence (EI) – Fisher Information Matrix

Effective Independence is based on an idea that the sensors in a modal test should be placed such that the obtained mode shapes on the measurement DOFs are spatially independent of each other. There are many papers dealing with this technique [35, 93, 73, and 23]. Each paper describes the technique in an individual way and the methods to perform the technique are also different between the papers. However, the basic logic of the technique for most of these methods is the same - choosing the sensor location candidates by the use of the Fisher Information Matrix.

The sensors’ response \( \{ u \}_N \) when concerning \( m \) modes of interest, can be given as:

\[
\{ u \}_N = [\phi]_{N\times m} \cdot \{ q \}_m + \{ v \}_N
\]  

(2.11)

where \( N \) is the number of DOFs in the initial FE model, \( \{ v \}_N \) is the measurement noise vector, \( \{ q \}_m \) is the mode partition vector for the \( m \) modes of interest.

On the assumption that the measurement noise is stationary random with zero mean, the estimate of the vector \( \{ q \}_m \), from the measurement \( \{ u \}_N \), can be solved by the formula:

\[
\{ \hat{q} \}_m = [\phi]_{m\times N}^{+} \cdot \{ u \}_N
\]  

(2.12)

where \([ \ ]^{+}\) means the pseudo inverse of the matrix \([ \ ]\). The estimated error covariance matrix will take the form as:

\[
[P] = E \left[ (\{ q \} - \{ \hat{q} \}) \cdot (\{ q \} - \{ \hat{q} \})^T \right] = \left( [\phi]_{m\times N}^{T} \cdot [\phi]_{N\times m} \right)^{-1} \cdot [R]^{-1}
\]  

(2.13)

where \([ R ]\) is the covariance intensity matrix of the measurement noise. The matrix \([ Q ]\) is called Fisher Information Matrix.

Furthermore, the matrix \([ R ]\) can be simplified as \([ R ] = \sigma^2 \cdot [I]\) for the condition that the sensor noise is equally distributed and non-correlated between the sensors. Then the Fisher Information Matrix \([ Q ]\) can be simplified as:

\[
[Q]_{m\times m} = \frac{1}{\sigma^2} \cdot [\phi]_{m\times N}^{T} \cdot [\phi]_{N\times m}
\]  

(2.14)
The best estimate of \( \{ q \}_m \) requires the minimisation of the matrix \([P]\) which results from the maximisation of the matrix \([Q]\) and thus the maximisation of a matrix:

\[
[A]_{m \times m} = [\phi]_{m \times N}^T \cdot [\phi]_{N \times m}
\]

The matrix \([A]\) can be considered as a sum of the contributions of each DOF:

\[
[A]_{m \times m} = \sum_{i=1}^{N} \lambda_i \cdot [\psi]_{m \times m}
\]

where \([\phi]_{m \times m} = [\phi_i, \phi_{i+1}, \ldots, \phi_m] \) is a row vector corresponding to the \( i^{th} \) DOF. Adding or subtracting a DOF will add or subtract the information to/from the information matrix. If the initial sensor location candidate set contains all the DOFs in the FE model, the mode shapes in the matrix \([\phi]_{N \times m}\) are linearly independent, and thus the matrix \([A]\) must be of full rank.

If only \( m \) \((<N)\) modes are considered, there will be some redundant DOFs which can be removed and will not affect the rank of the matrix \([A]\). According to the different optimisation criterion, there are several ways to judge which DOFs are redundant.

2.2.1.4.1 Method based on the Effective Independence Distribution

Kammer developed a way to divide the redundant DOFs from the others [35] through the eigen-solution of the matrix, \([A]\):

\[
([A]_{m \times m} - [\lambda]([I]))[\psi] = 0
\]

where \([\lambda]\) and \([\psi]\) are the eigenvalue and corresponding eigenvector matrices of the matrix \([A]\). Because the matrix \([A]\) is positive symmetric, the eigen-vectors in the matrix \([\psi]\) are orthogonal and can be considered as \( m \) orthogonal vectors in an \( m \)-dimension space.

\[
[\psi]^T \cdot [A] \cdot [\psi] = [\lambda]_1, \text{ and } [\psi]^T \cdot [\psi] = [I]
\]

The product of \([\phi][\psi]\) can be considered as the projection of the mode shape vector matrix \([\phi]\) onto the \( m \)-dimension space presented by the vectors in the matrix, \([\psi]\). Taking the squared value of each item in the product \([\phi][\psi]\), we can get a matrix in which each element represents the contribution of one DOF to one mode. If weighted by the inverse of the corresponding natural frequency of the mode, each element in the matrix will have all directions in the \( m \)-dimension space of equal importance. Adding up all elements corresponding to one DOF in the matrix results in a vector in which each element presents one DOF’s contribution to all modes of interest. This vector is referred to as the effective independence distribution vector of the candidate sensor set, and is denoted as \( \{E_d\} \). The element with the largest value in the vector \( \{E_d\} \) represents the
DOF that contributes most to the rank of the matrix $[A]$, and thus should be retained. The element with the smallest value in the vector represents the DOF that has the least contribution to the rank of the matrix $[A]$, and thus should be removed.

By repeating the process of removing the DOFs in the initial sensor location set with the least contributions to the rank of the matrix, $[A]$, and forming a new sensor location set, the number of DOFs can be reduced to the number which is set according to the requirement for the testing configuration.

The DOFs retained from this process are of importance to the rank of the mode shape matrix, and thus to the linear independence of the incomplete mode shapes. Some DOFs may be vital to this. If a DOF, which corresponds to the element of the largest value in the vector $\{E_d\}$ for a candidate sensor location set, is removed from that candidate sensor location set, the retained DOFs may not be able to make the mode shape to be linearly independent.

By examining the process of calculating the effective independence distribution vector, $\{E_d\}$, an easier way to do this can be found as follows:

The partition of the FE model eigenvector matrix, $[\phi]_{N\times m}$, is used to form the matrix, $[A]$:

$$[A]_{m\times m} = [\phi]^{T}_{m\times N} [\phi]_{N\times m}$$  \hspace{1cm} (2.19)

From equation (2.18)

$$[A]^{-1}_{m\times m} = [\psi]^{T}_{m\times N} \cdot [\lambda]^{-1}_{m\times m} \cdot [\psi]_{m\times N}$$  \hspace{1cm} (2.20)

The so-called Prediction Matrix, $[E]$, can be generated as:

$$[E]_{N\times N} = [\phi]_{N\times m} [A]^{-1}_{m\times m} [\phi]^{T}_{m\times N} = [\phi]_{N\times m} ( [\phi]^{T}_{m\times N} [\phi]_{N\times m} )^{-1} [\phi]^{T}_{m\times N}$$  \hspace{1cm} (2.21)

Taking the diagonal elements in the matrix $[E]$, the vector $\{E_d\}$, referred to as the effective independence distribution vector of the candidate sensor set, is obtained [35, 36, 37, 73].

Kammer developed this method [35], and considered the method in the cases of models with noise [40], measurements with noise [37], and sequentially assembled structures [36].

2.2.1.4.2 Method based on maximisation of the determinant of the Fisher Information Matrix

The determinant of the Fisher Information Matrix indicates the amount of information in the data retained at the reduced set of sensor locations. Kammer presented this idea [35, 40], but he did not propose a method to realize it. Yao et al [93] developed a program based on the idea of maximising the determinant of the Fisher Information Matrix. The program uses the genetic algorithm. The results presented in
[93] showed a slight improvement over the results from the effective independence distribution vector method. However, the results from this method were still sub-optimal in the sense of getting the maximum determinant of the Fisher Information matrix.

Park et al [74] developed a method to remove some DOFs from the candidate sensor location set in an iterative procedure in order to get the maximum determinant of the Fisher Information Matrix after removing DOFs. Several DOFs can be deleted at each iteration process and the number of DOFs that can be deleted can also be determined by the method.

Let $[A]$ be the matrix formed by using equation $[2.15]$ for all candidate sensor locations, and $[A_r]$ the matrix for the retained sensor locations:

$[A] = [\phi_0]^T [\phi_0]$  
$[A_r] = [\phi_m]^T [\phi_m]$  

(2.22)

where $[\phi_0]$ and $[\phi_m]$ are the eigenvector matrix for all candidate sensor locations and eigenvector matrix for the retained sensor locations, respectively. The singular value decomposition on the matrix $[\phi_0]$ can be represented as:

$[\phi_0] = [U_0] \cdot [\Sigma_0] \cdot [V_0]^T$  

(2.23)

The left singular vector matrix, $[U_0]$, can be expressed as:

$[U_0] = \begin{bmatrix} [U_0]_m \\ [U_0]_d \end{bmatrix}$  

(2.24)

where $[U_0]_m$ and $[U_0]_d$ correspond to the retained and deleted DOFs, respectively. Then the normalised determinant variation of the matrix, $[A]$ after deleting several DOFs in the candidate sensor locations can be expressed as:

$\frac{\text{Det}([A]) - \text{Det}([A_r])}{\text{Det}([A])} = 1 - \text{Det} \left( I - [U_0]_d^T [U_0]_d \right)$

$= \sum_{i=1}^{n} p_i$  

(2.25)

where

$p_i = \frac{1}{i} \left[ \text{tr} \left( B^i \right) - p_i \text{tr} \left( B^{i-1} \right) - \cdots - p_{i-1} \right]$  

(2.26)

$B = [U_0]_d \cdot [U_0]_d^T$

If $p_1$ equals a small value, $\rho$, less than 1, equation (2.25) can be simplified, after neglecting higher order terms.

$\frac{\text{Det}([A]) - \text{Det}([A_r])}{\text{Det}([A])} \approx p_1 = \text{tr} (B) = \sum_{k=1}^{n_d} [u_0]_k [u_0]_k^T$  

(2.27)

$n_d$ is the number of the deleted DOFs. $[u_0]_k$ is a row vector in the left singular vector
matrix \([U_0]\) which corresponds to a deleted DOF.

From equation (2.25) the smaller the value of \(p_i\), the larger the determinant of \([A_i]\) is. Thus, the DOFs with smaller values of \([u_0,k][u_0,k]^T\) among all DOFs in the candidate sensor locations might be deleted.

From equation (2.26) \(|p_i| \leq \rho|\) when \(\rho\) is a small number less than 1. Then:

\[
\frac{\text{Det}([A]) - \text{Det}([A_i])}{\text{Det}([A])} = \sum_{i=1}^{n} p_i \leq \sum_{i=1}^{n} |p_i| \leq \frac{\rho}{1 - \rho} \tag{2.28}
\]

When an allowance on the variation of the determinant of the matrix \([A]\) is given, the allowance on the value of \(\rho\) can be determined, and then the number of DOFs deleted in the iteration can be determined by checking the trace of the matrix \(B\). If, in one iteration, the trace of the matrix \(B\) is greater than the allowance value of \(\rho\), the number of deleted DOFs, \(n_{dl}\), should be decreased until the trace of the matrix \(B\) is less than the allowance value of \(\rho\).

By this method, the determinant variation of the matrix \([A]\) in each iteration step is kept as small as a previously set value, and the determinant of the matrix \([A_i]\) for the final sensor location set may achieve a maximum or a sub-maximum value.

**2.2.1.4.3 Effective Independence – QR decomposition**

Schedlinski and Link developed a method using QR decomposition for sensor location selection [84]. The method is based on the QR decomposition of the eigenvector matrix and to localise a subset of structural DOFs as sensor locations such that the linear independence of the mode shapes to be measured is maximised.

For a matrix \([A]\), the QR decomposition is given by:

\[
[A] \cdot [E] = [Q] \cdot [R] \tag{2.29}
\]

\([A] \in R^{m,N}, [E] \in R^{N,N}, [Q] \in R^{m,m}, [R] \in R^{m,N}\)

where \([Q]\) is a matrix orthogonal in columns, \([Q]^T [Q] = [I]\). \([R]\) is an upper triangular matrix with decreasing diagonal elements. \([E]\) is a permutation matrix that exchanges columns of \([A]\) in order to get maximum values on the diagonal elements of the \([R]\) matrix.

Due to the characteristic of the QR decomposition that the diagonal elements of the \([R]\) matrix are in descending order, the first columns of the matrix \([A] \cdot [E]\) can be considered as the ‘most linearly independent’ columns. This characteristic can be used for selecting sensor locations.

For using this method, the mode shape data, which are predicted from an FE model and with \(m\) modes and \(N\) DOFs, are arranged into a matrix such that the columns of the matrix correspond to the DOFs while the rows of the matrix correspond to the
modes. After performing QR decomposition on the mode shape matrix, the first $m$ columns of the matrix $[E]$ indicate the DOFs on which the $m$ modes will be the most linearly independent. If the number of the final set of sensor locations for the test is required to be greater than the number of modes, the columns in the mode shape matrix corresponding to the first $m$ columns in the matrix $[E]$ should be set to zero. Then QR composition can be performed again to get other DOFs that are the most linearly independent among the remaining DOFs. This procedure can be repeated until all sensor locations in the final set are selected.

2.2.1.4.4 Comparison of the EI methods

The basic idea of the Effective Independence family of methods is to make the experimental mode shapes distinguishable from each other. From this point of view, all three methods described in the previous sections can be used to select the measurement DOFs by which the experimental mode shapes are represented, if the number of the measurement DOFs is equal to the number of modes of interest and if the dynamic properties of the initial FE model are not too far away from those of the actual structure.

In test planning practice, the number of required measurement DOFs ($n$) is usually greater than the number of the modes of interest ($m$). The DOFs selected after the first $m$ DOFs should contribute as much as possible to the linear independence of the eigenvectors.

The first EI method, based on the Effective Independence Distribution, removes DOFs that have the least contribution to the modes of interest. There are no complicated calculations in this method. If there are two DOFs that are constrained in the initial FE model to have the same eigenvector values for each mode, the result from the method might contain only one of the two DOFs, although this result is not guaranteed when the number of measurement DOFs is greater than the number of modes (when $n>m$).

The second EI method, maximising the determinant of the Fisher Information matrix, removes DOFs that would not change the determinant significantly. If there are two DOFs that are constrained in the initial FE model to have the same eigenvector values for each mode, the method should definitely remove one of the two DOFs. However, in the approximate version of the method using the SVD of the eigenvector matrix, the final list of the measurement DOFs might contain both of the two DOFs.

The third EI method applies a QR decomposition to the transpose of the eigenvector matrix and retains the first $m$ DOFs indicated in the first $m$ columns of the permutation matrix that are considered to be the ‘most linearly independent’. When the number of DOFs, $n$, is greater than the number of modes, $m$, the selection of other DOFs after the $m^{th}$ DOF would not take account of the relation between the first $m$ DOFs and those that are left in the initial candidate DOF list. Therefore, if there are two DOFs that are constrained in the initial FE model to have the same eigenvector values for each
mode, it is very likely that one of these two DOFs is selected in the first group of $m$
DOFs and the other is selected in the second group.

From the above analysis, the EI distribution and maximising determinant
methods seem better, in the author’s opinion, than the QR method.

2.2.2 Optimum excitation point(s) selection

The purpose of optimum excitation point(s) selection is to determine the
excitation point(s) in the structure such that the all modes in the frequency range of
interest can be excited and the measurements on these modes can achieve a high
observability.

2.2.2.1 Optimum Driving Point(s) ($ODP$) and Non-Optimum Driving Point(s)
($NODP$)

This method was developed and used by Imamovic [31] for giving a guide to
test engineers in selecting excitation point(s) in the modal test. The basic idea of the
method is to select point(s) with large mode shape amplitudes for the modes of interest
and avoiding the nodal points or any points near the nodal line.

The $ODP$ parameter of a DOF, $i$, is defined as:

$$ ODP(i) = \prod_{r} \| \phi_{ir} \| $$

(2.30)

where $m$ is the number of modes of interest. All DOFs that have values of the $ODP$
parameter close to zero are close to or on a nodal line and, consequently, should not be
considered as possible excitation positions. In contrast to the above-mentioned DOFs,
the positions that have large values of the $ODP$ parameter should be considered as
possible excitation positions. The $ODP$ coefficient is also called Eigenvector
Component Product (EVP) [61, 10].

The Non-Optimum Driving Point ($NODP$) defines another parameter for each
DOF that describes how close each DOF is to any nodal line of any mode in a specified
frequency range. The $NODP$ parameter of the $i^{th}$ DOF is defined as:

$$ NODP(i) = \min_{r} \| \phi_{ir} \| $$

(2.31)

The $NODP$ parameter for a given DOF does not average out all modal constants
but shows only the lowest-valued one amongst all the modes in the specified frequency
range. The values of this parameter do not show how good a particular position is as a
driving point, but they show how bad each DOF is as a driving point. So, if a DOF has
a low value of the $NODP$ parameter, this means that that DOF is not suitable as possible
driving position. In contrast, if a DOF has high value of $NODP$, it does not mean that
this DOF is automatically the best driving position. The $NODP$ values show which
points are non-optimum rather than assessing an average value for all modes in
consideration.
ODP and NODP parameters of a DOF can also be divided by the various parameters of Average Driving DOF Residues to form the ODP-based and the NODP-based parameters. In this way, not only positions of nodal lines but also average response levels of the individual DOFs are taken into account for selecting excitation point(s). If a DOF has a high average response level, then this DOF may have relatively low values of the parameters from ODP-based methods and NODP-based methods. So ODP- and NODP-based methods can provide a reference to help selecting driving point. Any point selected will not be at or close to a nodal line of any modes in the specified frequency range. Also, it will not have so large an average response level as to impose unwanted effects on the excitation process.

2.2.2.2 QR decomposition

Schedlinski et al used QR decomposition not only for selecting sensor locations but also for selecting excitation point(s) [84]. From an initial FE model, the excitation force matrix \([F]\) can be obtained for the modes of interest:

\[
[F] = [M] \cdot [\Phi] \in R^{N \times m}
\]  

(2.32)

where \(N\) is the number of DOFs which are possible excitation points, \(m\) is the number of the modes of interest. By performing QR decomposition (equation (2.29)) on the transpose of the matrix \([F]\), the first \(s\) columns of the matrix \([E]\), the permutation matrix in equation (2.27), correspond to the DOFs which can be used as the excitation points. ‘s’ here is the number of the excitation points which is determined by the number of shakers that will be used in the test.

2.2.3 Selective sensitivity approach

The sensitivity matrix with respect to the uncertain parameters is used to select the reference modes and the DOFs that should be measured in the test. Gola et al [25] proposed this method.

The sensitivity matrix of an FE model to the uncertain parameters \((p_1, p_2, \ldots, p_n)\) is denoted as \([S]\).

\[
[S] = \begin{bmatrix}
\frac{\partial \omega_1}{\partial p_1} & \cdots & \frac{\partial \omega_1}{\partial p_i} & \cdots & \frac{\partial \omega_1}{\partial p_n} \\
\frac{\partial \omega_2}{\partial p_1} & \cdots & \frac{\partial \omega_2}{\partial p_i} & \cdots & \frac{\partial \omega_2}{\partial p_n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\partial \omega_i}{\partial p_1} & \cdots & \frac{\partial \omega_i}{\partial p_i} & \cdots & \frac{\partial \omega_i}{\partial p_n} \\
\frac{\partial \phi_{11}}{\partial p_i} & \cdots & \frac{\partial \phi_{1i}}{\partial p_i} & \cdots & \frac{\partial \phi_{1i}}{\partial p_n} \\
\frac{\partial \phi_{1i}}{\partial p_i} & \cdots & \frac{\partial \phi_{1i}}{\partial p_i} & \cdots & \frac{\partial \phi_{1i}}{\partial p_n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\partial \phi_{Nm}}{\partial p_i} & \cdots & \frac{\partial \phi_{Nm}}{\partial p_i} & \cdots & \frac{\partial \phi_{Nm}}{\partial p_n}
\end{bmatrix}
\]  

(2.33)

When the inverse sensitivity methods or other penalty function methods are used in model updating the parameter covariance depends on both the covariance of the
measurement quality and the covariance of the sensitivity matrix. The covariance of the sensitivity matrix can be written as:

\[
[cov] = [Cond]^{-1} = ([S]^T \cdot [W^{-2}] \cdot [S])^{-1}
\]  

(2.34)

where \([W]\) is a weighting matrix. In the best condition, the diagonal elements of the matrix \([cov]\) will become smaller when more modes are taken into account. The mode selection procedure based on the method is to indicate the modes to be taken into account to obtain a value of diag[\(cov\)] as close as possible to the lower limit. An alternative way is to omit the off-diagonal elements in the matrix \([cov]\). The diagonal elements of the matrix \([Cond]\) should get the maximum when the suitable modes are selected.

After the target modes have been selected, the measurement DOFs can also be selected by the use of the selective sensitivity. The diagonal elements of the matrix \([Cond]\) can be considered as the sum of the contribution of each mode and each DOF in the FE model. The contribution of the \(j^{th}\) DOF to the value of the \(i^{th}\) diagonal element can be written as:

\[
F_j = \sum_{r=1}^{m} \left( \frac{\partial \phi_{jr}}{\partial p_i} \right)^2
\]  

(2.35)

The DOF with the highest \(F_j\) value will be selected as a measurement DOF. After choosing the DOFs with the highest \(F_j\) values with respect to all uncertain parameters, the diag[\(cov\)] should be calculated again. If the diag[\(cov\)] can not be improved any more by adding another DOF, the DOFs selected can be considered as the final sensor location set.

The selective sensitivity approach is based on the prior selection of updating parameters in the initial FE model. Only when the parameters are chosen correctly, does this approach provide a prior test analysis of which the results from the test can help updating to the maximum.

2.2.4 Concluding remarks on test planning

Test planning techniques have been developed in the past decade, and are still under development. From viewpoint of FE model validation, the purpose of the test planning, or the purpose of the modal test, can be different from that when the test result will be used directly in the design of new products or modification of products. In the process of FE model validation, the role of the experimental data will be used (i) to verify the initial FE model, (ii) to compare (correlate) the predicted dynamic property data, and (iii) to update the FE model.

The methods for selecting excitation point(s) described in this chapter are intended to enable the test results to contain all the modes of interest. For a quasi-
axisymmetric structure, how to get all orthogonal modes excited out is still a problem in a single input multi-output test.

The sensor locations will define the information obtained from the test. The optimal sensor location methods described in this chapter are mainly for the purpose of (i) making the number of measurements to be as few as possible, and (ii) getting a good observability from the test data. Most updating algorithms available today have to use correlation results to select updating parameters and modify the parameters selected. According to this requirement, the optimal sensor location algorithms based on the Effective Independence (2.2.1.4) are suitable for being used in the process of model validation. However, that what is the sufficient information from test needed to validate an FE model is still a problem for both updating and test planning.

2.3 FE model verification

2.3.1 Definition of FE model verification

Natke gave a definition of model verification [69, 70], as “Verification is the reconstruction of the information used for modelling within the modelling process. … The verified model can represent the used data only within the applied quality criterion.” In this definition, model verification is to check the model with the respect to the information used for constructing the model.

Zimmerman et al presented another definition [94, 95], as “Model verification is considered to be the quantitative part of validation and verification. It is a substantiation of the accuracy of the model with reference to experimental data. This includes the quantitative estimation of parameter values required to ‘tune’ the model.” In this definition, model verification quantitatively proves and/or improves the model.

From these quotations, it can be seen that the definition of model verification is, in the moment, not as clear as that for model updating or model validation.

In the process of model validation as shown in Figure 1-1, the procedure of model verification is performed before a model is subjected to the procedure of model updating. The purpose of model verification is to make sure a model subjected to model updating is suitable for being updated by a normal updating program. Thus, FE model verification, of which the definition used in this thesis will be given in Chapter 4, does not modify the parameters in an FE model but debugs the model. According to the analysis on the three kinds of error in FE models, the procedure of model verification deals with discretisation errors and configuration errors.

2.3.2 Stress recovery and discretisation error estimating

The displacement predictions from an FE model usually have a precision higher than the predicted stress within elements of the model. In static analysis, the estimated
stress or strain energy is more important than the estimation of displacement in most applications. In dynamic analysis, errors in eigenvalue prediction are directly connected with errors in the strain energy prediction. Many papers dealing with discretisation errors proposed some methods for estimating errors of stress predicted by FE models.

The FE model technique with the displacement method uses shape functions to interpolate the displacement within elements (and rotating phase angles within high-order elements). The derivatives of these functions are usually not inter-element continuous. If estimating stress by differentiating the displacement estimation within an element, the estimated stresses on the nodes of the element have the lowest accuracy among all points within the element.

Hinton and Campbell [28] used the least-squares method for local and global smoothing of discontinuous finite element functions. Zienkiewicz and Zhu [98] presented a simple discretisation error estimator that is based on the global smoothing method. Besides this estimation method, Zienkiewicz and Zhu [98] also used an empirical correction multiplying factor to improve the error estimation.

Zienkiewicz and Zhu proposed another recovery method in [99, 100]. In the proposed method, the new recovered stress value on a node belongs to a polynomial expansion of the same order of the shape function for the model. The constants of the polynomial function are derived by curve-fitting the stress values on the optimal locations [4] on the patch of elements surrounding the assembly node considered. It was shown in [99, 100] that this recovery method was super-convergent. The convergent rate of the recovered nodal stress as mesh size was refined was one or two orders higher than the initial stress estimation in many types of element.

Mottershead et al [64] proposed a method for smoothing stress by redistribution of nodal loads. In this method, the stiffness matrix of FE model and the eigenvector of a mode is used to calculate the "equivalent point loads". The stress resultant at any point within an element is represented by a linear function. The stress within the element can be obtained from the redistributed stress. In this paper, it was pointed that the estimated strain energy was not necessarily going to be closer to the exact strain energy than the initial estimation from the FE model when regarding the assessment of errors in eigenvalues. The new estimated eigenvalues could be calculated from the Reyleigh quotient.

2.3.3 Discretisation error and model updating

Mottershead et al [65] presented a procedure to update FE models with discretisation errors. For a design model that will be further used after updating, an 'updating model' with a finer mesh than the design model is produced. The 'updating model' is fully convergent in the frequency range of interest. That means that there is no discretisation error in this model for predicting the dynamic properties of the structure.
in the frequency range of interest. There are two loops in the procedure, an inner loop and an outer loop. In the inner loop, the differences between the predictions from the design model and those from the updating model, which are the discretisation errors of the design model, are introduced into the experimental data. The adjusted experimental data are used to update the design model. In the outer loop, the obtained updating parameter values from the inner loop are used to modify the updating model. If the predictions from the modified updating model are converged to the initial experimental data, the whole updating process can be successfully ended. If not, the inner loop is re-entered.

From this procedure, the obtained updating parameter values can keep their physical meaning while the updated design model still has discretisation errors in its predictions.

2.3.4 Reducing discretisation errors

In order to reduce discretisation errors in an FE model, a finer model is generally needed. The error estimators developed with stress recovery methods can be used to determine the locations in the model where the estimations have large discretisation errors and thus further refinement is needed [98, 64].

Ahmadian et al proposed a method that does not refine the mesh size of an FE model but uses new mass and stiffness formulations to minimise the discretisation errors [3]. For doing this, some parameters are added to the formulations. By comparing the equations of motion for a continuous system and those for an FE model, the values of the parameters can be determined to minimise the difference between the equations.

2.3.5 Configuration errors and model updating

Some authors have mentioned uncertainties in the configuration of FE models [47, 51]. In [51], it was reported that the performance of model updating on a coarse model ended without improvement on correlation because the "distance" between the model predictions and experimental data was too large. However, few papers are found for dealing with configuration errors in the process of model validation.

2.3.6 Concluding remarks on model verification

Although model verification is an important step in the process of model validation, the technique for this step is still being developed, and is much behind the development of model updating methods. Many papers have been published dealing with discretisation errors whereas few are found concerning configuration errors.

Most methods for estimating discretisation errors are based on stress recovery. However, the most important properties of FE models for structural dynamics are eigenvalues, eigenvectors, and/or FRFs. In the model validation process, it is required not only to detect the existence and to estimate amplitude of discretisation errors in an
FE model, but also to minimise the errors or to reduce the effects of the errors on the dynamic properties of the model in the frequency range of interest.

A further requirement in the process of model validation is that minimisation of discretisation errors should be achieved without change of the mesh size of FE models. For an industrial structure, the number of DOFs in an FE model is usually very large. It is difficult, if not impossible, to rebuild another model with a finer mesh. Thus, the mesh size of the model has to be kept unchanged while reducing the effects of discretisation errors on the dynamic properties predicted by the model. For this requirement, any method of reducing discretisation errors by refining model mesh can be used when building the initial model, but not in the process of model validation.

There are few papers dealing with configuration errors in FE models. The possible reasons for this are as follows. (i) It is difficult to clearly define configuration errors of FE models. There are too many forms of configuration error. For example, both unsuitable types of element and loss of the key features of the structure in a model can cause inaccurate predictions from the model. (ii) In most situations, there are configuration errors and as well as discretisation errors and parameter errors. All of these errors deteriorate, more or less, the accuracy of the dynamic property predictions from FE models.

2.4 FE model updating methods

The purpose of FE model updating is to adjust an initial FE model so that either the modified model can reproduce a set of measured data or a modification is provided with a definite physical meaning to a deficient model by using test measurement. From these two different kinds of purpose, updating methods can be divided into two categories: (i) reference basis methods, by which the model after being updated will reproduce the reference data exactly; and (ii) design parameter methods, by which only the parameters in the initial model will be modified and the modified parameters will retain their physical meanings. The reference dynamic property data used in model updating can be both/either measured modal data and/or measured FRFs.

2.4.1 Direct updating methods using modal data (reference basis methods)

Most direct methods for FE model updating were introduced in the late 1970s and the early 1980s. Because of their advantages, these methods are still being used and developed. Due to advances in test measurement instruments, some disadvantages of the methods are being overcome.

2.4.1.1 Lagrange multiplier method

The basic idea of the Lagrange multiplier method is that one set of parameters among the measured modes, the analytical mass matrix and the analytical stiffness matrix are considered to be correct. The other two sets will be updated separately by
minimising a penalty function that is constrained by Lagrange multipliers [5, 7, 8, and 88].

In [39], Kenigdbuch and Halevi considered two problems: sequential mode shapes data and stiffness data updating, and sequential mass and stiffness data updating. The measured natural frequencies are considered as correct in both problems. For the optimisation criterion, general weighting matrices are used instead of the mass matrix. In this way, any prior knowledge on the error location and engineering practice can be incorporated into the updating process.

2.4.1.2 Error matrix method

The error matrix method estimates directly the error in the mass and stiffness matrices. It was developed in the 1980s and there are several papers presented for this method [79, 27, 29, 46, and 55].

In [55], the test data from the application of a laser vibrometer were used in updating an FE model of a cantilever-stepped plate. After using a smoothing technique on the test mode shape data, the updating procedure using the error matrix method obtained an improved model.

2.4.1.3 Eigenstructure assignment method

The eigenstructure assignment method is developed from control theory. In this method, a feedback matrix is to be found so that a looped system has the desired eigenvalues and eigenvectors. Then the feedback matrix perturbs the stiffness and damping matrices to obtain an updated model.

There are some limits on this method. For example, the mode shape data must be measured on all DOFs in the initial FE model. A special effort should be made to retain the symmetry properties of the stiffness and damping matrices. However, there are some papers which contribute to this method recently to enable the method to be used with less limitation [82, 96].

2.4.1.4 Other direct methods

The inverse eigenvalue method is addressed mainly by Starek and Inman [80, 81, 83, and 85]. The method is useful for low-order lumped parameter models of the type used for machines rather than for structures. However, there are some practical issues still unresolved, such as the effects of noise and incomplete test data.

The matrix mixing method was developed in the 1970s. The measured mode shape data and analytical eigenvectors are combined together to obtain a complete eigenvector set. Although the method can reproduce the measured mode shape data by the updated FE model, the stiffness and mass matrices returned from the updating procedure will be fully populated, and thus lose their physical meaning [58].
2.4.1.5 Conclusions on direct updating methods

The updated model from each of these methods can reproduce the reference data exactly and there is no iteration in the updating procedure. If the differences between the test data and the predictions from the initial FE model are small, direct updating methods can improve the FE model.

The test data used in the direct updating methods should be measured on all DOFs in the initial FE model. In most cases, the number of measured DOFs is much smaller than the number of DOFs in the FE model although the application of laser vibrometers will improve this situation. An alternative is to reduce the size of the initial FE model or to expend the test data, although both reduction and expansion will bring some erroneous information into the process.

Direct updating methods are all highly sensitive to noise in the measured data. Nowadays, measurements of natural frequencies can have high enough accuracy for model updating. However, the accuracy of measurements of mode shapes cannot usually satisfy the purpose of direct FE model updating.

Until the measurements on mode shapes get a significant improvement in both the accuracy and the number of measurement points, the application of direct methods on the structural FE model updating will be limited.

2.4.2 Iterative methods

There are many iterative methods for FE model updating. Some methods use only modal data or response functions. Some can use both of them. However, all iterative methods are involved in solving non-linear equations. Optimum solutions of the equations are achieved by iteration.

2.4.2.1 Penalty function methods

2.4.2.1.1 Basic idea of the methods

The objective of penalty function methods is to improve the correlation between the measured data and the analytical model. The basic idea concerned with all penalty function methods is as follows.

Suppose that the structural dynamic properties from the experiment are denoted as \( \{ Z_X \} \), which can be natural frequencies, mode shapes, frequency response functions, or a combination of these properties. The dynamic properties predicted by the FE model are denoted as \( \{ Z_i \} \) at the \( i^{th} \) iteration. \( \{ Z_X \} \) and \( \{ Z_i \} \) can also be considered as functions of the structure’s parameters, \( \{ p \} \). If modifying the parameters can update the initial FE model, the vectors \( \{ Z_X \} \) and \( \{ Z_i \} \) can be written as:

\[
\{ Z_X \} = \{ Z(\{ p \}) \}
\]

\[
\{ Z_i \} = \{ Z(\{ p_i \}) \}
\]  

(2.36)
The differences between the measured data and the analytical model can be expressed as:

\[
\{\delta Z\} = \{Z_x - Z_i\} = \{Z(p) - Z(p_i)\} = \{Z(p_i + \delta p_i) - Z(p_i)\}
\]  
(2.37)

When there are only small differences of the parameters between \(p\) and \(p_i\), by using derivatives of the properties with respect to the parameters, the difference between \(Z_x\) and \(Z_i\) can be approximated as:

\[
\{\delta Z\} = \{Z(p_i + \delta p_i) - Z(p_i)\} \approx \left[\frac{\partial Z}{\partial p}\right] \cdot \{\delta p_i\} = [S] \cdot \{\delta p_i\}
\]
(2.38)

where \([S]\) is the sensitivity matrix of the structure and contains the derivatives of the properties of interest with respect to the parameters. The elements in the sensitivity matrix are generally non-linear functions with variables of frequency and geometric position. When an FE model is being updated, the elements of the sensitivity matrix, \([S]\), at the \(i^{th}\) iteration will be different from what they are in the previous iterations.

The penalty function methods try to obtain the change of the parameters \(\{\delta p\}\) such that some kind of residual – the penalty function (or the objective function as appears in some papers) – is minimised.

From equation (2.38) a penalty function can be established as:

\[
J(p) = (\{\delta Z\} - [S] \cdot \{\delta p_i\})^T \cdot ([S] \cdot \{\delta p_i\})
\]
(2.39)

This penalty function is the norm of the difference between \(\{\delta Z\}\) and \([S] \cdot \{\delta p_i\}\). The vector \(\{\delta p_i\}\) is the approximate solution of equation (2.37) at the \(i^{th}\) iteration.

In order to avoid ill-conditioning when the penalty function is applied to get optimal solution of equation (2.37) or (2.38) weighting coefficients can be used in establishing the penalty function. One common kind of weighted penalty function is as follows:

\[
J(p) = ([\delta Z] - [S] \cdot [W] \cdot [\delta p_i])^T \cdot ([\delta Z] - [S] \cdot [W] \cdot [\delta p_i])
\]
(2.40)

The solution to the equation (2.36) or (2.37) by minimising the penalty function is an approximate one. The optimum solution of \(\{\delta p\}\) can only be obtained by an iterative procedure.

By differentiating equations (2.39) and (2.40) with respect to the parameter change \(\{\delta p_i\}\), and setting the result to be zero, the optimum solution of equation (2.37) can be obtained:

\[
\{\delta p_i\} = ([S]^T \cdot [S])^{-1} \cdot [S]^T \cdot [W] \cdot \{\delta Z\}
\]
(2.41)

or

\[
\{\delta p_i\} = ([S]^T \cdot [W] \cdot [S])^{-1} \cdot [S]^T \cdot [W] \cdot \{\delta Z\}
\]
(2.42)

When the number of measurements is greater than the number of parameters, the
parameter change, \( \{ \delta p_r \} \), can be calculated. However, the condition of the matrix \( ([S]^T[S]) \) is not always good enough to get stable inverse performance, even when some kind of weighting coefficients are added to the penalty function. Another problem that may occur is rank deficiency. If the information contained in the measurements is less than necessary to solve the parameters, the matrix \( ([S]^T[S]) \) is not of full rank.

### 2.4.2.1.2 Penalty methods using modal data

There are several forms of penalty function which use modal data.

Cobb et al. used a penalty function [101] as:

\[
J(p) = \sum_{r=1}^{m} a_r \left( \frac{(A_r^{\lambda_r}) - 1}{(A_r^{\lambda_r})} \right)^2 + \sum_{r=1}^{N} \sum_{i=1}^{m} b_{ir} \left( (A_r^{\phi_i}) - (A_r^{\phi_i}) \right)^2
\]

(2.43)

In this penalty function, the positive coefficients \( a_r \) and \( b_{ir} \) allow for individual weightings. The differences between the natural frequencies (real eigenvalues) of the experimental data and FE model predicted data are normalised. In this way, the difference of the weight on the natural frequencies can be scaled out. The different accuracies of natural frequency measurements and mode shape measurements are taken into account by the different weighting coefficients \( a_r \) and \( b_{ir} \).

Link et al. use the penalty function as [59, 52]:

\[
J(p) = ([\delta Z]^T \cdot [W] \cdot [\delta Z]) + ([\delta p]^T \cdot [W_p] \cdot [\delta p])
\]

(2.44)

In this penalty function, \([W]\) is a weighting matrix for the difference between the experimental data and FE model predictions, \([W_p]\) is another weighting matrix, which is for limiting the change of parameters in the updating step. This penalty function allows the updating parameters to be weighted in a selectable way. If the elements on the row and the column that correspond to a parameter in the vector \( \{ \delta p \} \) are set to be zero, the parameter will remain unchanged in the updating step. Some literature [68, 62, 59, and 52] presented proposals for the selection of the weighting matrix \([W_p]\). In [59 and 52], the \([W_p]\) was set as:

\[
[W_p] = \text{diag}([w_p] \cdot \text{diag}([S]^T \cdot [W] \cdot [S]))
\]

(2.45)

This weighting matrix adjusts the constraint to the parameters according to the sensitivity of the parameters. If the sensitivity of some parameters approach zero, the corresponding coefficients \( W_p \) approach zero and the parameters remain unchanged.

When modal data are used in FE model updating by a penalty function method, there is a problem connected with the amount of information from the experimental data. Measured natural frequencies have a higher accuracy than measured mode shapes have. In most cases, eigenvalue sensitivities are greater than eigenvector sensitivities. In this sense, more weighting should be put on the differences between the natural
frequencies of the experimental data and those predicted from the FE model. However, information from eigenvalues only is not enough for updating an FE model when the number of uncertain parameters in the model is greater than the number of modes measured. Sometimes, the information from mode shapes can also help to determine error locations. From this point of view, how to make the information from mode shape data more useful and sensitive to the updating process has become a critical problem in the penalty function methods when using modal data.

2.4.2.1.3 Penalty methods using FRF data

When FRF data are used for updating an FE model, the sensitivities of FRFs to the uncertain parameters should be calculated in order to obtain the relevant sensitivity matrix, \([S]\). The sensitivities of the FE model frequency response functions can be obtained from:

\[
\frac{\partial [H]}{\partial p} = -[H] \cdot \frac{\partial [B]}{\partial p} \cdot [H]
\]

(2.46)

where \([B(\omega)] = [H(\omega)]^{-1} = -\omega^2 [M] + j\omega[C] + [K]\).

There are two different types of penalty function used in model updating methods with FRF data [87, 19, 34]. The first one is based on the receptance residual \(\{\varepsilon\} = \{H_A\} - \{H_X\}\). This is often called output error. Its penalty function is

\(J(p) = (\{H_X\} - \{H_A\})^T (\{H_X\} - \{H_A\})\).

The second one is based on the force-balanced residual \(\{\varepsilon\} = [B_A] \cdot \{H_X\} - \{I\}\), the so-called equation error. Its penalty function is

\(J(p) = ([B_A] \cdot \{H_X\} - \{I\})^T ([B_A] \cdot \{H_X\} - \{I\})\).

By applying the sensitivity of the receptance with respect to the uncertain parameters, one can obtain:

\(\{H_A\} \equiv \{H_A\}_{i} + \frac{\partial \{H\}}{\partial p} \cdot \delta p\)

(2.47)

By applying the sensitivity of the dynamic stiffness with respect to the uncertain parameters, one can obtain:

\([B_A] \equiv [B_A]_{i} + \frac{\partial \{B\}}{\partial p} \cdot \delta p\)

(2.48)

Differentiating the penalty functions and setting the results to be zero, two optimal solutions can be obtained as follows:

\(\delta p = \left( \left\{ \begin{array}{c} \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial p} \end{array} \right\} \right)^{-1} \left( \left\{ \frac{\partial H}{\partial p} \right\} \left\{ \{H_X\} - \{H_A\}_{i} \right\} \right)\)

(2.49)
and 

\[
\delta p = \left( \{H_x\}^T \left[ \frac{\partial B}{\partial p} \right]^T \frac{\partial B}{\partial p} \{H_x\} \right)^{-1} \{H_x\}^T \left[ \frac{\partial B}{\partial p} \right]^T \left( \{I\} - \{B_A\}, \{H_x\} \right) \tag{2.50}
\]

The subscript \(i\) denotes the \(i^{th}\) iteration step.

Both penalty functions and corresponding solution methods have their advantages and drawbacks. By using receptance residuals in the penalty function, the comparison between test data and analytical data is made in a direct way. However, the analytical receptance, when as a function of the updating parameters, is non-monotonic. This may lead the updating process to a local minimum rather than to the global minimum. Another problem of this method concerns the stability of the solution. Due to the same reason that the analytical receptance is a non-monotonic function of updating parameters, the solution of the updating equation cannot be guaranteed to converge. Using dynamic stiffness in the penalty function has the advantage that the dynamic stiffness is monotonic and smooth for most updating parameters. The solution process of this method is far more stable than the method using receptance in the penalty function. However, the calculation of \(\{\epsilon\} = \left( [H_A] \cdot \{H_x\} - \{I\} \right)\) will result in relatively smaller values compared with the elements in the matrix \([B_A]\) and vector \(\{H_x\}\). Thus the method is very sensitive to noise in experimental FRF data, which may lead the solution to diverge.

Lammens et al proposed another residual for penalty function [44], which they called the indirect receptance residual.

\[
\{\epsilon\} = [H_A](\{B_A\} \cdot \{H_x\} - \{I\}) \tag{2.51}
\]

Although this residual is the same in value as the receptance residual, the sensitivity of the dynamic stiffness is used to obtain the optimum solution of the uncertain parameters. The corresponding penalty function is:

\[
J(p) = ([H_A] \cdot ([B_A] \cdot \{H_x\} - \{I\}))^T ([H_A] \cdot ([B_A] \cdot \{H_x\} - \{I\})) \tag{2.52}
\]

By differentiating the penalty function and setting the result to be zero, the optimal solution is as follows.

\[
\delta p = \left( [H_A]^T \frac{\partial [B_A]}{\partial p} \{H_x\} \right)^+ \left( [H_A^T], -\{H_x\} \right) \tag{2.53}
\]

where \((\cdot)^+\) denotes the pseudo-inverse of a matrix. The penalty function and its solution (equations (2.52) and (2.53)) keep the advantages of the dynamic stiffness residual. With replacement of \(\{\epsilon\} = \left( [B_A], \{H_x\} \right)\) by \(\{(H_A)_i, \{H_x\}\}\), the numerical condition of the calculation gets much better. It is worth noting here that the above equation is almost the same as that for the solution on the penalty function based using the receptance residual. When putting equation (2.46) into equation (2.49) and comparing
with equation \(2.53\) it is clearly to see the difference between these two equations. In equation \(2.53\) both the experimental FRFs, \(\{H_X\}\), and the analytical matrix, \([H_A]\), are used for calculation of \(\frac{\partial H}{\partial p}\), rather than using only the analytical matrix, \([H_A]\), as in equation \(2.49\). This improves the convergence of the updating process.

Lammens et al used this method on updating an FE model of an engine sub-frame [45, 53]. Although the initial FE model was very coarse and the experimental FRFs contained some noise, the updated model showed a good correlation with the experimental data. However, the updating results were not unique when different updating frequencies were selected.

One of the main advantages for using experimental FRF data in updating is that the damping data can be determined and updated from the process. When updating an FE model with modal data, the eigenvalues and mode shape data will be used. In most cases, the eigenvalues are not so sensitive to the damping data as the FRF data in the vicinity of resonance points are. Most of papers dealing with damping matrix updating use methods with FRF data [76, 48, 91].

When using experimental FRF data in updating, it seems that selecting enough points along the frequency range can help to solve the incompleteness problem of the measurements on both measure DOFs and measured modes. This is because the FRFs, in theory, contain the information of all modes, and the information taken from the different frequency points will not be completely correlated with each other. However, the practice of the methods gives some unsatisfactory results and reveals that the number of measured DOFs and the selection of the frequency points are critical in the performance of FE model updating when using FRF data. The points in the immediate vicinity of a resonance should be avoided because the information at these points is dominated by one or two modes and the damping effect will take a dominant role. Although the FRF data at the anti-resonant points should also not be selected because of low signal/noise ratio, the practice of this is impossible due to the shifting of anti-resonant frequency points from one FRF to another.

2.4.2.1.4 Concluding remarks on the penalty function methods

There are two key issues for all updating methods based on penalty functions: (i) the establishment of the penalty function; (ii) the methods used in optimising the penalty function.

The penalty function selected will determine how and where the updating performance will converge. The optimisation method mentioned in this section is the least-squares estimate method in the sense that the method is used in most of the literature. Other estimate methods can also be used here.

Not all parameters in an FE model can be updated by the penalty function.
methods in one procedure. It is important to select updating parameters correctly. There are some methods for selecting updating parameters and most of them are also connected with updating methods.

Because the penalty function methods modify only some parameters in FE models, the methods are also called ‘design parameter’ methods. The parameters selected as updating parameters can be any parameters in an FE model: stiffness, mass, or geometry data, etc. If the parameters selected are not the elements in the global matrices of mass, stiffness, or damping, the updating results will still have physical meaning. This is the main advantage of the methods over the direct updating methods.

All penalty functions are non-linear functions with respect to the updating parameters. The solution from one step of updating calculation will not obtain the final optimal solution in most cases. An iterative process is unavoidable for the methods using penalty functions and, as a consequence, the convergence problem appears in all these methods. Adding a weighting matrix or matrices in the penalty function will improve the problem but cannot guarantee the process to be convergent.

2.4.2.2 Minimum variance method

In most circumstances the experimental data will have a non-negligible amount of noise and real industry structures have, also, some randomness in their structural properties. By repeating the same experiment on several test pieces (identical in drawings but with tolerance differences in manufacturing), or repeating the same experiment several times on an individual structure, the experimental data obtained will exhibit statistical variations on the measured FRFs and estimated modal parameters. On the other side, the analysts have estimation on the error of the parameters in the initial FE model that they establish. Minimum variance methods take account of all these effects to improve the FE model prediction.

Suppose in the $i^{th}$ iteration of a model updating procedure, the updating parameters are \{$p_i$\} and the variance matrix of the parameters is $E[p_i \cdot p_i^T] = [V_i]$. The objective of the minimum variance methods is to minimise the variance of the parameters \{$p\$} at each iteration step.

For any prediction output \{$Z_i\$} at the $i^{th}$ iteration step and measurement output \{$Z_X\$}, if the difference between them is small, \[\{Z_X\} - \{Z_i\} = [S] \cdot (\{p\} - \{p_i\})\] (2.54)

\([S]\) is the sensitivity matrix of the output \{$Z\$} with respective to the updating parameters. The updating parameter estimate \{$p_{i+1}\$} can be represented in terms of the estimate at the previous iterative step: \[\{p_{i+1}\} - \{p_i\} = [T] \cdot (\{Z_X\} - \{Z_i\})\] (2.55)
The variance of parameter estimate \( \{p_{i+1}\} \) at the \((i+1)^{th}\) iteration step is:

\[
[V_i] = E[p_{i+1} \cdot p_i^T]
\]

\[
= [V_i] + ([D_i] - [V_i] \cdot [S_i]^T) [T]^T + [T] \cdot ([D_i] - [V_i] \cdot [S_i]) + [T] \cdot [V_{\infty}] \cdot [T]^T
\]

where \([D_i] = E[p_i \cdot \varepsilon]^T\) is the correlation between the parameter estimate and the measurement noise, \(\{\varepsilon\}\), and \([V_{\infty}] = E[(Z_X - \{Z_i\}) \cdot (Z_X - \{Z_i\})^T]\) is the output error variance.

By minimising the variance of the parameter estimate at the \((i+1)^{th}\) iteration step, matrix \([T]\) can be obtained as:

\[
[T] = ([V_i] \cdot [S_i]^T - [D_i]) \cdot [V_{\infty}]^{-1}
\] (2.56)

So, the parameter estimate \(\{p_{i+1}\}\) can be obtained:

\[
\{p_{i+1}\} = \{p_i\} + ([V_i] \cdot [S_i]^T - [D_i]) \cdot [V_{\infty}]^{-1} (\{Z_X\} - \{Z_i\})
\] (2.57)

When there is no correlation between the measurement error and the parameter estimate, the new parameter estimate can be calculated as:

\[
\{p_{i+1}\} = \{p_i\} + ([V_i] \cdot [S_i]^T) \cdot ([S_i] \cdot [V_i] \cdot [S_i]^T + [V_i])^{-1} (\{Z_X\} - \{Z_i\})
\] (2.58)

This will happen at the first iteration step, where the measured data and the parameter estimate are statistically independent. In subsequent iterations, the parameters are updated by use of the measurements and the independence will not be true.

The main advantage of the minimum variance method is the ability to use the information not only from experiment but also from the prior knowledge on the uncertain parameters. The amount of information from the experiment can be less than the number of uncertain parameters [19].

Lindholm et al used a Bayesian statistics approach in FE model updating. Prior knowledge of the uncertain parameters and the error estimate of the experimental data were utilised to add stability to the solution process and to improve the quality of the parameter estimate obtained [56, 57, 49]. They analysed four different types of experiment data. The comparing results showed that the sine-dwell response and frequency response functions were suitable for using the statistics methods of FE model updating.

Friswell et al proposed a method which took uncertainties of FE models and noise in measurement as bounds on uncertain parameters [21]. The updating parameter values were obtained as the ones that solve the following min-max problems:

\[
\min_p \{ \max \{(S + \delta S) P - (Z + \delta Z)\} : \|\delta S\|_e \leq \eta_s, \|\delta Z\|_e \leq \eta_d \}\}
\] (2.59)
2.4.3 Other methods for FE model updating

Besides the methods mentioned above, there are some other methods reported in the literature.

Using genetic algorithms for updating is one of the methods becoming popular now. The algorithms can help to search for the global maximum or global minimum. In this sense, the algorithms can be used for the minimisation of penalty functions. Larson and Zimmerman [60] developed a program using the genetic algorithm. They used their program to update an FE model of a six-bay truss with 25 DOFs. The updating results showed an improvement of the model after updating, even when the experimental mode shape data were added with some noise. Carlin and Garcia [12] studied the parameters for running the algorithm and used it to locate damage in structures. They compared the results from the genetic algorithm method with those from other methods, and the conclusion from these comparisons showed that the genetic algorithm method was able to correctly identify the damage cases with no ‘phantom’ damage appearing in other locations.

Using Perturbed Boundary Condition (PBC) test data in FE model updating is another new method. Although the concept of PBC test was proposed in 1980s, its use in model updating began only in the early 1990s [9, 11, 43, and 54]. The main purpose of using the PBC test data is to overcome one of the major problems in model updating – the lack of sufficient information – and to improve the ill-conditioning problem in updating equations. One of the problems concerned with the use of PBC test data is how to combine different test data sets into one updating procedure. Although it is simple to just increase the number of equations, the huge size of data will make the computation difficult. This problem can be addressed by condensing the PBC measurement database into a more complete dynamic model of the system [54]. Another problem concerned with the use of PBC test data is how to establish the different configurations for PBC tests in order to get the maximum amount of information from the tests. Yang and Brown [91] proposed a PBC configuration selection function to assist in selecting an optimal set of PBC configurations.

Instead of using several sets of experimental data as from PBC tests, Ibrahim et al used multi perturbed analytical models for performing model updating [102, 103]. Although the information of the structure comes from only one set of experimental data, the increase of the amount of information from the perturbed analytical models helps to improve the condition of updating equations.

In recent years, some researchers have used antiresonance frequencies together with natural frequencies, mode shapes or FRFs for performing model updating [13, 14, 66, 67, 77]. Natural frequencies and mode shapes of some modes of a structure can provide information of the structural dynamic properties of the structure in the
frequency range of these modes. Each FRF contains, in theory, the information of all modes of the structure. However, noise in practical measurements of FRFs makes the updating results dependent to the choice of the frequency points in updating performance. An antiresonance frequency of an FRF is a structural property which depends upon all modes of the structure although the sensitivity of the antiresonance frequency is dominantly contributed from the nearest eigenvalues and eigenvectors [66]. The frequencies of antiresonances of an FRF are different from those of other FRFs. Each antiresonance frequency of measured FRFs can provide independent information in model updating performance. This helps to make updating equations to be over-determined.

2.4.4 Concluding remarks on FE model updating methods

Although there are many updating methods, none of them can be said to be mature and reliable for validating FE models of all types of industrial structure. There are some successful cases, but they only show that the success is case-dependent and the solutions are not unique.

The incompleteness of experimental data, both in the experimental frequency range (number of modes) and measured DOFs, determines that only some of the parameters in the FE model can be updated. How to select the best parameters for updating in the FE model should be determined not only by mathematical calculation but also, and perhaps even more importantly, from the modelling engineers’ judgement.

Different reference data (natural frequencies, mode shapes, frequency response functions, and/or antiresonances) have different sensitivities to the updating parameters. It is possible to define some new residuals that combine all the reference data in such a way that makes use of the advantages of individual reference data and compensates its disadvantage by other reference data.

The form of the objective functions (penalty functions) will influence the updating performance in a great extent. A good form of objective function should be monotonic and smooth with respect to the updating parameters to be studied.

Once the kind of residual and the form of objective function are established, the optimisation method used plays a key role in finding the global minimum. The least-squares estimation method is just one of those available. When the objective function is not smooth, it is often a local minimum rather than the global minimum that this method possibly obtains. This is also one of the reasons for the non-uniqueness of the FE model updating results even when the updating parameters are selected correctly. Some modern estimation methods, such as the genetic algorithm, may improve the searching results from the optimisation process.
2.5 Summary of literature review

The literature on test planning, FE model verification and model updating has been reviewed on this chapter.

For test planning, the most important issue is what should be provided from experimental data when the data are used to validate an FE model for structural dynamics. All methods for test planning mentioned in this chapter are for the purpose of ensuring that all modes in the frequency range of interest should be measured and all measured mode shapes should be observable (or linearly independent). However, the information from the experimental data that fulfils these requirements may not be sufficient to update an FE model. The Perturbed Boundary Condition (PBC) test method is a good approach to alleviate this problem.

For model verification, there are several methods for getting more accurate stress estimates than the predictions from FE model calculations and they are helpful to estimate the discretisation errors in FE models. Few papers have been found to tackle the problem with configuration errors and their effects on model validation.

For model updating, to find a suitable kind of residual and a suitable optimisation method is of the most important for achieving satisfactory results. Most methods for model updating mentioned in this chapter use either modal data or frequency response function data. Both of these two types of data have their advantages that bring benefits to model updating.
Chapter 3

Finite Element Model Updating

3.1 Introduction

Finite element (FE) model updating is one of the procedures in the process of model validation for structural dynamic analysis. In this procedure, some parameters of initial FE models are modified in order to minimise the discrepancies between the model predicted dynamic properties and the reference properties (usually, measured data).

Before an FE model is subjected to an updating procedure, some information must be prepared. The first set of information includes the reference data for model updating – certain structural dynamic properties that are expected to be predicted by the updated FE model with an acceptable accuracy. The reference data are usually provided by a set of measured data that are obtained from tests performed on the actual structure. The second set of information is an understanding of the model – the information about the uncertainties within the model that may cause the discrepancies in dynamic properties. This information can be obtained by examining the modelling process, and/or from comparing model predictions with the reference data. The acquisition of this information is critical for selecting updating parameters and for preparing the settings for model updating.

The FE model updating procedure is usually undertaken by a computer program that reads in all the relevant information and performs numerical calculations to determine the parameter values that could minimise the discrepancies.

Although many papers have been published in the area of model updating, the methodology of model updating is still being developed. Some basic issues for model updating need to be explored further. For example, what kind of information from a structure is necessary for updating the FE model of a structure for dynamic analysis? What will happen if an FE model with all three kinds of error (see Section 1.1.1 of this
thesis) is subjected to an updating procedure?

In this chapter, a theoretical background for model updating will be described in section 3.2 and followed by some case studies in section 3.3. From these case studies, a perspective of model updating (section 3.4) will be given in which the questions in the last paragraph will be answered.

3.2 Theoretical background for model updating

3.2.1 Model updating procedure

An FE model updating procedure can be considered as a series of operations performed on an FE model. The inputs to the operations are the initial FE model and the reference data. The output from the operations is an updated model.

When an analyst undertakes a model updating procedure, he/she has to decide the settings for the procedure. In Figure 3-1, three main considerations are listed: (i) Updating parameters, (ii) Updating methods, and (iii) Weighting factors.

![Figure 3-1 Diagram for model updating](image-url)
3.2.1.1 Updating parameter selection

For performing a model updating procedure, every parameter in an FE model can be considered as a candidate updating parameter. In an FE model for a continuous structure, the number of the independent parameters is equal to the number of DOFs of the model. This number is usually much greater than the number of the independent information provided in the reference data. However, from the understanding of the model, some parameters may have a high confidence while others may not. Furthermore, some uncertain parameters can be grouped together according to their physical meaning, and not all uncertain parameters in the model will significantly affect the structural dynamic properties in the frequency range of interest. Thus, the number of updating parameters used in the updating procedure must be much less than the number of DOFs in the model.

Many papers propose methods for selecting updating parameters [19, 20, 41, 42, 30]. Most methods are based on sensitivity analysis and some numerical manipulations of the sensitivity matrix. The parameters selected by these methods are not unique and their results are sometimes case sensitive [92].

Although it was not a part of this thesis to study the choice of updating parameters, the author is aware of the importance of this choice for the success of a model updating procedure and of the fact that considerable care is needed when choosing the updating parameters.

In the case studies of model updating in this thesis, in which the reference data are obtained from reference models (rather than practical test data), the updating parameters will be selected according to the known difference between the reference models and the analytical models. In the case studies of updating industrial models, the updating parameters will be selected using both sensitivity analysis and engineering judgment. The details of the selection method will be described in each case study.

3.2.1.2 Updating method and weighting factors

There are so many model updating methods published that it is impossible to compare all these methods and to determine which one is the best.

Although an updated FE model is expected to predict structural dynamic responses in certain situations, the reference structural dynamic properties in model updating procedures are usually those from representative modal tests. FRF-based model updating methods use residuals between experimental FRFs and model-predicted FRFs in updating equations. Since an experimental FRF is a measured quantity, errors due to modal parameter extraction could be avoided. A large amount of test data from frequency points can be used to alleviate the ill-condition problem of the updating equations. Modal data based model updating methods use residuals of eigenvalues and
eigenvectors to form the right hand side in updating equations. Modal data have a clear physical meaning in describing structural dynamic properties. After extraction of modal data from experimental FRFs, the obtained natural frequencies can have a high precision. The precision of mode shape data is usually considered lower than that of natural frequencies. However, including mode shape data in updating equations may help to increase the amount of information and to alleviate the ill-condition problem of the updating equations. The anti-resonance frequencies may also be used in model updating, although the method using anti-resonance frequencies needs to be further developed for practical application.

The essential part of a model updating program is to establish the format of the updating equation and the solution for this equation. Iterative methods based on inverse sensitivity seem more encouraging than others [32] because the output of these methods have physical meaning and this is critical for its application in industry. In the case studies in this thesis, the inverse eigen-sensitivity method with eigenvalues/eigenvectors is used. The theoretical background of this method will be described in the next sub-chapter.

The addition of weighting factors to the updating equation can significantly affect the updating results. For modal data based model updating methods, the accuracy of measured natural frequencies and mode shapes is the main consideration for determining the weighting factor values. Different weighting factors are used in the case studies of model updating in the thesis and will be discussed in each case study.

3.2.1.3 Model updating procedure

When an initial FE model is subjected to model updating together with a set of reference data, it cannot be guaranteed that the modified model from the first trial of a model updating procedure will give an acceptable prediction for the dynamic properties of structure. There are so many choices for the options in the model updating procedure. It is also extremely difficult to find a set of updating parameters by which updating the initial model can get the best updated model. Thus, a general model updating procedure is undertaken with two loops as shown in Figure 3-2.

In the inner loop is a model updating session which is usually undertaken by running a computer program with settings for updating calculations. In each iteration within the session, the updating parameter values are modified gradually. At the end of a session after several iterations, a modified model is obtained. The modified model may not be satisfactory in the light of the physical meaning of modified parameter values and/or the improvement of correlation between the model predictions and the reference data. Then, the outer loop of the procedure will be undertaken, where the previous updating sessions are assessed in order to get a better understanding of the model and the structure. From the assessment, new settings for running the updating
program will be established and a new session of model updating will be undertaken. When a modified model satisfies the requirements of keeping the physical meaning of the modified parameters and of improving the correlation between the predictions from a modified model and the reference data, this modified model can be considered as an updated model. It is often the case that several updating sessions with different settings for the model updating program will be undertaken to achieve an updated model.

Most papers dealing with model updating focus on the methods for the inner loop in Figure 3-2 – a session of model updating. The outer loop, assessing model updating performance and proposing new settings for new model updating sessions, has been hardly documented. The assessment of model updating performance is mainly based on experience and engineering judgment. In some model updating case studies in this thesis, the whole process of model updating with several sessions will be described in detail in order to illustrate the relationship between the inner loop and outer loop.

![Figure 3-2 FE model updating procedure](image-url)
3.2.2 Theoretical background of the inverse eigen-sensitivity method

3.2.2.1 Sensitivities of an eigenvalue and an eigenvector

The inverse eigen-sensitivity method \cite{101, 59, 52} uses modal data for updating FE models. The sensitivity of eigenvalues and eigenvectors to the updating parameters should be calculated in order to obtain the sensitivity matrix \([S]\).

From the matrix form of the equation of motion of an FE model:

\[
[M]\{\ddot{x}\} + [K]\{x\} = \{0\}
\]  

\[(3.1)\]

The eigen solution of this equation for the \(i^{th}\) mode is:

\[
[K]\{\phi_i\} - \lambda_i \cdot [M]\{\phi_i\} = 0
\]

\[(3.2)\]

where \(\lambda_i\) and \(\{\phi_i\}\) are the eigenvalue and the mass normalised eigenvector of the mode respectively. Performing partial differentiation of the above equation to an updating parameter \(p_j\), it is shown that:

\[
\frac{\partial K}{\partial p_j} \phi_i + K \frac{\partial \phi_i}{\partial p_j} - \lambda_i \frac{\partial M}{\partial p_j} \phi_i - \lambda_i M \frac{\partial \phi_i}{\partial p_j} = 0
\]

\[(3.3)\]

Left-multiplying with the transpose of the eigenvector gives:

\[
-\frac{\partial \lambda_i}{\partial p_j} \phi_i^T M \phi_i + \phi_i^T \left(\frac{\partial K}{\partial p_j} - \lambda_i \frac{\partial M}{\partial p_j}\right) \phi_i + \left(\phi_i^T K - \lambda_i \phi_i^T M\right) \frac{\partial \phi_i}{\partial p_j} = 0
\]

\[(3.4)\]

Because \(\phi_i^T M \phi_i = 1\) and \(\phi_i^T K - \lambda_i \phi_i^T M = 0\),

\[
\frac{\partial \lambda_i}{\partial p_j} = \phi_i^T \left(\frac{\partial K}{\partial p_j} - \lambda_i \frac{\partial M}{\partial p_j}\right) \phi_i
\]

\[(3.5)\]

This is the formula for the eigenvalue sensitivity of the \(i^{th}\) mode to the \(j^{th}\) updating parameter. From this formula, it can be seen that the sensitivity of an eigenvalue to an updating parameter can be calculated from the eigenvalue, the corresponding eigenvector, and the sensitivities of the stiffness and mass matrices to the updating parameter.

Rearranging equation \[(3.3)\] gives:

\[
(K - \lambda_i M) \frac{\partial \phi_i}{\partial p_j} = \frac{\partial \lambda_i}{\partial p_j} M \phi_i - \left(\frac{\partial K}{\partial p_j} - \lambda_i \frac{\partial M}{\partial p_j}\right) \phi_i
\]

\[(3.6)\]

This is an equation for the eigenvector sensitivity. Because the coefficient matrix of the equation is singular, there are an infinite number of solutions, when a solution

* For convenience, in the following algebraic manipulations in this chapter, some matrices and vectors will be simplified without brackets when there is no confusion.
exists at all.

If the $i^{th}$ mode is not a double mode, there is only one eigenvalue that is equal to $\lambda_i$ among all eigenvalues of equation (3.1). In this case, the solution of equation (3.6) can be obtained as follows [71].

Suppose the position of the element with the maximum absolute value in the eigenvector, $\phi_i$, is $l$. Let all the elements in the $l^{th}$ row and the $l^{th}$ column of the coefficient matrix of equation (3.6) be set to zero, but the diagonal element in the $l^{th}$ row be set to one. The corresponding $l^{th}$ element in the right hand side of equation (3.6) is also set to zero. Then an equation with a non-singular coefficient matrix is established.

$$ l \rightarrow \begin{bmatrix} k_{i1} - \lambda_i m_{i1} & 0 & k_{i3} - \lambda_i m_{i3} \\ 0 & 1 & 0 \\ k_{31} - \lambda_i m_{31} & 0 & k_{33} - \lambda_i m_{33} \end{bmatrix} \begin{bmatrix} \xi_{ij} \\ \xi_{ij} \\ \xi_{ij} \end{bmatrix} = \begin{bmatrix} b_l \\ 0 \\ b_3 \end{bmatrix} \iff l \tag{3.7} $$

The solution for $\{\xi_{ij}\}$ obtained from the above equation can be considered as one of the solutions for equation (3.6).

The eigenvector, $\phi_i$, is a null space of the coefficient matrix of equation (3.6). Thus any solution for equation (3.6) can be written in the form of:

$$ \frac{\partial \phi_i}{\partial p_j} = \xi_{ij} + \phi_i \cdot c_{ij} \tag{3.8} $$

Considering the orthogonality property of the eigenvector, $\phi_i$,

$$ \phi_i^T M \phi_i = 1 \tag{3.9} $$

and partial-differentiating this equation with respect to the updating parameter, $p_j$, it can be obtained that:

$$ 2\phi_i^T M \frac{\partial \phi_i}{\partial p_j} + \phi_i^T \frac{\partial M}{\partial p_j} \phi_i = 0 \tag{3.10} $$

Combining equations (3.8) and (3.10), it is shown that:

$$ 2\phi_i^T M (\xi_{ij} + \phi_i \cdot c_{ij}) + \phi_i^T \frac{\partial M}{\partial p_j} \phi_i = 0 \tag{3.11} $$

The coefficient $c_{ij}$ can be obtained by the following formula:

$$ c_{ij} = -\frac{1}{2} \frac{\partial M}{\partial p_j} \phi_i \tag{3.12} $$

With $c_{ij}$ from the above formula and $\{\xi_{ij}\}$ from equation (3.7), the eigenvector sensitivity of the $i^{th}$ mode to updating parameter $p_j$ can be calculated from equation (3.8).

Compared with the formula for eigenvalue sensitivity, the eigenvector sensitivity
calculation is more complicated and takes more computing time. For the solution of equation [3.7] the computing time depends on the number of DOFs in the FE model. From equation [3.7] it can also be seen that, for the eigenvector sensitivity of a mode to all updating parameters, the coefficient matrix of the equation will remain unchanged. If the sensitivity of an eigenvector is calculated for one updating parameter after another, the equation has to be solved several times although the coefficient matrix is same. When there are \( m \) modes in the updating objective and \( l \) updating parameters, this equation needs to be solved \((m \times l)\) times for all modes and all updating parameters.

Forming equation [3.7] for the sensitivity of an eigenvector to all \( l \) updating parameters in matrix format gives:

\[
(K - \lambda_i M) \begin{bmatrix} \frac{\partial \phi_i}{\partial p_1} & \frac{\partial \phi_i}{\partial p_2} & \cdots & \frac{\partial \phi_i}{\partial p_l} \end{bmatrix} = [B_{i1}, B_{i2}, \cdots, B_{il}] 
\]

(3.13)

where vector \( B_i = \begin{bmatrix} \frac{\partial K}{\partial p_j} - \lambda_i \frac{\partial M}{\partial p_j} - \frac{\partial \lambda_i}{\partial p_j} M \end{bmatrix} \phi_i \). Following the same procedure described above, a matrix with each column as the sensitivity of the eigenvector to an updating parameter can be obtained. In this way, solving a linear equation with a coefficient matrix of possible huge size is only performed once for an eigenvector.

### 3.2.2.2 Inverse eigen-sensitivity method and weighting factors

The basic inverse eigenvalue sensitivity method minimises the discrepancy between the reference eigenvalues and the analytical eigenvalues predicted by the initial FE model. The penalty function for the method is:

\[
J(p) = \sum_{i=1}^{m} ((\lambda_{\lambda_i})_i - (\lambda_{x_i})_i)^2
\]

(3.14)

where \((\lambda_{\lambda_i})_i\) and \((\lambda_{x_i})_i\) are the analytical eigenvalue from the FE model and the reference eigenvalue of the \(i^{th}\) correlated mode pair, respectively. Each analytical eigenvalue can be presented as a function of updating parameters. Using a Taylor series to expand the function and retaining only the first order terms, the penalty function is as follows:

\[
J(p) = \sum_{i=1}^{m} ((\lambda_{\lambda_i})_i(p) - (\lambda_{\lambda_i})_i)\]

\[
\approx \sum_{i=1}^{m} \left( (\lambda_{\lambda_i})_i \bigg|_{p=p_0} + \frac{\partial (\lambda_{\lambda_i})_i}{\partial p} \delta p - (\lambda_{x_i})_i \right)^2
\]

(3.15)

where \(p_0\) represents the initial values of the updating parameters, \((\lambda_{\lambda_i})_i \bigg|_{p=p_0}\) is the analytical eigenvalue predicted by the initial FE model, and \(\delta p\) is the increment of the values of the updating parameters, which should be determined in the updating
procedure. Partial-differentiation of the penalty function with respect to $\delta p$, and setting it to zero, it can be obtained that:

$$\frac{\partial J(p)}{\partial (\delta p)} \equiv 2 \sum_{i=1}^{m} \left( (\lambda_i)_{p=p_0} + \frac{\partial (\lambda_i)}{\partial p} \delta p - (\lambda_i)_{i} \right) \frac{\partial (\lambda_i)}{\partial p} = 0$$

(3.16)

which results in:

$$\left( \lambda_i \right)_{p=p_0} + \frac{\partial (\lambda_i)}{\partial p} \delta p - (\lambda_i)_{i} = 0 \quad (i = 1, 2, \cdots m)$$

(3.17)

Writing the above equation in the matrix form gives:

$$[S_{\lambda}] [\delta p] = [R_{\lambda}]$$

(3.18)

where $[S_{\lambda}]$ is the sensitivity matrix of the eigenvalues with respect to all updating parameters. Each element in the sensitivity matrix $[S_{\lambda}]$ is derived as described in the previous sub-section. $\{R_{\lambda}\} = \{\lambda_{X}\} - \{\lambda_{X}\}$ is the residual vector between the reference eigenvalues and FE model predicted eigenvalues.

Usually, eigenvalues are non-linear functions with respect to almost all kinds of updating parameter. In equations (3.15) and (3.16), the eigenvalues are represented with only the first order of $\{\delta p\}$. Thus, the solution from equation (3.18) cannot, in general, minimise the penalty function of equation (3.14). The minimisation of the penalty function can only be achieved in an iterative way of solving equation (3.18). Figure 3-3 shows a block diagram of the method.

Figure 3-3 Block diagram of inverse eigenvalue sensitivity method

In order to avoid the possibility of a divergent situation in the iterative cycles for obtaining the optimum solution of the updating parameters, the solution of the updating equation is usually multiplied by a release factor, $C_{Rel}$, before it is added to the initial parameters. The amplitude of the release factor should be smaller than 1.0.

When applying the inverse eigenvalue sensitivity method, the reference and FE
model predicted eigenvectors are used to correlate the modes. However, only the discrepancies between the eigenvalues of each correlated mode pair are used in the optimisation process.

The inverse eigenvector sensitivity method minimises the discrepancies between the reference eigenvectors and FE model predicted eigenvectors. Following the same manipulation as for the inverse eigenvalue sensitivity method, the updating equation for the basic inverse eigenvector sensitivity method is:

\[
S_\phi \delta p = R_e
\]  

(3.19)

where \( S_\phi \) is the sensitivity matrix of the eigenvectors with respect to the updating parameters. Figure 3-4 shows a block diagram of the method.

![Figure 3-4 Block diagram of inverse eigenvector sensitivity](image)

From the last sub-section, which describes sensitivity calculations, the sensitivities of eigenvalues and eigenvectors depend on the eigenvalues and eigenvectors as well as the mass matrix and stiffness matrix. When some parameters in the FE model are changed, not only the structural dynamic properties but also the sensitivities of the properties to the updating parameters are changed. Thus, in each iteration of a model updating session, the eigenvalues/vectors of the FE model and their sensitivities to the updating parameters need to be calculated.

In equations (3.18) and (3.19) either eigenvalues or eigenvectors are used in the updating equations. We can put both of them and weighting factors into an updating equation as follows:

\[
W_S \cdot [S] \cdot \delta p = W_R \cdot \{R\}
\]  

(3.20)

where \( \delta p \) is a vector of updating parameters, \([S]\) is a sensitivity matrix, \( \{R\} \) is the residual vector, and \( W_S \) and \( W_R \) are weighting factor matrices applied to the sensitivities and the residuals respectively. With different combinations of eigenvalues, eigenvectors and weighting factors in an updating equation, the inverse eigen-sensitivity
method has several variations. Some of them are listed in Table 3-1.

<table>
<thead>
<tr>
<th>Method:</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inverse eigenvalue sensitivity</td>
<td>( [S_\lambda] \cdot { \delta p } = { R_\lambda } )</td>
</tr>
<tr>
<td>Weighted inverse eigenvalue sensitivity</td>
<td>( [W_\lambda] \cdot [S_\lambda] \cdot { \delta p } = [W_\lambda] \cdot { R_\lambda } )</td>
</tr>
<tr>
<td>Inverse eigenvector sensitivity</td>
<td>( [S_\phi] \cdot { \delta p } = { R_\phi } )</td>
</tr>
<tr>
<td>Weighted inverse eigenvector sensitivity</td>
<td>( [W_\phi] \cdot [S_\phi] \cdot { \delta p } = [W_\phi] \cdot { R_\phi } )</td>
</tr>
<tr>
<td>Inverse eigenvalue and eigenvector sensitivity</td>
<td>( [S_\lambda \ S_\phi] \cdot { \delta p } = \begin{bmatrix} R_\lambda \ R_\phi \end{bmatrix} )</td>
</tr>
<tr>
<td>Weighted inverse eigenvalue and eigenvector sensitivity</td>
<td>( [W_\lambda \ W_\phi] \begin{bmatrix} S_\lambda \ S_\phi \end{bmatrix} \cdot { \delta p } = [W_\phi] \begin{bmatrix} R_\lambda \ R_\phi \end{bmatrix} )</td>
</tr>
</tbody>
</table>

Table 3-1 Variations of inverse eigen-sensitivity methods

The methods mentioned in section 2.4 of this thesis can be easily represented in the form of equation (3.20). For example, for the penalty function proposed by Link et al [59, 52] (the penalty function proposed by Cobb et al [101] can be easily converted to the same form) is:

\[
J(p) = \{ \delta Z \}^T \cdot [W] \cdot \{ \delta Z \} + \{ \delta p \}^T \cdot [W_p] \cdot \{ \delta p \}
\]

(3.21)

Partial-differentiating the function with respect to the updating parameter vector \( \{ \delta p \} \) and converting to the form of equation (3.20) it can be obtained that:

\[
[W_p] = [W_\lambda] \left[ S_\lambda \right]^{-1} + [S_\lambda]^T [W] \\
[W_\phi] = [S_\phi]^T [W]
\]

(3.22)

The weighting factors are used to improve the solution of the updating equation. In most of cases using the inverse eigen-sensitivity methods, the number of the independent equations is greater than the number of the updating parameters. Setting different kinds of weighting factor in an updating equation will result in different solutions for the updating parameters. For example, if only eigenvalues are used in the updating equation and unit weighting is applied to all eigenvalues, the eigenvalue residuals of the modes with higher natural frequencies will give more contribution in numerical calculation to the updating results than do those with lower eigenvalues. If the inverse of the eigenvalue \( (W_{\lambda_i} = (\lambda_{\lambda_i})^{-1}) \) is applied to each mode, it results in the relative eigenvalue differences on the right hand side of the equation, and the eigenvalue residuals of all modes will have the same weight in the equation’s solution.

For eigenvectors, the different amplitudes between analytical eigenvectors
would cause different weighting effects between the modes. When the inverse of the square root of the norm of each eigenvector is used for weighting the eigenvector sensitivity and the eigenvector residual, each mode will have the same weight in the equation, although the weighting effect for each DOF may be different in this case.

In some cases, the modal scale factors ($MSF$) for some correlated mode pairs are not equal to or near unity. This can happen for the following reasons: (i) the unit systems for the reference (experimental) data set and the analytical data set are different, (ii) the calibration of measurement instrumentation is not perfect, and (iii), more commonly, the measurement of the point FRF has a relatively low accuracy. When the eigenvectors are used in the updating equation, all these factors will bring unwanted effects to the updating results. In order to avoid these effects, $MSFs$ should be included in the residual for eigenvectors. This means that, for the $i^{th}$ correlated mode pair:

$$\{R\}_i^{(MSF)} = MSF_i \cdot \{\phi_x\}_i - \{\phi_\lambda\}_i,$$

and

$$MSF_i = \frac{\{\phi_x\}_i^T \cdot \{\phi_\lambda\}_i}{\{\phi_x\}_i^T \cdot \{\phi_x\}_i}.$$  \hspace{1cm} (3.24)

Several FE models can be updated together. This is especially useful for updating an FE model with Perturbed Boundary Conditions (PBC) test data. Different boundary conditions make an FE model become several different models. However, they all have same updating parameters in an updating procedure. The updating equation for the updating procedure can be written as (e.g. for the inverse eigenvalue sensitivity only):

$$\begin{bmatrix} S_{\lambda}^{(1)} \\ S_{\lambda}^{(2)} \\ \vdots \\ S_{\lambda}^{(n)} \end{bmatrix} \cdot \{\delta p\} = \begin{bmatrix} R^{(1)}_\lambda \\ R^{(2)}_\lambda \\ \vdots \\ R^{(n)}_\lambda \end{bmatrix}.$$  \hspace{1cm} (3.25)

The superscripts in parentheses are the model numbers.

**3.2.3 Solution of updating equation**

The solution of an updating equation is to find a set of parameter values that can either satisfy the equation completely or minimise the given residuals. The selection of a method used to obtain the solution of an updating equation depends on the condition of the equation.

When the number of updating parameters of an updating equation is equal to the number of rows in the coefficient matrix (the product of the weighting factor matrix and the sensitivity matrix) and the rank of the coefficient matrix is equal to the number of updating parameters, the equation is determined. The solution of the equation can be
obtained by using the formula:

\[
\{\delta p\} = ([W_s][S])^{-1}[W_K]\{R\}
\]  

The solution \{\delta p\} satisfies equation (3.20) completely in each iteration of an updating session.

When the number of updating parameters of an updating equation is less than the number of rows in the coefficient matrix and the rank of the coefficient matrix is equal to the number of updating parameters, the equation is over-determined. The solution of the equation can be obtained by using the least-squares method:

\[
\{\delta p\} = \left( ([W_s][S])^T([W_s][S]) \right)^{-1}([W_s][S])^T[W_K]\{R\}
\]

This formula minimises the value of \|[W_s][S]\{\delta p\} - [W_K]\{R\}\| in each iteration of an updating session. At the end of the updating session, the value of \|[W_K]\{R\}\| approaches a local (or the global) minimum.

When the number of updating parameters of an updating equation is greater than the number of rows in the coefficient matrix or the rank of the coefficient matrix is less than the number of the updating parameters, the equation is under-determined. Solution of the equation in this case can also be obtained by using the least-squares method:

\[
\{\delta p\} = (([W_s][S])^T([W_s][S])([W_s][S])^T)^{-1}[W_K]\{R\}
\]

This formula minimises the value of \|[\delta p]\| and the solution \{\delta p\} satisfies equation (3.20) completely in each iteration of an updating session. However, this is not “the” solution, but simply one of many. At the end of the updating session, the value of \|[W_K]\{R\}\| approaches zero.

Equation (3.20) can also be solved using the SVD method. The SVD method represents a matrix \[A\] of dimension \(n \times m\) with three matrices:

\[
[A] = [U]^T[\Sigma][V]
\]

where the matrices \[U\] and \[V\] are orthogonal matrices. The matrix \[\Sigma\] is a diagonal matrix with its diagonal elements in descending order as follows:

\[
[\Sigma] = \begin{bmatrix}
\sigma_1 & & \\
& \ddots & \\
& & \sigma_l
\end{bmatrix}
\]

with \(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_l > 0\) and \(l \leq \min(n, m)\) is the rank of the matrix \[A\].
For equation (3.20), applying an SVD on the product of the sensitivity matrix and the weighting matrix $[W_S]$, the solution can be expressed as:

$$\{\delta p\} = [V]^T \left[\Sigma^{-1}\right] [U] \left[W_r\right] \{R\}$$

(3.31)

where

$$[\Sigma]^{-1} = \begin{bmatrix}
\sigma_1^{-1} & 0 \\
0 & \sigma_i^{-1}
\end{bmatrix}$$

(3.32)

With the SVD method, equation (3.20) can be solved in a unique form no matter if the equation is over-determined or under-determined.

### 3.2.4 Concluding remarks on the theoretical background of model updating

In this sub-chapter, the theoretical background for model updating has been discussed with the emphasis focused on the inverse eigen-sensitivity method. Although there are many other methods for model updating, the principles of most of these methods are the same as that have been discussed here.

In section 3.2.1 a general model updating session and a model updating procedure with several sessions are described. When a computer program for model updating calculation is available, undertaking a model updating session is a matter of running the program with prepared settings for the program. Because the preparation of the settings for a model updating program cannot, at the moment, be independent from engineering judgement, the operation of the model updating procedure is not a black box.

The descriptions of model updating theory in section 3.2.2 from the formulae of the sensitivity calculation to the solution of updating equations, are purely mathematical. When this theory is applied to a purely mathematical model, it can be expected that an updating session will obtain a modified model that is exactly the same as the reference model. In the practice of updating models in industry, there are many uncertainties in the models and in the experimental dynamic properties which are used as the reference data. Thus, performing model updating in practice cannot follow a fixed procedure and the understanding of model updating problems from the engineer’s viewpoint may play a more important role than the pure mathematical formulae.

### 3.3 Case studies of FE model updating

In this sub-chapter, several case studies of model updating will be presented in detail in order to demonstrate each operation in an updating procedure. First group of case studies describe the practice of updating simple numerical models, each of these models has one or two kinds of error amongst parameter errors, discretisation errors and configuration errors. The FE models in the second group of case studies are from
industry – they are large in size and complicated in geometry and there are possibly all three kinds of errors in these models.

3.3.1 Case studies of model updating on simple numerical models

In order to examine the effects of different settings on model updating, the case studies in the first group use several variations of the inverse eigen-sensitivity method to update simple numerical models of plate. In each case study, reference structural dynamic properties are provided by an FE model, which is referred to as the reference model. The FE model that is to be updated in each case study is called the analytical model. The differences between the parameter values of the reference model and those of the analytical model are known before performing model updating and, thus, can be taken as the target of the model updating procedure.

Three cases will be described in this section. The analytical models in the cases have (i) parameter errors (Case 1), (ii) parameter errors and/or discretisation errors (Case 2), and (iii) parameter errors and/or configuration errors (Case 3), respectively. In the case studies for Case 1, there are two sets of reference data – with and without measurement noise.

3.3.1.1 Case 1 – FE model with parameter errors and reference data with noise

In Case 1, the reference model and the analytical model have the same mesh but different parameter values. Figure 3-5 shows the mesh of the models in this case.

Figure 3-5 Mesh for the FE models in Case 1

In each of the models, there are 150 shell elements divided into three groups, as shown in Figure 3-5. All elements in one group have the same physical parameter values. However, the parameter values of elements of one group are different from those of other groups. Table 3-2 lists the physical parameter values of the models in Case 1.
Three macro-parameters were selected as updating parameters – each of which represents the change ratio of the stiffness of all the elements in one group. Suppose the Young's moduli for the elements in Group $i$ are $E_i^0$ for the initial model and $E_i^u$ after model updating and the updating parameter for this group of elements is $\delta p_i$. The relation between these parameters is:

$$\delta p_i = \frac{E_i^u - E_i^0}{E_i^0}.$$

The target values of the updating parameters are also listed in Table 3-2 as $\delta p^0$. When using the inverse eigenvector sensitivity method, only two updating parameters were selected (if all three macro parameters were selected, the sensitivity matrix would become singular).

The first eight flexible modes predicted by the reference model were used as the reference dynamic properties in the case studies. The reference mode shape data were represented by 28 DOFs that were determined using the Effective Independent (EI) method [35, 93, 73, and 23] on the analytical eigenvectors.

In order to examine the effects of noise in reference data, two sets of reference data were used, one without noise and the other with artificial noise. The noise added to the reference data was obtained from the MATLAB RANDN() function, which creates a set of random data whose elements are normally distributed with mean value equal to 0 and variance value equal to 1. Multiplying the eigenvalues produced by the reference model with a fraction of the random data set (multiplied by a constant $C_\lambda$) created a set of noise data, which are added to the eigenvalues to form a set of noise-contaminated eigenvalue data. In the same way, with another set of random data and constant $C_\phi$, a noise-contaminated eigenvector matrix can also be obtained. In the case study in this section, the constants $C_\lambda$ and $C_\phi$ were set to 0.02 and 0.30 – this is to assume that the noise-contaminated reference data have 2% and 30% noise on average for the eigenvalues and eigenvectors, respectively.

The correlation between the initial model prediction and the reference data is shown in Table 3-3. Although there is a significant difference between the parameter values of the reference model and those of the analytical model, the correlation results
are acceptable in the general sense, even with the noise contaminated reference data.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ref. Freq</td>
<td>NFD (%)</td>
</tr>
<tr>
<td>27.73</td>
<td>27.61</td>
<td>0.43</td>
</tr>
<tr>
<td>30.55</td>
<td>30.30</td>
<td>0.82</td>
</tr>
<tr>
<td>64.55</td>
<td>62.77</td>
<td>2.85</td>
</tr>
<tr>
<td>63.74</td>
<td>63.95</td>
<td>-0.33</td>
</tr>
<tr>
<td>79.19</td>
<td>80.05</td>
<td>-1.07</td>
</tr>
<tr>
<td>90.79</td>
<td>91.29</td>
<td>-0.55</td>
</tr>
<tr>
<td>117.26</td>
<td>116.93</td>
<td>0.28</td>
</tr>
<tr>
<td>130.60</td>
<td>130.74</td>
<td>-0.11</td>
</tr>
<tr>
<td>Average*</td>
<td></td>
<td>0.81</td>
</tr>
</tbody>
</table>

Table 3-3 Initial correlation in Case 1

Eight model updating sessions were conducted in this case study: four using the reference data without noise, and four using the noise-contaminated reference data. The results from all these sessions are listed in Table 3-4. In Session 5 there are only two updating parameters, because of the ill-condition in the sensitivity matrix when three updating parameters are selected.

In the first four sessions where the analytical model was updated with the noise-free reference data, the differences between the natural frequencies predicted by the modified models and those of the correlated modes in the reference data set approached zero and the MAC values of the correlated mode pairs approached 100%. The final updating parameter values were almost equal to the target values that are listed in Table 3-2. The modified models were approximately the same as the target model – the reference model.

In the last four sessions where the reference data used in model updating have noise on both eigenvalues and eigenvectors, the correlations between the reference data and the predictions from the modified models were improved for the dynamic properties that were used in the updating equations.

In Session 5 where only the eigenvalues were used in the updating equation with the weighting factor $W_i = 1.0$ for each mode, the average natural frequency difference was not improved and the eigenvector correlation was worse than the correlation of the initial eigenvectors.

---

* The average values for NFD in all tables in the thesis are the average values of absolute relative natural frequency differences in percent.
In Session 6, only the eigenvalues were used in the updating equation and each eigenvalue was weighted by a weighting factor $W_{\lambda_i} = \lambda_i^{-1}$. The correlation of both eigenvalues and eigenvectors was improved.

In Session 7 where only the eigenvectors were used in the updating equation, the correlation of eigenvectors was improved while that of eigenvalues was worse than the initial correlation.

In Session 8, both eigenvalues and eigenvectors were used in the updating equation and weighting factors were set to balance the contribution from each eigenvalue and each eigenvector. From this session, the correlation on both eigenvalues and eigenvectors was improved.

<table>
<thead>
<tr>
<th>Session</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weighting $W_{\lambda}$</td>
<td>1.0</td>
<td>$\lambda_i^{-1}$</td>
<td>0.0</td>
<td>$\lambda_i^{-1}$</td>
</tr>
<tr>
<td>Weighting $W_{\phi}$</td>
<td>0.0</td>
<td>0.0</td>
<td>1</td>
<td>$</td>
</tr>
<tr>
<td>Noise</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>$\delta p_1$</td>
<td>-0.1974</td>
<td>-0.1988</td>
<td>-0.1979</td>
<td>-0.1977</td>
</tr>
<tr>
<td>$\delta p_2$</td>
<td>-0.0014</td>
<td>0.0000</td>
<td>---</td>
<td>-4.0E-04</td>
</tr>
<tr>
<td>$\delta p_3$</td>
<td>0.2856</td>
<td>0.2885</td>
<td>0.2877</td>
<td>0.2865</td>
</tr>
<tr>
<td>Average $</td>
<td>NFD</td>
<td>$</td>
<td>0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>Average $MAC$</td>
<td>100.0</td>
<td>100.0</td>
<td>100.0</td>
<td>100.0</td>
</tr>
</tbody>
</table>

Table 3-4 Updating results from case study with parameter errors in Case 1

3.3.1.2 Case 2 – FE models with discretisation errors and with/without parameter errors

In this second case, two analytical models were subjected to the model updating procedure. They have different meshes from the reference model. The mesh of the analytical models is the same as that of the models in Case 1, as shown in Figure 3-5.
The mesh of the reference model is shown in Figure 3-6. There are 3750 shell elements in this model, of which the mesh size is 5 times finer in one dimension than that in the analytical models.

One analytical model, analytical model A, has the same physical parameter values as those of the reference model. There are no parameter errors but discretisation errors in this model when compared with the reference model. Another analytical model, analytical model B, has different physical parameter values from those of the reference model. There are both parameter errors and discretisation errors in this model. Table 3-5 lists the element physical parameter values of the reference model and the analytical models in the case study.

![Figure 3-6 Reference model with fine mesh for Case 2](image)

<table>
<thead>
<tr>
<th>Group</th>
<th>Reference model</th>
<th>Analytical model A</th>
<th>Analytical model B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 1</td>
<td>$1.41 \times 10^{11}$</td>
<td>$1.41 \times 10^{11}$</td>
<td>$0.0$</td>
</tr>
<tr>
<td>Group 2</td>
<td>$2.11 \times 10^{11}$</td>
<td>$2.11 \times 10^{11}$</td>
<td>$0.0$</td>
</tr>
<tr>
<td>Group 3</td>
<td>$3.17 \times 10^{11}$</td>
<td>$3.17 \times 10^{11}$</td>
<td>$0.0$</td>
</tr>
</tbody>
</table>

Table 3-5 Physical parameter values of the models in Case 2

As in Case 1, three macro parameters were selected as updating parameters – each of which represents the change ratio of the stiffness of all the elements in one group. The first eight flexible modes from the reference model were used as the reference dynamic properties in the case study. The reference mode shape data were represented by 28 DOFs, the same DOFs as in Case 1. The correlation between the initial model predictions and the reference data is shown in Table 3-6.
The eigenvector correlation between the reference data with those predicted by analytical model A has a very high $MAC$ value for each mode while there are significant differences between the natural frequencies from the reference model and analytical model A. For the correlation between the reference model and analytical model B, although the eigenvector correlation has the $MAC$ values lower than those for analytical model A, all eight modes can still be considered as highly correlated. The natural frequency differences between the reference model and analytical model B are from about 0.5% to 5% for these modes.

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>$NFD$ (%)</th>
<th>$MAC$ %</th>
<th>Mode No.</th>
<th>$NFD$ (%)</th>
<th>$MAC$ %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.247</td>
<td>100.00</td>
<td>1</td>
<td>-0.823</td>
<td>99.46</td>
</tr>
<tr>
<td>2</td>
<td>-1.394</td>
<td>100.00</td>
<td>2</td>
<td>-0.584</td>
<td>99.78</td>
</tr>
<tr>
<td>3</td>
<td>-2.871</td>
<td>99.98</td>
<td>3</td>
<td>-0.102</td>
<td>95.08</td>
</tr>
<tr>
<td>4</td>
<td>-2.198</td>
<td>100.00</td>
<td>4</td>
<td>-2.522</td>
<td>99.06</td>
</tr>
<tr>
<td>5</td>
<td>-2.851</td>
<td>99.55</td>
<td>5</td>
<td>-3.894</td>
<td>89.75</td>
</tr>
<tr>
<td>6</td>
<td>-2.679</td>
<td>99.80</td>
<td>6</td>
<td>-3.216</td>
<td>95.29</td>
</tr>
<tr>
<td>7</td>
<td>-3.447</td>
<td>99.95</td>
<td>7</td>
<td>-3.176</td>
<td>99.39</td>
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<tr>
<td>8</td>
<td>-4.404</td>
<td>99.99</td>
<td>8</td>
<td>-4.507</td>
<td>97.21</td>
</tr>
<tr>
<td>Average</td>
<td>2.636</td>
<td>99.91</td>
<td></td>
<td>2.353</td>
<td>96.88</td>
</tr>
</tbody>
</table>

Table 3-6 Correlation with the initial model predictions in Case 2

Four updating sessions were conducted on analytical model A, which has only discretisation errors, and four sessions on analytical model B, which has both discretisation errors and parameter errors. The results from these eight updating sessions are listed in Table 3-7.

From both Sessions 1 and 5, where two analytical models were updated with the eigenvalue differences at the right hand side of the updating equation, the norm of the residuals of the eigenvalues was minimised. However, this does not mean that the norm of the relative natural frequency differences was minimised. The minimisation of the norm of the relative natural frequency differences was achieved from both Sessions 2 and 6 that took the inverse of each eigenvalue as the weighting factor for the eigenvalue in the updating equation.

From both Sessions 3 and 7, higher average $MAC$ values were achieved for the modified models than those from Sessions 1, 2, 5, and 6. However, the average absolute relative natural frequency differences from Sessions 3 and 7 are larger than those from Sessions 2, 4, 6 and 8. Among all sessions in this case, Sessions 4 and 8 have achieved the highest average $MAC$ values and a relatively small average absolute relative natural frequency differences.
frequency differences.

Comparing between the results from four sessions for an analytical model, the results from Session 4 (or Session 8) should be considered as the best when the improvement of the correlations for both eigenvalues and eigenvectors is concerned.

From Table 3-7 it can also be seen that two updating sessions with the same settings for running the updating program but with different initial analytical models (such as Sessions 1 and 5) can arrive at the same correlation results between the reference data and the data predicted from the modified models.

For model updating on analytical model A, the target value is zero for each updating parameter, because there is no parameter error in the initial analytical model. However, none of the four sessions resulted in the final updating parameter values being equal to zero. For model updating on analytical model B, the target values for the updating parameters are the differences between the physical parameter values of the reference model and analytical model B, which are listed in Table 3-5. Comparing the target values and the results from the four sessions of model updating on analytical model B, it can be seen that none of the sessions resulted in the updating parameter values being equal to the target ones. Among all these sessions in this case, the results from Sessions 3 and 7 are the closest to the target values.

<table>
<thead>
<tr>
<th>Session</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_\lambda$</td>
<td>1.0</td>
<td>$\lambda_1^{-1}$</td>
<td>0</td>
<td>$\lambda_2^{-1}$</td>
</tr>
<tr>
<td>$W_\phi$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$</td>
</tr>
<tr>
<td>$\delta p_1$</td>
<td>0.3507</td>
<td>0.0977</td>
<td>0.0317</td>
<td>0.0537</td>
</tr>
<tr>
<td>$\delta p_2$</td>
<td>-0.2788</td>
<td>-0.0307</td>
<td>0.0097</td>
<td></td>
</tr>
<tr>
<td>$\delta p_3$</td>
<td>0.1390</td>
<td>0.0576</td>
<td>0.0603</td>
<td>0.0854</td>
</tr>
<tr>
<td>Average</td>
<td>1.90</td>
<td>0.57</td>
<td>0.7311</td>
<td>0.605</td>
</tr>
<tr>
<td>Average MAC</td>
<td>99.34</td>
<td>99.89</td>
<td>99.93</td>
<td>99.94</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Session</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_\lambda$</td>
<td>1.0</td>
<td>$\lambda_1^{-1}$</td>
<td>0</td>
<td>$\lambda_2^{-1}$</td>
</tr>
<tr>
<td>$W_\phi$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$</td>
</tr>
<tr>
<td>$\delta p_1$</td>
<td>0.0824</td>
<td>-0.1205</td>
<td>-0.1733</td>
<td>-0.1559</td>
</tr>
<tr>
<td>$\delta p_2$</td>
<td>-0.2790</td>
<td>-0.0307</td>
<td>0.0097</td>
<td></td>
</tr>
<tr>
<td>$\delta p_3$</td>
<td>0.4677</td>
<td>0.3629</td>
<td>0.3665</td>
<td>0.3987</td>
</tr>
<tr>
<td>Average</td>
<td>1.90</td>
<td>0.57</td>
<td>0.7311</td>
<td>0.605</td>
</tr>
<tr>
<td>Average MAC</td>
<td>99.34</td>
<td>99.89</td>
<td>99.93</td>
<td>99.94</td>
</tr>
</tbody>
</table>

Table 3-7 Updating results from the case study with discretisation errors in Case 2
3.3.1.3 Case 3 – FE models with parameter errors and with/without configuration errors

In this third case, two analytical models were subjected to updating sessions with the same reference data, which was produced from a reference model. Both the reference model and the analytical models have 70 shell elements representing two plates joined by some connecting elements as shown in Figure 3-7. The connecting elements are scalar spring elements in the reference model and analytical model A, whereas they are MultiPoint Constraints (MPCs) in analytical model B. Because of the difference in the type of the connecting elements, analytical model B has configuration errors when compared with the reference model. Parameter errors were also introduced into the analytical models. Table 3-8 lists the parameter values of the elements in the reference model and the analytical models. From the table, it can be seen that both analytical models have parameter errors in the elements of Group A and there are configuration errors in analytical model B.

![Figure 3-7 Model mesh in Case 3](image)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Reference model</th>
<th>Analytical model A</th>
<th>Analytical model B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E$ (N/m²)</td>
<td>$2.06 \times 10^{11}$</td>
<td>$3.17 \times 10^{11}$</td>
<td>$3.17 \times 10^{11}$</td>
</tr>
<tr>
<td>$\rho$ (kg/m³)</td>
<td>7860</td>
<td>7860</td>
<td>7860</td>
</tr>
<tr>
<td>Group B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E$ (N/m²)</td>
<td>$2.06 \times 10^{11}$</td>
<td>$2.06 \times 10^{11}$</td>
<td>$2.06 \times 10^{11}$</td>
</tr>
<tr>
<td>$\rho$ (kg/m³)</td>
<td>7860</td>
<td>7860</td>
<td>7860</td>
</tr>
<tr>
<td>Connecting Elements</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_x$ (N/m)</td>
<td>$1.0 \times 10^{6}$</td>
<td>$4.0 \times 10^{6}$</td>
<td>MPC</td>
</tr>
<tr>
<td>$K_y$ (N/m)</td>
<td>$1.0 \times 10^{6}$</td>
<td>$4.0 \times 10^{6}$</td>
<td></td>
</tr>
<tr>
<td>$K_z$ (N/m)</td>
<td>$2.0 \times 10^{4}$</td>
<td>$8.0 \times 10^{4}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3-8 Parameter values of the models in Case 3
The first 20 modes predicted from the reference model are used as the reference data in the updating sessions for this case study. The eigenvectors are represented by all translational DOFs in the model. The correlation between the dynamic properties predicted from the analytical models and those from the reference model are listed in Table 3-9 and plotted in Figure 3-8.

<table>
<thead>
<tr>
<th>CMP</th>
<th>Analytical Model A</th>
<th>Analytical Model B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mode_X</td>
<td>NFD %</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>13.50</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>8.22</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2.58</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4.61</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>15.83</td>
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<td>6</td>
<td>6</td>
<td>7.43</td>
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<td>7</td>
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<td>5.55</td>
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<td>8</td>
<td>8</td>
<td>2.76</td>
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<td>9</td>
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<td>10.67</td>
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<td>9.08</td>
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<td>11</td>
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<td>4.87</td>
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<td>25.03</td>
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<td>15</td>
<td>8.60</td>
</tr>
<tr>
<td>14</td>
<td>16</td>
<td>4.54</td>
</tr>
<tr>
<td>15</td>
<td>17</td>
<td>8.41</td>
</tr>
<tr>
<td>16</td>
<td>18</td>
<td>20.14</td>
</tr>
<tr>
<td>17</td>
<td>20</td>
<td>9.09</td>
</tr>
<tr>
<td>Average</td>
<td>9.47</td>
<td>90.12</td>
</tr>
</tbody>
</table>

Table 3-9 Correlation of the initial analytical models in Case 3

Three updating parameters were selected in the updating sessions on analytical model A. They represent the stiffness change ratio of the elements in Group A, Group B, and the Connecting Elements, respectively, as shown in Figure 3-7. In the updating sessions on analytical model B, only two updating parameters were selected because the stiffness of the MPC elements is infinite and cannot be changed. Six updating sessions were undertaken on the analytical models – Sessions 1 to 3 were for analytical model A, and Sessions 4 to 6 for analytical model B. The results from these sessions are listed in Table 3-10.
From the table, it can be seen that all three sessions on analytical model A yielded perfect results. The final updating parameter values were almost equal to the target values. The correlation between the modified models and the reference model achieved almost zero natural frequency differences and 100% MAC values. However, the results from the sessions on analytical model B were not so good. Some modes in the reference data set were not correlated with any modes from the modified models. The eigenvalue correlation for the correlated mode pairs was improved more or less in
the sessions. However, none of the sessions on analytical model B has produced a modified model with an average absolute natural frequency difference less than 4% when compared with the reference eigenvalues. The average MAC values for the correlated mode pairs after updating in sessions 4 and 6 were worse than those before updating. In Session 5, the average MAC value for the correlated mode pairs was improved but not significantly. The final values of all updating parameters in these three sessions are far away from the target values which can be determined from the physical parameter values of the reference model and analytical model B.

3.3.1.4 Concluding remarks on the case studies on the plate models

In this section, several case studies of model updating on numerical plate models have been discussed. Because all the reference data for performing model updating come from reference models, the known differences between the physical parameter values of the reference models and those of the analytical models are the target of model updating. The updating parameters were also selected according to these parameter differences.

In the case studies in this section, each analytical model has one or two types of error among parameter errors, discretisation errors, and configuration errors when compared with the reference model in the same session. In Case 1, noise was added to the reference data for some sessions. Thus, all possible reasons (except factors of non-linearity) for discrepancies between experimental data and FE model predictions in practical cases have been included in these case studies.

From these case studies, it can be seen that different errors in analytical models have different effects on model updating results. A model updating session may improve an FE model with some types of error while it may not improve a model with other types of error. The results from these model updating sessions will be further analysed in the next sub-chapter – "3.4 Perspective on model updating".

3.3.2 Case studies of model updating on industrial models

In this section, the experience of updating two industrial FE models will be presented. The initial FE models in both case studies were provided from industry. They may contain parameter errors, configuration errors and discretisation errors. The objective of updating these FE models is to reduce the discrepancies between the model-predicted dynamic properties and the reference structural dynamic properties which are represented by data from experiments that were conducted on the actual structures.

The updating parameters in each case study in this section were selected from the analysis of the sensitivity of eigenvalues to candidate parameters combined with engineering judgement. Although some parameters with incorrect values may not be
selected as updating parameters by this method, it is an available and efficient method applicable to industrial models. The inverse eigen-sensitivity method was used in the case studies in this section. The comparison between the FE model predictions and the structural dynamic properties was represented by eigenvalue and eigenvector correlation.

3.3.2.1 Case 4 – case study on the Rear Bearing Housing (RBH) component of an aero-engine

In this case, the experience of updating an FE model of the Rear Bearing Housing component of an aero engine will be presented. The component has three parts – the outer case, the inner case, and the vanes. The model of the component is shown in Figure 3-9. It has 288 shell elements for the outer case and 48 beam elements in each end of the outer case. For each vane there are eight shell elements and eight beam elements in the model. For the inner case there are ten rings and 48 shell elements in each ring.

A modal test was undertaken on the component. The vibration response due to a hammer input was measured in three orthogonal directions at each of 24 measurement points marked as purple spots in Figure 3-9. They are all located on the outer case. A total of 20 modes were identified from the measured FRFs. The correlation between the experimental modal data and the dynamic properties predicted by the FE model are shown in Table 3-11. The Auto-MAC matrix of the experimental mode shapes and the MAC matrix between the experimental mode shapes and eigenvectors predicted by the initial FE model are shown in Figure 3-10.
Table 3-11 Correlation of the initial FE model in Case 4

<table>
<thead>
<tr>
<th>CMP No.</th>
<th>Exp. Mode</th>
<th>Ana. Mode</th>
<th>MAC (%)</th>
<th>NFD (%)</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>98.01</td>
<td>-8.52</td>
<td>2ND out</td>
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<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>98.27</td>
<td>-9.14</td>
<td>2ND out</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>5</td>
<td>89.13</td>
<td>11.11</td>
<td>3ND out</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>6</td>
<td>85.37</td>
<td>0.36</td>
<td>2ND in</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>7</td>
<td>80.45</td>
<td>-0.19</td>
<td>2ND in</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>11</td>
<td>93.07</td>
<td>18.15</td>
<td>4ND out</td>
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<td>7</td>
<td>10</td>
<td>90.71</td>
<td>17.77</td>
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<tr>
<td>8</td>
<td>10</td>
<td>9</td>
<td>83.72</td>
<td>-3.38</td>
<td>3ND in</td>
</tr>
<tr>
<td>9</td>
<td>11</td>
<td>8</td>
<td>88.34</td>
<td>-3.78</td>
<td>3ND in</td>
</tr>
<tr>
<td>10</td>
<td>12</td>
<td>14</td>
<td>71.74</td>
<td>5.00</td>
<td>4ND in</td>
</tr>
<tr>
<td>11</td>
<td>13</td>
<td>13</td>
<td>60.74</td>
<td>3.98</td>
<td>4ND in</td>
</tr>
<tr>
<td>12</td>
<td>18</td>
<td>21</td>
<td>77.59</td>
<td>15.50</td>
<td>5ND out</td>
</tr>
<tr>
<td>13</td>
<td>19</td>
<td>20</td>
<td>67.28</td>
<td>15.00</td>
<td>5ND out</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td></td>
<td>83.42</td>
<td>8.61</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3-10 Auto-MAC (left) of the experimental modes and MAC (right) between the experimental modes and the analytical modes in Case 4

From the correlation results, it can be seen that 13 modes identified from the experimental FRFs are closely correlated with some modes predicted by the initial FE model. The natural frequency differences of the correlated mode pairs are about –10% to +20%. Some modes in the experimental data set are not correlated with any mode in the analytical data set, and vice versa. In Table 3-11, the mode shapes of the correlated modes are briefly described. "nND in" and "nND out" mean the n\(^{th}\) family nodal diametral modes with the mode shapes at the two ends of the outer case in phase and out-of-phase, respectively.

Before conducting model updating sessions, the sensitivities of the analytical
eigenvalues of the correlated modes to all candidate updating parameters need to be calculated and analysed. For doing this, all elements in the initial FE model are divided into 34 groups according to element locations and element types. There are six groups of shell elements and two groups of beam elements for the outer case. In the inner case, there are ten groups of shell elements. The elements in the vanes are in 16 groups – 8 groups of shell elements and 8 groups of beam element. Because the structure is axisymmetric, in order to keep this property in the modified models, the locations of elements in each group are symmetric about the axis of the structure. The numbers shown in Figure 3-9 are the group numbers. The sensitivities of the eigenvalues of the correlated modes with respect to the mass and stiffness of the elements in each group are plotted in Figure 3-11 (the sensitivities of eigenvalues to mass properties are plotted with their absolute values).

![Figure 3-11 Sensitivities of eigenvalues to mass (left) and stiffness (right) in Case 4](image)

From this figure, it can be seen that the eigenvalues of the correlated modes are sensitive to the mass and stiffness of the elements in the outer case. The mass and stiffness of the elements in the vanes are the least sensitive with the correlated modes. By examining the sensitivity data on the elements in the outer case, it can be seen that the eigenvalues of the "in" modes are more sensitive to the elements in Groups 1 to 3 than to the elements in Groups 5 to 8, while the eigenvalues of the "out" modes are more sensitive to the elements in Groups 5 to 8 than to the elements in Groups 1 to 3.

After the sensitivity analysis, the settings for three model updating sessions were determined. The updating parameters in the sessions were the modification ratios of the stiffness of some elements and the inverse eigenvalue sensitivity method was used with the inverses of the eigenvalues as the weighting factors. The groups of elements in each updating parameter for the updating sessions are listed in Table 3-12.
<table>
<thead>
<tr>
<th>Session</th>
<th>Number of Parameters</th>
<th>Parameter Number</th>
<th>Group numbers of elements</th>
<th>Location of the elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1 – 3</td>
<td>Outer case</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>5 – 8</td>
<td>Outer case</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1 – 3</td>
<td>Outer case</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>5 – 8</td>
<td>Outer case</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>25 – 34</td>
<td>Inner case</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>1</td>
<td>1 – 3</td>
<td>Outer case</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>5 – 8</td>
<td>Outer case</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>9 – 24</td>
<td>Vanes</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>25 – 34</td>
<td>Inner case</td>
</tr>
</tbody>
</table>

Table 3-12 Elements in updating parameters of model updating sessions in Case 4

<table>
<thead>
<tr>
<th>CMP No.</th>
<th>Exp. Mode</th>
<th>Session 1</th>
<th>Session 2</th>
<th>Session 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Ana. Mode</td>
<td>MAC (%)</td>
<td>NFD (%)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>98.25</td>
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<td>3</td>
<td>4</td>
<td>97.15</td>
<td>6.06</td>
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</tr>
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<td>7</td>
<td>82.51</td>
<td>-0.15</td>
</tr>
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<td>6</td>
<td>6</td>
<td>9</td>
<td>94.65</td>
<td>5.07</td>
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<td>92.10</td>
<td>4.73</td>
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<tr>
<td>9</td>
<td>10</td>
<td>10</td>
<td>84.86</td>
<td>-2.14</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>11</td>
<td>86.40</td>
<td>-2.54</td>
</tr>
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<td>11</td>
<td>12</td>
<td>16</td>
<td>75.21</td>
<td>7.76</td>
</tr>
<tr>
<td>12</td>
<td>13</td>
<td>15</td>
<td>63.32</td>
<td>6.71</td>
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<td>14</td>
<td>19</td>
<td>13</td>
<td>84.23</td>
<td>-3.87</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td>87.11</td>
<td>4.45</td>
<td></td>
</tr>
</tbody>
</table>

Table 3-13 Correlation of the modified models in Case 4
The correlation between the experimental data and the modal data predicted by the modified models from the updating sessions is listed in Table 3-13. The MAC matrices for the modified models are plotted in Figure 3-12. The final values of the updating parameters in the updating sessions are listed in Table 3-14.

From Table 3-13 and Figure 3-12 it can be seen that the average eigenvalue correlation for the correlated mode pairs has improved a little in all three modified models. However, eigenvector correlation does not show any significant improvement.

The final updating parameter values from the updating sessions show the modification on the initial model. The values from the first two sessions are more reasonable than those from the third session which tried to modify the stiffness of the elements in the vanes of the component. The value of the third updating parameter in Session 3 is far beyond the acceptable range to engineers.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Element Group</th>
<th>Session 1</th>
<th>Session 2</th>
<th>Session 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 – 3</td>
<td>-0.4889</td>
<td>-0.4924</td>
<td>-0.5088</td>
</tr>
<tr>
<td>2</td>
<td>5 – 8</td>
<td>0.1432</td>
<td>0.1582</td>
<td>-0.1531</td>
</tr>
<tr>
<td>3</td>
<td>9 – 24</td>
<td>--</td>
<td>--</td>
<td>18.3806</td>
</tr>
<tr>
<td>4</td>
<td>25 – 34</td>
<td>--</td>
<td>0.0291</td>
<td>-0.2888</td>
</tr>
</tbody>
</table>

Table 3-14 Updating parameter values after model updating in Case 4

The results from these sessions indicate that the model may not be able to be updated with a normal computer program. The unusual value of the third updating parameter in Session 3 indicates that there may be configuration errors in the initial model especially in the elements for the vanes.

3.3.2.2 Case 5 – case study on Combustion-Case-Outer-Casing (CCOC) component of an aero-engine

The structure of the CCOC component in this case study is composed of two cases and an inner flange. Figure 3-13 shows the FE model. The experimental data for the component were obtained from a modal test conducted on an actual component structure. There are three rings with a total of 36 grid points in the experimental mesh.
which is also shown in Figure 3-13.

At each of the grid points in the far end (numbered from 85 to 96 in the figure) vibration was measured in two directions – radial and axial. At all other grid points, measurements were made only in the radial direction. Thus, the total number of DOFs in the experimental data set was 48. Below 430Hz, 11 modes (except for the rigid body modes) were identified from the experimental FRFs and were taken as the reference properties for model updating. Using the normal mode extraction with the Lanczos method in NASTRAN, 19 modes were obtained from the initial FE model in the frequency range of 50 – 550Hz.

![Figure 3-13 FE model mesh (left) and experimental mesh (right) of CCOC component](image)

<table>
<thead>
<tr>
<th>CMP No.</th>
<th>Exp. Mode</th>
<th>Exp. Freq.</th>
<th>Ana. Mode</th>
<th>MAC (%)</th>
<th>NFD (%)</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>64.73</td>
<td>1</td>
<td>84.04</td>
<td>4.91</td>
<td>2ND in</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>65.29</td>
<td>2</td>
<td>88.85</td>
<td>5.08</td>
<td>2ND in</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>99.90</td>
<td>3</td>
<td>79.12</td>
<td>-10.34</td>
<td>2ND out</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>100.83</td>
<td>4</td>
<td>70.08</td>
<td>-10.16</td>
<td>2ND out</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>170.52</td>
<td>5</td>
<td>89.96</td>
<td>6.58</td>
<td>3ND in</td>
</tr>
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<td>6</td>
<td>6</td>
<td>171.01</td>
<td>6</td>
<td>86.76</td>
<td>6.42</td>
<td>3ND in</td>
</tr>
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<td>7</td>
<td>7</td>
<td>262.13</td>
<td>7</td>
<td>83.58</td>
<td>-2.96</td>
<td>3ND out</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>263.00</td>
<td>8</td>
<td>83.94</td>
<td>-3.15</td>
<td>3ND out</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>308.39</td>
<td>9</td>
<td>97.70</td>
<td>6.86</td>
<td>4ND in</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>309.02</td>
<td>10</td>
<td>97.59</td>
<td>6.97</td>
<td>4ND in</td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>384.20</td>
<td>11</td>
<td>87.75</td>
<td>8.44</td>
<td>5ND in</td>
</tr>
</tbody>
</table>

| Average |           |           |           | 86.53   | 6.53    |        |

Table 3-15 Correlation of the initial model of the combustion case
The correlation between the reference data and the analytical predictions from the initial model are listed in Table 3-15 in which brief descriptions for the mode shapes are also given that will be helpful for sensitivity analysis. The MAC matrix between the reference data and the analytical predictions is plotted in Figure 3-14.

![Figure 3-14 Initial mode shape correlation in case 5](image)

From Table 3-15 and Figure 3-14 it can be seen that these two data sets have high correlation for the eigenvectors of the first 11 modes, but the correlation on the eigenvalues is not acceptable. The natural frequency differences of modes 3 and 4 between these two data sets are greater than 10%. The smallest difference of the natural frequencies between the correlated mode pairs is about 3%.

The correlation results of these two data sets provide a good starting point for the initial FE model to be updated. From Table 3-15 it can also be seen that the natural frequency differences of the "in-phase" modes (Modes 1, 2, 5, 6, 9, 10 and 11) vary from about 5% to 8.5% – there are 3.5 percentage points in difference. However, the natural frequency differences of the "out-of-phase" modes (Modes 3, 4, 7 and 8) are from -10% to -3% – there are 7 percentage points in difference. The variations of the natural frequency differences will make the updating procedure difficult, especially for those "out-of-phase" modes.

The sensitivity analysis was performed on the initial FE model. All candidate updating parameters were the ratios between the increment/decrement of either the mass densities or the Young's moduli of the elements and their initial values. A total of 38 parameters – each of them containing elements located on one or several rings – were selected as candidate updating parameters and are shown in Figure 3-15. The numbers in the figure are the parameter numbers. The sensitivities of eigenvalues to the stiffness and mass properties, separately, of the elements in each parameter are shown in Figure 3-16. For convenience, the sensitivities of eigenvalues to the mass properties are plotted with the absolute values.
Several conclusions can be drawn from the sensitivity calculation results.

(1) Some parameters are less sensitive to all eigenvalues concerned than other parameters: for example, the stiffness of the elements in Groups 15 – 18;

(2) Some parameters have high sensitivities for the in-phase modes and low sensitivities for the out-of-phase modes. For example, the stiffness of the elements in Group 34 is very sensitive for the three pairs of in-phase modes. However, the sensitivity data of this parameter with the two pairs of out-of-phase modes are almost only 10% of those for the in-phase modes. In contrast, some parameters, such as the
stiffness of the elements in Group 23, have high sensitivities for the out-of-phase modes and low sensitivities for the in-phase modes;

(3) Only the stiffness and mass of parameters 24 –30 have high sensitivity data to Mode 11.

The conclusions observed from the analysis of the sensitivity data can be used in the selection of updating parameters that may change the eigenvalues of some modes while the eigenvalues of other modes remain unchanged. According to the sensitivity analysis, model updating on the FE model was carried out with the following settings.

(1) Combining the sensitivity analysis and the experience of the previous updating sessions, three sets of updating parameters were selected. These updating parameters are listed in Table 3-16. “K” and “M” in the table denote the stiffness and the mass properties, respectively, of elements.

<table>
<thead>
<tr>
<th>Parameter No.</th>
<th>Session 1</th>
<th>Session 2</th>
<th>Session 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-3 (K)</td>
<td>1-3,34 (K)</td>
<td>1-3,34 (K)</td>
</tr>
<tr>
<td>2</td>
<td>11-20 (M)</td>
<td>23 (K)</td>
<td>22, 23, 38 (K)</td>
</tr>
<tr>
<td>3</td>
<td>21-23 (K)</td>
<td>19,20,35 (K)</td>
<td>11-20, 35-37 (K)</td>
</tr>
<tr>
<td>4</td>
<td>26,27 (K)</td>
<td>27,28 (M)</td>
<td>27,28 (M)</td>
</tr>
<tr>
<td>5</td>
<td>27-30 (M)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>34 (K)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3-16: Updating parameters and their elements for updating sessions in Case 5

(2) Both eigenvalues and eigenvectors were included in the updating equation. The eigenvalues were put into the equation in their relative differences. The eigenvectors were weighted by the inverses of their modules and Modal Scale Factors (MSFs) were used to scale the experimental eigenvectors.

(3) The weighting factors for eigenvalues can be adjusted to put more or less weight on eigenvalues than on eigenvectors. After several trials, the weighting factor for the eigenvalue of each mode was assigned to ten times the inverse of the eigenvalue.

Table 3-17 lists the natural frequency differences and MAC values for the correlated mode pairs after the updating sessions. From the table, it can be seen that the natural frequency differences between the experimental modal data set and the modal data predicted by the modified FE models from Sessions 1 and 3 are smaller than 2% for all the modes in the table. The MAC values for all correlated mode pairs in the updating frequency range are greater than 80%. The MAC values for most of the modes in the table increased from the updating sessions, although a few MAC values decreased a little. Comparing with the correlation results between the experimental modal data set and the modal data from the initial FE model as listed in Table 3-15, the correlation results for all the modified FE models from these three updating sessions show the
improvement on the predictions of both eigenvalues and eigenvectors. Table 3-18 lists the final values of the updating parameters in all three updating sessions. Before doing the model updating, a limit of $-0.8$ in the negative direction and $+2.0$ in the positive direction was set for the updating parameters. In each of these three updating sessions, there are one or two updating parameters that have the final value close to the limit. All other parameters have reasonable final values.

Parameters 2 and 3 in all three sessions contain the candidates that are sensitive for the eigenvalues of the out-of-phase modes. Although different combinations of the candidates were used in the sessions, one or two of the updating parameters have final values close to the limitations as represented in red in Table 3-18.

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Session 1</th>
<th>Session 2</th>
<th>Session 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MAC %</td>
<td>NFD %</td>
<td>MAC %</td>
</tr>
<tr>
<td>1</td>
<td>85.93</td>
<td>-0.19</td>
<td>82.13</td>
</tr>
<tr>
<td>2</td>
<td>91.13</td>
<td>-0.06</td>
<td>87.81</td>
</tr>
<tr>
<td>3</td>
<td>82.72</td>
<td>-1.28</td>
<td>80.37</td>
</tr>
<tr>
<td>4</td>
<td>80.76</td>
<td>-1.22</td>
<td>77.10</td>
</tr>
<tr>
<td>5</td>
<td>90.81</td>
<td>0.75</td>
<td>92.36</td>
</tr>
<tr>
<td>6</td>
<td>89.57</td>
<td>0.63</td>
<td>91.11</td>
</tr>
<tr>
<td>7</td>
<td>95.85</td>
<td>-0.25</td>
<td>87.73</td>
</tr>
<tr>
<td>8</td>
<td>96.34</td>
<td>-0.38</td>
<td>89.27</td>
</tr>
<tr>
<td>9</td>
<td>96.71</td>
<td>0.31</td>
<td>97.86</td>
</tr>
<tr>
<td>10</td>
<td>96.58</td>
<td>0.41</td>
<td>97.83</td>
</tr>
<tr>
<td>11</td>
<td>88.03</td>
<td>0.87</td>
<td>83.26</td>
</tr>
<tr>
<td>Average</td>
<td>90.40</td>
<td>0.6</td>
<td>87.89</td>
</tr>
</tbody>
</table>

Table 3-17 Correlation of the modified models

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Session 1</th>
<th>Session 2</th>
<th>Session 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.4968</td>
<td>-0.0913</td>
<td>-0.1737</td>
</tr>
<tr>
<td>2</td>
<td><strong>-0.7919</strong></td>
<td>-0.6410</td>
<td><strong>-0.7856</strong></td>
</tr>
<tr>
<td>3</td>
<td>-0.5242</td>
<td><strong>1.9324</strong></td>
<td><strong>1.8729</strong></td>
</tr>
<tr>
<td>4</td>
<td>-0.4931</td>
<td>0.1750</td>
<td>0.3854</td>
</tr>
<tr>
<td>5</td>
<td>-0.1490</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.9770</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3-18 Updating parameter values after model updating in Case 5
Examining the out-of-phase modes in the initial analytical data set, it can be found that the natural frequency differences for the 2ND modes are about \(-10\%\), while those for the 3ND modes are about \(-3\%\). The change of any parameter in the model that is sensitive for the eigenvalues of these modes will modify the eigenvalues of these modes in the same direction and almost the same amplitudes. This is the reason why the model cannot be updated with both improved correlation for these modes and a reasonable modification of the parameters.

In a model updating procedure, the results from an updating session should be assessed in order to get a better understanding of the model and the structure. In the case study for updating the CCOC model, the results from Session 1 show that Parameters 1 and 6 have opposite signs although both have high sensitivities for the in-phase modes. These two parameters were merged into one parameter in Sessions 2 and 3, and the results from these sessions showed that the correlation of the in-phase modes was still improved while the values for this updating parameter were in a reasonable range. The same results were found when Parameters 4 and 5 in Session 1 were merged to Parameter 4 in Sessions 2 and 3 and the correlation of Mode 11 was still improved.

From the assessment of the updating results for these three sessions, all three modified models can be considered as updated models in the sense that they can predict the dynamic properties of the CCOC in the frequency range of the reference modes. However, the modified model from Session 3 is considered as the best one when the physical meaning of the model parameters is taken into account.

### 3.4 Perspective on model updating

From the case studies in the previous sub-chapter, it can be seen that it is possible for an FE model to be modified in an updating session in such a way that it may be able to predict the dynamic properties of a structure with an acceptable accuracy compared with the reference data. However, not all models can be updated to fulfil this objective, and not all updating sessions can produce a modified model that has improved correlations and with all the parameters in the model having acceptable physical meaning. Thus, in this sub-chapter, the procedure of model updating will be explored in detail in order to get a clear idea about: (i) what kinds of error can be eliminated in model updating and what kinds of error cannot, and (ii) when performing model updating sessions with the inverse eigen-sensitivity method, how the information of eigenvalues and eigenvectors contributes to the results of model updating.

#### 3.4.1 Updating models with three kinds of error

Cases 1 – 3 in the previous sub-chapter describe model-updating sessions with different kinds of error in the initial analytical models. Comparing the updating results from these cases, the effects of the different kinds of error in an initial model on model
updating can be explained.

3.4.1.1 Reproduction of reference data and/or improvement of correlations

In all the cases presented in the previous sub-chapter, only the first four sessions of Case 1 and the first three sessions of Case 3 lead to modified models which are capable of reproducing the reference data exactly. In these case studies, the initial analytical models have only model parameter errors. The configuration and the mesh size of the models are the same as those of the reference models and, furthermore, the reference data used in these sessions are noise-free.

When there are other errors in the initial models, as in all the sessions of Case 2 and in the last three sessions of Case 3, or when the reference data used for model updating are noise-contaminated, as in some sessions of Case 1, the modified models cannot reproduce the reference data exactly. The correlation between the predictions from the modified models and the reference data is improved more or less depending on the settings for the updating procedures. Using the information from both eigenvalues and eigenvectors in the updating equation, as in those sessions with both the weighting factors $W_\lambda$ and $W_\phi$ not equal to zero, both eigenvalue correlation and eigenvector correlation can be improved.

If two initial analytical models with different parameter values are subjected to two model updating sessions with the same settings and the same reference data, as in some sessions in Case 2, the modified models from the sessions will be the same.

3.4.1.2 Updating parameter values

Comparing the final values of the updating parameters with the target values that are the differences between the reference models and the initial analytical models, it can be seen that different kinds of error in the initial model have different effects on the updating parameter values.

When there are only parameter errors in an initial analytical model and the reference data are noise-free, as in some sessions in Cases 1 and 3, the updating parameters will approach the target values, although the speeds of convergence are different when different settings are used in updating sessions.

When there are other kinds of error in the initial models besides parameter errors, or when the reference data are contaminated with noise, the obtained updating parameter values will not be equal to the target values. In some sessions, the obtained updating parameter values are far away from the target values of the sessions. The differences between the target values and the obtained updating parameter values vary according to the types of error in the models and the settings for updating sessions.

If two analytical models with different parameter values are subjected to two model updating sessions with the same settings and the same reference data, the
obtained updating parameter values will indicate the parameter differences in the initial models. For example, in Case 2 in the previous sub-chapter, the Young’s moduli of the elements of analytical model A are different from those of analytical model B. The obtained updating parameter values from the sessions – the ratios between the increments of the stiffness of some elements and their initial values – are listed in Table 3-19. Supposing the relative difference between the value of a physical parameter in analytical model A and that in analytical model B is denoted as $\delta p^{(0)}$, the relation between $\delta p^{(0)}$ and the updating parameter values for these two analytical models, $\delta p^{(A)}$ and $\delta p^{(B)}$, will be:

$$
\left(1 + \delta p^{(0)}\right) \times \left(1 + \delta p^{(A)}\right) = \left(1 + \delta p^{(B)}\right)
$$

(3.33)

Thus:

$$
\delta p^{(0)} = \frac{\delta p^{(B)} - \delta p^{(A)}}{1 + \delta p^{(A)}}
$$

(3.34)

The values of $\delta p^{(0)}$ are listed in Table 3-19. From the table it can be seen that the value of $\delta p^{(0)}$ for each parameter from a pair of sessions is almost equal to that from the other pairs of session.

<table>
<thead>
<tr>
<th>Parameter No.</th>
<th>Sessions 1 and 5</th>
<th>Sessions 2 and 6</th>
<th>Sessions 3 and 7</th>
<th>Sessions 4 and 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta p^{(A)}$ for analytical model A</td>
<td>0.3507</td>
<td>0.0977</td>
<td>0.0317</td>
<td>0.0537</td>
</tr>
<tr>
<td>1</td>
<td>-0.2788</td>
<td>-0.0308</td>
<td></td>
<td>-0.0097</td>
</tr>
<tr>
<td>2</td>
<td>0.1390</td>
<td>0.0576</td>
<td>0.0603</td>
<td>0.0854</td>
</tr>
<tr>
<td>3</td>
<td>-0.2790</td>
<td>-0.0308</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta p^{(B)}$ for analytical model B</td>
<td>0.0824</td>
<td>-0.1205</td>
<td>-0.1734</td>
<td>-0.1559</td>
</tr>
<tr>
<td>1</td>
<td>-0.2790</td>
<td>-0.0308</td>
<td></td>
<td>-0.0097</td>
</tr>
<tr>
<td>2</td>
<td>0.4677</td>
<td>0.3629</td>
<td>0.3665</td>
<td>0.3987</td>
</tr>
<tr>
<td>3</td>
<td>-0.1987</td>
<td>-0.1989</td>
<td>-0.1988</td>
<td>-0.1989</td>
</tr>
<tr>
<td>$\delta p^{(0)} = \frac{\delta p^{(B)} - \delta p^{(A)}}{1 + \delta p^{(A)}}$</td>
<td>-0.0003</td>
<td>0.0000</td>
<td></td>
<td>0.0000</td>
</tr>
<tr>
<td>1</td>
<td>0.2886</td>
<td>0.2886</td>
<td>0.2887</td>
<td>0.2886</td>
</tr>
<tr>
<td>2</td>
<td>0.3665</td>
<td>0.3665</td>
<td>0.3987</td>
<td>0.3987</td>
</tr>
<tr>
<td>3</td>
<td>-0.1987</td>
<td>-0.1989</td>
<td>-0.1988</td>
<td>-0.1989</td>
</tr>
</tbody>
</table>

Table 3-19 Updating parameter values in Case 2

Because analytical model A in Case 2 has only discretisation errors when compared with the reference model in the case study, the obtained updating parameter values for this model can be considered as compensations to the discretisation errors. Then, the updating parameter values for analytical model B in the same case can be considered as combinations of the corrections for the parameter errors and compensations for the discretisation errors as described with equation (3.33).

From the above analysis, it is clear that an FE model of a structure without errors in configuration or discretisation and having errors only in model parameters can be
correctly modified through a model updating session to have the parameters in the modified model equal to the exact parameter values of the structure. If an FE model with both discretisation errors and parameter errors is subjected to a model updating session, the final updating parameter values will not be equal to the corrections of the parameter errors but will be combinations of the corrections and the compensations for the discretisation errors.

3.4.1.3 Updating models of industrial structures

In Cases 4 and 5, which are case studies on updating FE models for industrial structures, the full information about the differences between the analytical models and the structures could not be obtained. The results of updating these models show that the reference data have not been reproduced exactly.

In Case 5 – the case study for updating the CCOC model – the initial eigenvector correlation showed that the initial FE model can predict eigenvectors with a high accuracy. This encourages the use of eigenvectors in updating. The final updating parameter values in the modified models from three updating sessions seem to be quite far away from the range of acceptable adjustment. However, all three updating sessions have modified the initial model with improved correlation to the reference data both in eigenvalues and in eigenvectors.

In the case study of updating the RBH model, the correlation of eigenvalues and eigenvectors for the correlated mode pairs was not significantly improved. Some experimental modes are neither correlated to modes predicted by the initial model nor to those by the modified models from any updating session. None of these modified models can be taken as an acceptably updated model. Through several updating sessions on the model, among which the results from only three sessions are presented in this thesis, the initial model of the structure was considered as unsuitable for model updating. There are several reasons for the inability to update the initial model of the RBH component through a normal numerical updating program.

The first reason is the possible existence of configuration errors in the model. Table 3-20 lists the brief descriptions for the eigenvectors predicted by the initial RBH model. "nND in" or "nND out" modes are the $n^{th}$ nodal diametral modes of the outer case with the deformation at the two ends of the case in phase or out of phase, respectively. "Inner xx" modes means the modes are dominated by the deformation (displacement) of the inner case. "$R_z$" means rotating about the Z axis, which is along the axis of symmetry of the structure. The modes with "N" in the column of "Match" of the table are not correlated to any mode in the experimental data set.

By examining these uncorrelated modes, it can be found that most of these modes have a large part of the modal strain energy located in the vanes of the component. The actual structure of the vane is a thin hollow box, while in the model it
is represented with a layer of plate and beam elements as in Figure 3-9. The simplification of the vane structure in the model makes it unable to represent the dynamic properties of the vanes properly. In Session 3 of the case study, the stiffness of the elements in the vanes was modified with the intention of extending the limitation of the parameter. The parameter reached its final value of 18.38 – the Young's modulus of the material for the vanes is increased by the 18.38 times of the initial data. This value does not mean that the Young's modulus of the material should be increased so much but indicates that the improper modelling of the vane may be the main source of the discrepancies between the model predictions and the experimental data.

<table>
<thead>
<tr>
<th>No.</th>
<th>Mode</th>
<th>Match</th>
<th>No.</th>
<th>Mode</th>
<th>Match</th>
<th>No.</th>
<th>Mode</th>
<th>Match</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2ND out</td>
<td>Y</td>
<td>8</td>
<td>3ND in</td>
<td>Y</td>
<td>15</td>
<td>5ND in</td>
<td>N</td>
</tr>
<tr>
<td>2</td>
<td>2ND out</td>
<td>Y</td>
<td>9</td>
<td>3ND in</td>
<td>Y</td>
<td>16</td>
<td>5ND in</td>
<td>N</td>
</tr>
<tr>
<td>3</td>
<td>Inner R&lt;sub&gt;z&lt;/sub&gt;</td>
<td>N</td>
<td>10</td>
<td>4ND out</td>
<td>Y</td>
<td>17</td>
<td>6ND in</td>
<td>N</td>
</tr>
<tr>
<td>4</td>
<td>3ND out</td>
<td>N</td>
<td>11</td>
<td>4ND out</td>
<td>Y</td>
<td>18</td>
<td>Inner R&lt;sub&gt;x&lt;/sub&gt;</td>
<td>N</td>
</tr>
<tr>
<td>5</td>
<td>3ND out</td>
<td>Y</td>
<td>12</td>
<td>Inner Z&lt;sub&gt;0&lt;/sub&gt;</td>
<td>N</td>
<td>19</td>
<td>Inner R&lt;sub&gt;y&lt;/sub&gt;</td>
<td>N</td>
</tr>
<tr>
<td>6</td>
<td>2ND in</td>
<td>Y</td>
<td>13</td>
<td>4ND in</td>
<td>Y</td>
<td>20</td>
<td>5ND out</td>
<td>Y</td>
</tr>
<tr>
<td>7</td>
<td>2ND in</td>
<td>Y</td>
<td>14</td>
<td>4ND in</td>
<td>Y</td>
<td>21</td>
<td>5ND out</td>
<td>Y</td>
</tr>
</tbody>
</table>

Table 3-20 Brief description of eigenvectors predicted by the initial RBH model

The second reason for the model not suitable of being updated is the improper selection of the measurement DOFs. In Figure 3-10 (Page 76), the auto-MAC shows that some modes in the experimental data set are not linearly independent according to the measurement DOFs. Because there was no measurement taken on either the inner case or the vanes in the experiment, the modes dominated with the displacement or deformation of these two parts cannot be well represented in the measurement. This is a possible reason why both the experimental data set and the analytical data set have some modes not correlated and why the model cannot be updated to improve the correlation for these modes.

Another possible reason for the inability to update the initial model with a normal numerical updating program is the presence of discretisation errors in the model. The effects of the discretisation errors on the out-of-phase modes of the RBH component can also be observed from the natural frequency differences for these modes. These vary from about -9% to +18%, while those for the in-phase modes are only from -0.2% to +5%.

3.4.1.4 Conclusions for updating models with different errors

Most FE models for industrial structures have all three kinds of error and the effects of these errors on model predictions should be taken into account when updating the models and assessing the model updating results. From the case studies in this
chapter and the analysis of the results from the updating sessions, the following conclusions can be drawn:

(1) model updating can eliminate parameter errors correctly provided that the reference data are noise-free;

(2) model updating can improve the correlation between the model predictions and the reference data when there is noise in the reference data with updating parameter values different from the "actual" data;

(3) model updating can reduce the effects of discretisation errors on structural dynamic predictions with updating parameter values different from the "actual" data;

(4) model updating cannot function in the presence of configuration errors if the required structural dynamic properties are affected significantly by these errors.

3.4.2 Effects of eigenvalues and eigenvectors on model updating

When using the inverse eigen-sensitivity method for model updating, both/either eigenvalues and/or eigenvectors of the structure can be used in the updating equation. From Cases 1 to 3 in the previous sub-chapter, it can be seen that the influence of eigenvalues on model updating is different from that of eigenvectors.

The key calculation in an updating session is to solve an updating equation. In the left hand side of the equation, there is a sensitivity matrix. From the sensitivity formulae (see Section 3.2.2.1 of this thesis) and the practice of model updating, it is known that because the eigenvectors are usually mass normalized, the amplitude of the eigenvalue sensitivity to an updating parameter is almost always greater than that of an element of the eigenvector sensitivity to the same parameter. In the right hand side of the equation, there is a vector containing the residuals of the eigenvalues and eigenvectors. The amplitudes of the eigenvalue residuals are almost always greater than those of the eigenvector residuals. The amplitude of the eigenvalue residual of a mode with a higher natural frequency is greater than that with a lower natural frequency if both modes have the same amplitudes in the relative eigenvalue differences.

Because of the different amplitudes between the sensitivities and the residuals, the contribution of an eigenvalue to the rank of the sensitivity matrix is usually greater than that of an eigenvector element and the contribution of an eigenvalue residual to the solution of the updating equation is usually greater than that of an element of an eigenvector residual if no weighting factors are applied to the updating equation.

In Case 1 in this chapter, there are four sessions with the reference data contaminated with noise. Different weighting factors were applied in these sessions, but none of these sessions achieved a modified model which was capable of reproducing the reference data, and the obtained updating parameter values were distant from the target values. By analysing the results from these sessions, the different effects of eigenvalues
and eigenvectors on the updating procedure can be revealed.

When only eigenvalues are included in the updating equation of an updating session and the weighting factors for eigenvalues are the same for all modes, as in Session 5 of Case 1, the weighting factor $W_\lambda$ is 1.0 and $W_\phi$ is 0.0. Table 3-21 lists the correlation of eigenvalues and eigenvectors for the initial model and the modified model from the session. At the end of the updating session, the eigenvalue correlation of Modes 4, 6, and 7 has been improved. However, although Mode 2 has the greatest relative natural frequency difference before model updating, the eigenvalue correlation of this mode was not improved in the session. This is because the objective function of the updating session was not composed of the relative natural frequency differences but the norm of the eigenvalue residuals. The modes with greater eigenvalue residuals contributed more to the solution of the updating equation and their eigenvalue correlation was improved. In this session, the norm of the eigenvalue residuals was minimized while the average absolute natural frequency difference became worse. The eigenvectors were not included in the updating equation in this session and the eigenvector correlation for all eight modes was not improved.

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Exp. Freq.</th>
<th>Initial</th>
<th>Session 5</th>
<th>Session 6</th>
<th>Session 8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>MAC</td>
<td>NFD</td>
<td>MAC</td>
<td>NFD</td>
</tr>
<tr>
<td>1</td>
<td>27.37</td>
<td>93.94</td>
<td>1.31</td>
<td>90.77</td>
<td>1.96</td>
</tr>
<tr>
<td>2</td>
<td>29.29</td>
<td>88.77</td>
<td>4.30</td>
<td>87.64</td>
<td>4.77</td>
</tr>
<tr>
<td>3</td>
<td>62.92</td>
<td>89.99</td>
<td>2.59</td>
<td>62.92</td>
<td>3.37</td>
</tr>
<tr>
<td>4</td>
<td>64.32</td>
<td>89.97</td>
<td>-0.90</td>
<td>85.29</td>
<td>-0.09</td>
</tr>
<tr>
<td>5</td>
<td>78.21</td>
<td>84.27</td>
<td>1.25</td>
<td>65.96</td>
<td>2.13</td>
</tr>
<tr>
<td>6</td>
<td>93.47</td>
<td>92.94</td>
<td>-2.87</td>
<td>84.77</td>
<td>-1.97</td>
</tr>
<tr>
<td>7</td>
<td>119.71</td>
<td>95.17</td>
<td>-2.05</td>
<td>92.85</td>
<td>-1.17</td>
</tr>
<tr>
<td>8</td>
<td>130.64</td>
<td>84.36</td>
<td>-0.03</td>
<td>71.37</td>
<td>0.82</td>
</tr>
<tr>
<td>Average</td>
<td>89.92</td>
<td>1.91</td>
<td>80.20</td>
<td>2.04</td>
<td>92.83</td>
</tr>
</tbody>
</table>

Table 3-21 Correlation before and after updating from sessions in Case 1

In order to achieve a balanced contribution from the information for each mode to the solution of the updating equation, weighting factors for the eigenvalues can be included in an updating session. In Session 6 of Case 1, the weighting factor for the eigenvalue of each mode was set to the inverse of the eigenvalue itself while the weighting factors for all eigenvectors were set to 0.0. In this way, the right hand side of the updating equation was a vector containing the relative eigenvalue differences. Modes with the same relative eigenvalue difference had the same contribution to the solution of the updating equation. In Table 3-21 there are MAC values and NFD values
for the modified model from Session 6 of Case 1. From these correlation parameters, it can be seen that although the eigenvalue correlation of some modes from Session 6 is not as good as that of the same modes from Session 5, the eigenvalue correlation for most modes from Session 6 is better than that from Session 5 and the average absolute relative eigenvalue difference from Session 6 is much smaller than that from Session 5. Although the information of eigenvectors was not included in the updating equation in this session, the eigenvector correlation was improved from the modified model.

Although the amplitudes of an element of the eigenvector sensitivity to an updating parameter may be smaller than that of the eigenvalue sensitivity to the same updating parameter, the information in the eigenvectors can still contribute to the solution of the updating equation in an updating session, especially when using weighting factors to balance the effects of eigenvalues and eigenvectors. In Session 8 of Case 1, the weighting factor for the eigenvalue of each mode was set to the inverse of the eigenvalue and that for each element of an eigenvector was set to the inverse of the norm of the eigenvector. In this way, the discrepancies of both eigenvalues and eigenvectors are normalized. The correlation results for the modified model from this session are also listed in Table 3-21. From these correlation parameters, it can be seen that a balanced improvement on correlation has been obtained not only among the eigenvalues of all modes but also between eigenvalues and eigenvectors.

Besides comparing the results of updating, the individual effects of eigenvalues and eigenvectors on an updating session can also be observed through the condition number of the coefficient matrix (the product of the weighting factor matrix and the sensitivity matrix) of the updating equation. Figure 3-17 shows the condition numbers of the coefficient matrices at each iteration step in some sessions in Case 1.
The curve marked with Session 5 (3 parameters) shows the condition numbers of the coefficient matrices when three updating parameters the same as those in Sessions 6 and 8 were used in an updating session with the same weighting factors as those for Session 5. From this picture, it can be seen that using weighting factors to balance the contribution of each mode can improve the ill-condition of the coefficient matrix of the updating equation. Furthermore, adding eigenvector sensitivity information into the updating equation improves not only eigenvector correlations but also the ill-condition of the coefficient matrix of the updating equation. From Figure 3-17, it can be seen that the coefficient matrices of Session 8 are the least ill-conditioned among all the sessions in Case 1.

From the above discussion of the updating results, the relationship between performing an updating session with eigenvalues only or with eigenvalues and eigenvectors together in the updating equation can be listed below.

(1) Using the sensitivities and the residuals of eigenvalues in the updating equation without weighting factors applied to them may minimise the norm of the eigenvalue residuals. However, if there are errors other than parameter errors in the initial model or the reference data are noise-contaminated, improvement in either the eigenvalue correlation or the eigenvector correlation for some modes, especially for those with lower natural frequencies, cannot be obtained.

(2) By using the inverse of the eigenvalue of each mode as the weighting factor applied to the eigenvalue sensitivity and to the eigenvalue residual of the mode, it is possible to improve the condition number of the coefficient matrix of the updating equation and to get improved eigenvalue and eigenvector correlation for most modes. The norm of the relative eigenvalue differences may be minimized through the updating session.

(3) Although the amplitudes of the eigenvector sensitivity and the eigenvector residual of a mode may be much smaller than those of the eigenvalue sensitivity and the eigenvalue residual of the mode, including eigenvectors in the updating equation can improve the condition number of the coefficient matrix of the equation. The improvement in the eigenvalue correlation may not be the best possible. However, a balanced improvement on correlation of eigenvalues and eigenvectors could be achieved by putting proper weighting factors on the updating equation to balance the contributions of eigenvalues and eigenvectors.

3.4.3 Requirements for an FE model to be updated

Updating an FE model for structural dynamics is to modify the model for reducing the discrepancies between model predictions and reference data. Usually, model updating is performed with a numerical calculating program. Based on the analysis of the updating results of the case studies in this chapter, some requirements
could be proposed for initial models and the reference data in order to perform the updating calculation successfully.

3.4.3.1 Requirement on the initial model

Usually, an FE model may have all three kinds of error – discretisation errors, configuration errors, and parameter errors. A numerical computation program for model updating can only deal properly with parameter errors and partly with discretisation errors in models. Thus, before performing a model updating session, the initial model must satisfy some requirements.

(1) When the initial model is for a continuous structure, there will be discretisation errors, more or less. Although model updating can minimize the discrepancies between the model predictions and the reference data caused by the discretisation errors, the final updating parameter values may be a distance away from the "true" values of the parameters. In order to get an updated model without losing the physical meaning of its parameters, the effects of the discretisation errors on the structural dynamic properties of interest should not be significant. The distortion of the parameter values due to the discretisation errors should be estimated before performing the model updating session.

(2) When the initial model is for an industrial structure that has complicated geometrical shapes or complicated connections between its components or parts, the model may have configuration errors due to the simplification in the model for these geometrical shapes and joints. If the configuration errors in the model have no significant effects on the structural dynamic properties of interest, the model may be updated to reduce the discrepancies due to these errors. However, if the configuration errors in the model affect significantly the structural dynamic properties of interest, the discrepancies caused by the configuration errors cannot be reduced in model updating sessions with a numerical model-updating program. Thus, for an FE model subjected to an updating procedure with a numerical calculation program, the configuration errors in the model should have as little as possible effects on the structural properties of interest.

(3) Although a model updating procedure with a numerical calculation program can update a model with parameter errors, it does not guarantee that any model with only parameter errors can be updated. In general, the modelling engineer has information about the structure for creating an FE model. The values he/she has assigned to the parameters in the model have a tolerance that can be set as the limitation for the change of the parameter values. If some parameters in an FE model have errors which are too large that the dynamic properties of the model are far away from the reference properties, the numerical updating program may not be able to establish the correlation between the predicted and the reference properties. In this case, model updating cannot be performed properly.
3.4.3.2 Requirements on reference data

Although there are other kinds of data that can be used as the reference data in model updating, all case studies for model updating in this chapter use modal properties as the reference data. The discussion about the requirements for reference data in model updating will focus on modal data in this section.

(1) Requirement on natural frequencies.

To structural dynamic analysis engineers, the natural frequencies of a structure are of the most important among all structural dynamic properties. The improvement on eigenvalue correlation is usually set as the primary criterion for assessing model updating results. Thus, the information (sensitivities and residuals) of the natural frequencies of all the modes in the frequency range of interest must be included in the updating equation.

For symmetric or quasi-symmetric structures, such as the structures in Cases 4 and 5, it is not easy to identify all the natural frequencies of double modes (modes with identical or close natural frequencies and orthogonal mode shapes) from experimental measurements. Usually, one mode of a pair of double modes can be identified while the other is likely to be missing from the experiment. In Case 4, one mode of the pair of "3ND out" modes was missing in the experimental data set. Because the natural frequencies of a pair of double modes are almost the same in the structure, and are exactly the same in the model predictions, the information for the eigenvalues of a pair of double modes is almost the same. Thus, missing the information of the eigenvalue of one mode in a pair of double modes will not affect model updating. However, for a quasi-axisymmetric structure, two modes with the same nodal diametral order but orthogonal eigenvectors may have different natural frequencies. In this case, the information of the eigenvalues of both modes in the pair is important to model updating for improving the eigenvalue correlation for them.

(2) Requirement on mode shapes.

In general, mode shapes are only measured for selected points (DOFs) on a structure. The number of the measurement DOFs is much smaller than the number of total DOFs in an FE model for the structure. The requirements on mode shape data for model updating should consider the number and the places of the mode shape measurements.

The information of mode shapes is used in a model updating session at two points. The first one is to correlate modes in the reference data set with the modes in the analytical data set. Only after a mode in the analytical data set is correlated with a mode in the reference data set, can residuals of the eigenvalue and the eigenvector of this mode be calculated. Only the sensitivities and the residuals of the correlated modes will
be included in the updating equation. For this usage, the mode shape measurements are required to be able to distinguish the mode shapes from each other using the measurement DOFs. This can be fulfilled by selecting measurement DOFs using the existing Effective Independent (EI) test planning technique. The second point to include eigenvectors in a model updating session is to make use of the information of reference eigenvectors and analytical eigenvectors in the updating equation. Including the eigenvectors in an updating equation can alleviate ill-condition of the coefficient matrix. For this purpose, the requirement of the reference eigenvector is that the mode shapes for the measurement DOFs have a greater contribution to the rank of the coefficient matrix than other possible choices for the number and the places of the measurement DOFs. The fulfilment of this requirement for the reference eigenvectors is not the same as that for the purpose of correlation and will be discussed in Chapter 6 – Testing for Model Validation.

(3) Requirement concerning noise in reference data.

From the practice of modal testing and model updating, it is well known that the experimental modal data obtained from a test will contain a certain amount of noise and the precision of measured natural frequencies is much higher than that of measured mode shapes. In this chapter, there are four updating sessions with the reference data contaminated by artificial noise. The effects of noise in the reference data on model updating can be explained from the analysis of the updating results from these sessions (Table 3-4). In all these sessions, the reference data are the same. They have an average 2% noise on the natural frequencies and 30% noise on the mode shapes.

Comparing the updating results listed in Table 3-4, it can be seen that the modified model from Session 8 is the closest to the reference model among all the modified models from the updating sessions with the same noise-contaminated reference data. In Sessions 6 and 8, the weighting factors for eigenvalues were the same and the eigenvectors were included into the updating equation in Session 8 but not in Session 6. Although the level of the noise added to the reference eigenvectors was much greater than that added to the reference eigenvalues, including the eigenvectors in the updating equation improved the results.

There are two reasons for this. The first reason is the improvement of the ill-condition of the coefficient matrix by adding more data into the matrix which has been discussed in the previous sub-section. The second reason is the statistic property of the noise which contaminates the reference data. One of the updating results – the final value of an updating parameter – can be represented by the sum of two parts:

\[ p = p^0 + p^N \]

where \( p^0 \) is the true value that corresponds to the target value of this parameter and \( p^N \) is
the disturbed value that is due to the noise in the reference data.

It is supposed (although it is difficult to give a theoretical proof) that if the noise in the reference data is of normal distribution and with a zero mean-value, the un-biased estimation of $p^N$ is zero. In a model updating session which uses only eigenvalues in the updating equation, the small number of eigenvalues may result in a non-zero mean value of the noise added to the eigenvalues and thus a non-zero value of $p^N$ could be obtained. Including eigenvector data into the calculation means that a large number of data can be used as samples for estimation. This improves the estimation of the mean value of the noise and therefore improves the estimation for the value of $p^N$.

From the above discussion, the requirement on the reference data when concerning noise in the data is to increase the number of the data used in a model updating session. Including mode shape data in the updating equation is the best way to fulfil this requirement.

### 3.5 Summary on FE model updating for structural dynamics

In this chapter, the theoretical background for model updating has been described with an emphasis on the inverse eigen-sensitivity method. Through case studies on some simple plate models and two industrial structural models, the model updating procedure has been examined in detail. The analysis of the results from the case studies has introduced a perspective on model updating. The influences of different errors in an FE model to the updating sessions performed to the model have been discussed and followed by the analysis of the effects of eigenvalues and eigenvectors on model updating. The requirements for an FE model to be updated successfully with a numerical calculation program have been proposed. The methods to fulfil some of the requirements will be discussed in the following chapters.
Chapter 4

FE Model Verification

4.1 Introduction – FE model verification and validation

By modifying the values of some parameters in an FE model to improve the correlation between dynamic property predictions from the model and reference data as described in the previous chapter, the model updating procedure plays a central role in the process of model validation. However, it is often found that either the parameters in the modified models obtained from updating sessions lose their physical meaning, or there is no significant improvement in the accuracy of the dynamic property predictions from the modified models. There may be several reasons for this and one of them is that the model is not a verified model and is thus not capable of representing the true structural behaviour.

In general, there may be as many as three kinds of error in an FE model that can cause the discrepancies between dynamic property predictions and reference data (usually, experimental data). Model updating, through modifying the values of updating parameters, can only eliminate the discrepancies caused by parameter errors. When performing model updating on an FE model with discretisation errors and/or configuration errors, it cannot be guaranteed that the discrepancies caused by these errors are reduced. Accordingly, it is required that a model being subjected to a model updating procedure is a verified model or, in other words, an updatable model.

The definition of FE model verification in this thesis refers to the procedure that checks the dynamic properties of FE models in such a way that the form/configuration of a verified model makes it possible to have acceptably similar dynamic behaviour to that of the actual structure under consideration with only its parameter values modified if necessary. From the discussion on the three kinds of error of FE model and the flowchart of the model validation process in Chapter 1 of the thesis, and from the discussion about the requirements for an FE model being able to be updated in the
previous chapter, the model verification procedure for an FE model should contain, at least, the following contents.

(1) Checking the convergence of the model. The number of modes (or the frequency range) of convergent prediction should be determined at this stage. The application of model updating has to be limited to those convergent-predicted modes. Furthermore, the effect of the discretisation errors on updating parameters should be estimated.

(2) Checking the effect of uncertainties in the models on the possible results of model updating. The effect of parameter errors in an FE model can be minimised in a model updating procedure. However, the effects of other uncertainties, especially configuration errors, cannot expect to be minimised by model updating.

If an FE model has passed through the verification procedure listed above, the model is ready to be subjected to a model updating procedure. This means that model is a verified model. In order to verify an FE model, one or more sets of experimental data have to be used as the reference data to determine the extent to which the model is verified.

In this chapter, discretisation errors and configuration errors in FE models will be studied and followed by the proposal of two methods for model verification, one deals with discretisation errors and the other deals with configuration errors.

4.2 Model verification via convergence check

4.2.1 Discretisation errors of FE models

In general, an FE model, denoted as the analytical model, represents a continuous structure. In theory, the structure can also be considered as a model, denoted as the structural model, that has an infinite number of DOFs and an infinitesimal mesh size. The data obtained from experiments on the structure can be considered as the output from this structural model. The discrepancies between the analytical model predictions and the experimental data can, then, be taken as the differences between the calculated results from the analytical model and the results from the structural model. No matter how small the mesh size of the analytical model is, it can never be exactly the same as that of the structural model. Thus, the discrepancies of the predictions from the analytical model caused by the different mesh size can never be eliminated completely but be reduced to small enough for practice. Furthermore, the amplitude of this kind of discrepancy is also a function of the frequency range over which the behaviour of the structure is compared with the analytical model predictions. In general, the higher the frequency to which the comparison is made, the greater the discrepancies will be.

It is often thought that the discrepancies caused by discretisation errors can be “easily” reduced by making a new FE model with a finer mesh and, thus, this kind of
discrepancy is much smaller than those caused by other factors. In practice, building an FE model for a typical complicated industrial structure is not a simple task. To build a second FE model with finer mesh is expensive in most cases at the moment. In this sense, it cannot necessarily be assumed that the discrepancies caused by discretisation errors are much smaller than those caused by other factors and that it is easy to eliminate or ignore the amplitude of the discrepancies caused by an over-coarse mesh.

When an FE model is subjected to a model updating procedure, no matter what algorithm or method is used, the discrepancies between FE model predictions and the reference data for the procedure are used as an objective function. When considering the discrepancies as a combination of those caused by each of the four factors (discretisation errors, configuration errors, parameter errors, and noise in experimental data), each factor has a contribution to solutions of the updating equation in the procedure. In the previous chapter, a case study with discretisation errors in an analytical model has shown that the discretisation errors in the model can make the updating parameter values different from the target values.

In order to make an FE model suitable for being subjected to an updating procedure, the discretisation errors in the model should be checked and the effects of the errors to the updating results from the procedure should be estimated before the updating procedure can be carried out.

4.2.1.1 Discretisation errors of FE models with bar elements

An FE model with only bar elements is a simple model. The dynamic properties of the structure of a bar in extension can be calculated with mathematical formulae. Thus, the discretisation errors can be easily examined for this type of model.

4.2.1.1.1 Solutions for a structure of the bar in extension

The equation of motion for a bar in extension is:

$$E \frac{d^2 u}{dx^2} - \rho \ddot{u} = 0$$  \hspace{1cm} (4.1)

where $E$ and $\rho$ are the Young’s modulus and the mass density, respectively, of the material of the bar, $x$ is the coordinate along the axle of the bar, and $u(x)$ is the vibration of the bar in the $x$ direction.

The general solution of the above equation is:

$$u(x,t) = u(x) \cos \omega t$$  \hspace{1cm} (4.2)

and

$$u(x) = a \sin \frac{x}{l} + b \cos \frac{x}{l}$$  \hspace{1cm} (4.3)

with

$$\lambda = \sqrt{\rho \cdot l^2 / E \cdot \omega}$$  \hspace{1cm} (4.4)

where $\lambda$ is the characteristic root of equation 4.1 and $l$ is the length of the bar.

For the free-free case, the characteristic roots of the equation of motion
(equation 4.1) are:
\[ \lambda_k = k\pi, \quad k = 1, 2, \ldots \] (4.5)
Then, the natural frequency, \( \omega_k \), can be defined from equation (4.4):
\[ \omega_k = \frac{E}{\sqrt{\rho \cdot l^2}} \cdot k\pi \] (4.6)
Corresponding to the natural frequency of \( \omega_k \), the mode shape, \( u_k(x) \), is:
\[ u_k(x) = b \cos \left( k\pi \frac{x}{l} \right) \] (4.7)
The value of \( b \) can be determined by the mass normalisation calculation.
\[ \int_0^l (u_k(x) u_k(x) \rho A) \, dx = \int_0^l b^2 \cos^2 \left( k\pi \frac{x}{l} \right) \rho A \, dx = b^2 \frac{\rho A l}{2} = 1 \] (4.8)
Thus,
\[ b = \pm \frac{2}{\rho A l} \] (4.9)
The strain energy in a small section \( dx \) is \( dU_\sigma = \frac{1}{2} \sigma \varepsilon A dx = \frac{1}{2} EA \left( \frac{du}{dx} \right)^2 \, dx \). The strain energy in the whole length of the bar is:
\[ U_\sigma = \int_0^l \frac{1}{2} EA \left( \frac{du}{dx} \right)^2 \, dx = EA \left( \frac{b k \pi}{l} \sin \left( \frac{k\pi}{l} x \right) \right)^2 \, dx = \frac{1}{2} \frac{E}{\rho l^2} (k\pi)^2 \] (4.10)
The solutions for the bar in extension under three different boundary conditions are listed in Table 4-1

<table>
<thead>
<tr>
<th>( \lambda_k )</th>
<th>Free-free</th>
<th>Clamped-free</th>
<th>Clamped-clamped</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k\pi )</td>
<td>( \frac{2k-1}{2}\pi )</td>
<td>( k\pi )</td>
<td></td>
</tr>
<tr>
<td>( \omega_k )</td>
<td>( \frac{E}{\sqrt{\rho \cdot l^2}} \cdot k\pi )</td>
<td>( \frac{E}{\sqrt{\rho \cdot l^2}} \cdot \frac{2k-1}{2}\pi )</td>
<td>( \frac{E}{\sqrt{\rho \cdot l^2}} \cdot k\pi )</td>
</tr>
<tr>
<td>( u_k(x) )</td>
<td>( b \cos \left( k\pi \frac{x}{l} \right) )</td>
<td>( a \sin \left( \frac{(2k-1)\pi x}{2 l} \right) )</td>
<td>( a \sin \left( \frac{k\pi x}{l} \right) )</td>
</tr>
<tr>
<td>Constant</td>
<td>( b = \pm \frac{2}{\rho A l} )</td>
<td>( a = \pm \frac{2}{\rho A l} )</td>
<td>( a = \pm \frac{2}{\rho A l} )</td>
</tr>
</tbody>
</table>

Table 4-1 Solutions for the bar in extension at different boundary conditions
4.2.1.1.2 **Solutions for an FE model of the bar in extension using consistent mass matrices**

Suppose an FE model for the bar in extension has \( N \) elements, each of which has the same mesh size. The equation of motion of the \( i^{th} \) element (between the \((i-1)^{th}\) and the \(i^{th}\) nodes) in the model is:

\[
[M_i]\{\ddot{u}_{i-1}\} + \{F_{i-1}\} = 0
\]  \hspace{1cm} (4.11)

\( F_{i-1} \) and \( F_i \) are the forces applied to the element. In free vibration, there is no external force. Thus, \( F_{i-1} \) and \( F_i \) are the internal forces between this element and its neighbouring elements:

\[
\begin{bmatrix}
F_{i-1} \\
F_i
\end{bmatrix} = \begin{bmatrix}
K_i
\end{bmatrix}\begin{bmatrix}
u_{i-1} \\
u_i
\end{bmatrix} = \frac{EA}{\Delta x}\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}\begin{bmatrix}
u_{i-1} \\
u_i
\end{bmatrix}. \hspace{1cm} (4.12)
\]

When using the consistent mass-matrix approach for forming the mass matrices of elements, the mass matrix for the \( i^{th} \) element is:

\[
[M_i] = \frac{\Delta x}{6} \rho A \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]  \hspace{1cm} (4.13)

After assembling all elements, the equation of motion for the whole model is:

\[
[M]\begin{bmatrix}
\ddot{u}_0 \\
\ddot{u}_1 \\
\ddot{u}_2 \\
\vdots \\
\ddot{u}_N
\end{bmatrix} + [K]\begin{bmatrix}
u_0 \\
u_1 \\
u_2 \\
\vdots \\
u_N
\end{bmatrix} = \{0\}
\]  \hspace{1cm} (4.14)

with \( [M] = \frac{\Delta x}{6} \rho A \) and \( [K] = \frac{EA}{\Delta x} \).

The general solution of equation (4.14) can be written as:

\[
u_i(t) = u_i \sin \omega t
\]  \hspace{1cm} (4.16)

and

\[
u_i = a \sin \mu t + b \cos \mu t
\]  \hspace{1cm} (4.17)

where \( \mu \) is the characteristic root of equation (4.14).

At node \( i \), the free vibration equilibrium equation takes the form:

\footnote{Taking equation (4.14) as a second-order linear differential equation, \( \mu \) is the solution of its characteristic equation. It is different from what is meant by the term "eigenvalue" that is used generally as a solution, \( \lambda \), to the equation of \( ( [K] - \lambda [M] )\{\phi\} = 0 \).}
Thus
\[ EA \left( \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x} \right) + \omega^2 \frac{\rho A \Delta x}{6} \left( \frac{u_{i-1} + 4u_i + u_{i+1}}{\Delta x} \right) = 0 \] (4.18)

Thus
\[
\frac{EA}{\Delta x} \left[ a \left( \sin(i-1) \mu - 2 \sin \dot{\mu} + \sin(i+1) \mu \right) + b \left( \cos(i-1) \mu - 2 \cos \dot{\mu} + \cos(i+1) \mu \right) \right] + \omega^2 \frac{\rho A \Delta x}{6} \left[ a \left( \sin(i-1) \mu + 4 \sin \dot{\mu} + \sin(i+1) \mu \right) + b \left( \cos(i-1) \mu + 4 \cos \dot{\mu} + \cos(i+1) \mu \right) \right] = 0
\] (4.19)

Substituting
\[
\begin{cases}
\sin(i \pm 1) \mu = \sin \dot{\mu} \cos \mu \pm \cos \dot{\mu} \sin \mu \\
\cos(i \pm 1) \mu = \cos \dot{\mu} \cos \mu \mp \sin \dot{\mu} \sin \mu
\end{cases}
\] (4.20)

into equation (4.19) yields:
\[
\frac{EA}{\Delta x} \left[ 2a \sin \dot{\mu} \left( \cos \mu - 1 \right) + b \cos \dot{\mu} \left( \cos \mu - 1 \right) \right] + \omega^2 \frac{\rho A \Delta x}{6} \left[ 2a \sin \dot{\mu} \left( \cos \mu + 2 \right) + b \cos \dot{\mu} \left( \cos \mu + 2 \right) \right] = 0
\] (4.21)

With either \( a \) or \( b \) not equal to zero,
\[
\frac{EA}{\Delta x} \left( \cos \mu - 1 \right) + \omega^2 \frac{\rho A \Delta x}{6} \left( \cos \mu + 2 \right) = 0
\] (4.22)

Thus
\[
\omega^2 = \frac{6E}{\rho A x^2} \left( \frac{1 \mp \cos \mu}{2 + \cos \mu} \right)
\] (4.23)

Characteristic root and natural frequency in the case of the free-free condition

For the free-free case, in the first line of equation (4.14)
\[
\frac{EA}{\Delta x} \left( \frac{u_0 - 2u_i + u_{i+1}}{\Delta x} \right) + \omega^2 \frac{\rho A \Delta x}{6} \left( \frac{2u_0 + u_i}{\Delta x} \right) = 0
\] (4.24)

combining equations (4.17) and (4.24) it can be shown that,
\[
\frac{EA}{\Delta x} \left( b - a \sin \mu - b \cos \mu \right) - \omega^2 \frac{\rho A \Delta x}{6} \left( 2b + a \sin \mu + b \cos \mu \right) = 0
\] (4.25)

Replacing \( \omega^2 \) with the expression in equation (4.23)
\[
b \left( 1 \mp \cos \mu \right) - a \sin \mu \mp \frac{1 \mp \cos \mu}{2 + \cos \mu} (b(2 + \cos \mu) + a \sin \mu) = 0
\]

After algebraic manipulation:
\[
a \sin \mu \left( 1 \pm \frac{1 \mp \cos \mu}{2 + \cos \mu} \right) = 0
\]

Thus
\[
a = 0.
\] (4.26)

For the same free-free case, in the last line of equation (4.14)
\[
\frac{EA}{\Delta x} \left( \frac{u_N - u_{N-1}}{\Delta x} \right) - \omega^2 \frac{\rho A \Delta x}{6} \left( 2u_N + u_{N-1} \right) = 0
\] (4.27)

Taking account of equations (4.17) and (4.26),
\[ b \cos(N-1)\mu - b \cos N\mu + \frac{1 - \cos \mu}{2 + \cos \mu} [b \cos(N-1)\mu + 2b \cos N\mu] = 0 \quad (4.28) \]

After algebraic manipulation:

\[ \sin \mu \sin N\mu \frac{1 - \cos \mu}{2 + \cos \mu} = 0 \quad (4.29) \]

From

\[ \sin N\mu = 0 \quad (4.30) \]

the characteristic root of the equation of motion is:

\[ \mu = k \pi / N, \quad k = 1, 2, \ldots \quad (4.31) \]

Substituting this into equation (4.23), the eigenvalue of the \( k \)th mode is given by

\[ \omega^2_k = \frac{6E}{\rho \Delta x^2} \frac{1 - \cos(k \pi / N)}{2 + \cos(k \pi / N)} \quad (4.32) \]

Making use of the series expansions of \( \cos(x) \) and of \( \frac{1}{1+x} \):

\[ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{and} \quad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \]

equation (4.32) becomes:

\[ \omega^2_k = \frac{6E}{\rho \Delta x^2} \left( \frac{(k\pi)^2}{6N^2} \left( 1 + \frac{1}{12} \left( \frac{k\pi}{N} \right)^2 + \frac{1}{24} \left( \frac{k\pi}{N} \right)^4 + \cdots \right) \right) \]

Now, with the total length of the bar: \( l = N \cdot \Delta x \)

the natural frequency of the model is:

\[ \omega_k = \sqrt{\frac{E}{\rho l^2}} \cdot k \pi \left( 1 + \frac{1}{24} \left( \frac{k\pi}{N} \right)^2 + \frac{1}{24} \left( \frac{k\pi}{N} \right)^4 + \cdots \right) \quad (4.33) \]

Amplitude of the eigenvector and strain energy

The amplitude of the eigenvector is determined by the mass normalisation condition: \( \{ \phi \}^T [M] \cdot \{ \phi \} = 1 \). For the \( k \)th mode, the element of the eigenvector at the \( i \)th node is \( u_{ki} \). Because \( a = 0 \) (equation (4.26)),

\[ u_{ki} = b \cos(k \mu) = b \cos(i \cdot k \pi / N) \quad (4.34) \]

Putting the above into the mass normalisation condition and after algebraic manipulations, it can be shown that:

\[ b = \pm \sqrt{\frac{6}{2 + \cos(k \pi / N)}} \times \frac{1}{\rho Al} \quad (4.35) \]

The strain energy of the \( i \)th element can be calculated with the formula:

\[ U_i = \frac{EA}{2\Delta x} \begin{bmatrix} u_{i+1} \\ u_i \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{i+1} \\ u_i \end{bmatrix} \quad (4.36) \]

Adding together the strain energy in all elements, it can be shown that:
\[ U_a = \frac{1}{2} \cdot \frac{6E}{\rho \cdot \Delta x^2} \cdot \frac{1-\cos(k\pi/N)}{2+\cos(k\pi/N)} \]  
(4.37)

4.2.1.1.3 Solutions for an FE model of the bar in extension with lumped mass matrices and with coupled mass matrices

When using the lumped mass-matrix approach for forming the mass matrices of elements, the mass matrix for the \( i^{th} \) element is:

\[
[M_i] = \frac{\Delta x}{2} \rho A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]  
(4.38)

and the system mass matrix is:

\[
[M] = \frac{\Delta x}{2} \rho A \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{bmatrix}
\]  
(4.39)

The general solution of equation (4.4) can be written as in equations (4.16) and (4.17). At node \( i \), the free vibration equilibrium equation takes the form:

\[
\frac{EA}{\Delta x} (u_{i-1} - 2u_i + u_{i+1}) + \omega^2 \rho A \Delta x u_i = 0
\]  
(4.40)

After algebraic manipulation, the relationship between the eigenvalue, \( \omega^2 \), and the characteristic root, \( \mu \), is given by:

\[
\omega^2 = \frac{2E}{\rho \Delta x^3} (1 - \cos \mu)
\]  
(4.41)

Considering the free-free case, the characteristic root of the equation of motion can be obtained as:

\[
\mu = \frac{k}{N} \pi
\]  
(4.42)

Substituting it into equation (4.41), the eigenvalue is given by

\[
\omega_k^2 = \frac{2E}{\rho \Delta x^3} \left[ 1 - \cos \left( \frac{k}{N} \pi \right) \right]
\]  
(4.43)

Thus, the natural frequency of the model is:

\[
\omega_k = \sqrt{\frac{E}{\rho l^2 \pi^2}} \cdot k \pi \left[ 1 - \frac{1}{24} \left( \frac{k\pi}{N} \right)^2 + O \left( \frac{k\pi}{N} \right)^4 \right]
\]  
(4.44)

In some FE model calculation programs, such as NASTRAN, the coupled mass-matrix approach forms the mass matrix of an element of the bar in extension by averaging the consistent mass matrix and the lumped mass matrix. The mass matrix for an element by this approach is:

\[
[M] = \frac{\Delta x}{12} \rho A \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}
\]  
(4.45)
The assembled mass matrix for the whole model is:

\[
[M] = \frac{\Delta x}{12} \rho A \begin{bmatrix}
5 & 1 & 0 \\
1 & 10 & 1 \\
\ddots & \ddots & \ddots \\
1 & 10 & 1 \\
0 & 1 & 5
\end{bmatrix}
\]

At node \(i\), the free vibration equilibrium equation takes the form:

\[
\frac{EA}{\Delta x} (u_{i-1} - 2u_i + u_{i+1}) + \omega^2 \frac{\rho A \Delta x}{12} (u_{i-1} + 10u_i + u_{i+1}) = 0
\]  

After algebraic manipulation, \(\omega^2 = \frac{12E}{\rho \Delta x^2} \left( \frac{1 - \cos \mu}{5 + \cos \mu} \right)\) (4.48)

For the free-free case, \(\mu = k\pi/N\) (4.49)

Substituting this into equation (4.48), the eigenvalue is given by

\[
\omega_k^2 = \frac{12E}{\rho \Delta x^2} \frac{1 - \cos \left( \frac{k\pi}{N} \right)}{5 + \cos \left( \frac{k\pi}{N} \right)}
\]  

Making use of the series expansions of \(\cos(x)\) and of \(\frac{1}{1+x}\), the natural frequency of the model is:

\[
\omega_k = \sqrt{\frac{E}{\rho l^2}} \cdot k\pi \left( 1 - \frac{1}{180} \left( \frac{k\pi}{N} \right)^4 + 0 \left( \frac{k\pi}{N} \right)^6 + \cdots \right)
\]  

(4.51)

The amplitude of the eigenvector and the strain energy of a mode from the models with the lumped mass approach and the coupled mass approach can be calculated in the same way as in the case with the consistent mass approach.

Table 4-2 lists the solutions for the FE models of the bar in extension under free-free boundary conditions.

With the same process of algebraic manipulations, the dynamic properties of the FE models of the bar in extension under other boundary conditions can be calculated. Table 4-3 lists the solutions for the FE models of the bar in extension under two boundary conditions.
### Case: Free-free case

<table>
<thead>
<tr>
<th>Case</th>
<th>Free-free case</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary</td>
<td>$a = 0$ and $\sin N\mu = 0$</td>
</tr>
<tr>
<td>Characteristic roots</td>
<td>$\mu = k\pi/N$</td>
</tr>
<tr>
<td>Eigenvectors</td>
<td>$u_{i,k} = b\cos (i \cdot k\pi/N)$</td>
</tr>
</tbody>
</table>

#### Consistent mass

| Mass matrix | $[M_i] = \frac{\Delta x}{6} \rho A \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ |
| Eigenvalue | $\omega^2 = \frac{6E}{\rho \Delta x^2} \frac{1 - \cos \mu}{2 + \cos \mu}$ |
| Eigenvector amplitude | $b = \pm \frac{6}{\sqrt{2 + \cos (k\pi/N) \rho A l}} \frac{1}{\rho A l}$ |

#### Lumped mass

| Mass matrix | $[M_i] = \frac{\Delta x}{2} \rho A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Eigenvalue | $\omega^2 = \frac{2E}{\rho \Delta x^2} (1 - \cos \mu)$ |
| Eigenvector amplitude | $b = \pm \frac{2}{\sqrt{\rho A l}}$ |

#### Coupled mass

| Mass matrix | $[M_i] = \frac{\Delta x}{12} \rho A \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ |
| Eigenvalue | $\omega^2 = \frac{12E}{\rho \Delta x^2} \frac{1 - \cos \mu}{5 + \cos \mu}$ |
| Eigenvector amplitude | $b = \pm \frac{12}{\sqrt{5 + \cos (k\pi/N) \rho A l}} \frac{1}{\rho A l}$ |

Table 4-2 FE model results for bar in extension in the free-free condition
### Table 4-3 FE model results for the bar in extension with two boundary conditions

#### 4.2.1.1.4 Comparison between the structural dynamic properties and the model predictions

Comparing the dynamic properties of the structure of the bar in extension and those from the FE models, differences can be found between them.
Natural frequency discrepancies

From Tables 4-2 and 4-3 it can be seen that the eigenvalues (natural frequencies) predicted by the FE model are directly related to the characteristic roots. From the formulae for eigenvalues, the natural frequencies predicted by the FE model with three mass-matrix approaches and their discrepancies from the natural frequencies of the structure can be given as listed in Table 4-4. In the table, \((\omega_x)_k\) and \((\omega_s)_k\) are the natural frequencies of the structure and the FE model respectively, \(\mu\) is the characteristic root of the equation of motion for the FE model, which depends on boundary conditions and is listed in Tables 4-2 and 4-3.

Under a boundary condition, the value of \(\mu\) is a function of \(k\), the mode number, and \(N\), the number of the elements in the model. Thus, for a mode predicted from the model, the natural frequency discrepancy is strongly related with the number of the elements in the model. The more elements a model has, the smaller of the value of \(\mu\) is, and thus the smaller the natural frequency discrepancy of the mode will be. This natural frequency discrepancy shows an effect of the discretisation errors of the model.

<table>
<thead>
<tr>
<th>Mass-matrix approach</th>
<th>Natural frequency</th>
<th>Relative natural frequency discrepancy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consistent mass</td>
<td>((\omega_x)_k = \sqrt{\frac{E}{\rho}} \cdot \frac{\mu}{\Delta x} \left(1 + \frac{1}{24} \mu^2 + 0(\mu^4)\right))</td>
<td>(\frac{(\omega_s)_k - (\omega_x)_k}{(\omega_x)_k} = \frac{1}{24} \mu^2 + 0(\mu^4))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(= \frac{1}{6} \left(\frac{\pi}{L}\right)^2 + 0\left(\frac{\pi}{L}\right)^4)</td>
</tr>
<tr>
<td>Lumped mass</td>
<td>((\omega_x)_k = \sqrt{\frac{E}{\rho}} \cdot \frac{\mu}{\Delta x} \left(1 - \frac{1}{24} \mu^2 + 0(\mu^4)\right))</td>
<td>(\frac{(\omega_s)_k - (\omega_x)_k}{(\omega_x)_k} = \frac{1}{24} \mu^2 + 0(\mu^4))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(= -\frac{1}{6} \left(\frac{\pi}{L}\right)^2 + 0\left(\frac{\pi}{L}\right)^4)</td>
</tr>
<tr>
<td>Coupled mass</td>
<td>((\omega_x)_k = \sqrt{\frac{E}{\rho}} \cdot \frac{\mu}{\Delta x} \left(1 - \frac{1}{480} \mu^4 + 0(\mu^6)\right))</td>
<td>(\frac{(\omega_s)_k - (\omega_x)_k}{(\omega_x)_k} = \frac{1}{480} \mu^4 + 0(\mu^6))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(= -\frac{1}{30} \left(\frac{\pi}{L}\right)^4 + 0\left(\frac{\pi}{L}\right)^6)</td>
</tr>
</tbody>
</table>

Table 4-4 Natural frequency discrepancies for FE models of the bar in extension

From Table 4-4 it can be seen that the relative natural frequency discrepancy of a mode predicted by an FE model of the bar in extension is a function of the characteristic root, \(\mu\), only. By performing algebraic manipulations as follows, the relative natural frequency discrepancy of a mode can be written in another form.

Under the free-free condition, the eigenvector of the \(k^{th}\) mode is:
\[ u_{k,i} = a \cdot \cos \left( \frac{i}{N} k \pi \right) = a \cdot \cos \left( \frac{i}{2N/k} 2\pi \right) = a \cdot \cos \left( \frac{i}{L} (2\pi) \right) \]

where the parameter \( L \) can be considered as the number of the elements in a wavelength. For example, for the first mode, the mode shape represented by the eigenvector data

\[ u_{k,i} = a \cdot \cos \left( \frac{i}{N} \pi \right) = a \cdot \cos \left( \frac{i}{2N} 2\pi \right), \quad i = 0, \ldots, N \]

forms a half of a wave-cycle. Thus, the number of elements in a wavelength is twice the number of elements in the model: \( L = 2N \). Substituting \( L \) into the formula for the characteristic root, \( \mu \), gives:

\[ \mu = \frac{k}{N} \pi = 2 \cdot \frac{\pi}{L} \]

Thus, the relative natural frequency discrepancy of a mode with the lumped mass-matrix approach can be expressed as:

\[ \left( \frac{\omega_{n,h}}{\omega_{n,k}} \right)_{h} = \frac{1}{6} \left( \frac{\pi}{L} \right)^4 + 0 \left( \frac{\pi}{L} \right)^6 \]  

(4.52)

Following the same algebraic manipulations, the natural frequency discrepancies of modes predicted by an FE model of the bar in extension with the consistent mass-matrix approach and with the coupled mass-matrix approach can also be expressed as functions of the number of elements in a wavelength. The equations for the functions are also listed in Table 4-4.

From Table 4-4 it can also be seen that the natural frequency of a mode predicted by a model with a mass-matrix approach is different to the natural frequency of the mode by the same model with another mass-matrix approach. The natural frequency of a mode predicted with the consistent mass-matrix approach is higher than that of the structure. The natural frequency predicted with the lumped mass-matrix approach or the coupled mass-matrix approach is lower than that of the structure. Among all three mass-matrix approaches, the predicted natural frequency with the coupled mass-matrix approach is the closest to the natural frequency of the structure when the value of the characteristic root, \( \mu \), is smaller than 1.

Amplitude of eigenvector

From Table 4-2 and Table 4-3 it can be seen that under the same boundary conditions the eigenvector of a mode predicted by the FE model, no matter which mass-matrix approach is used, has the same format as that of the structure. Thus, if a correlation is performed between the eigenvector of a mode of the structure and the eigenvector of the same mode predicted by an FE model with only discretisation errors, a high MAC value can be expected.
However, the amplitude of a predicted eigenvector depends on the normalisation method and the mass-matrix approach used. When eigenvectors are normalized to mass matrices, the amplitudes of the eigenvectors of a mode predicted by the same model with three mass-matrix approaches are different. Only the amplitude of the eigenvector predicted by the model with the lumped mass-matrix approach is the same as that of the structure.

**Strain energy**

The strain energy of a mode of the bar in extension is calculated by equation \[(4.10)\] where the deformation of the system is continuous and is described by a trigonometric function. In the predictions from an FE model of the structure, the strain energy of the mode is calculated by equation \[(4.36)\] where the deformation within an element is interpreted as a linear function.

When the lumped mass-matrix approach is used, the amplitude of the eigenvector of a mode predicted by the model is the same as that of the structure. However, the difference between the function of the deformation within an element and that within the same section of the structure makes the strain energy of the mode in the element different from that in the section of the structure. The eigenvector amplitude of a mode is also different when different mass-matrix approaches are used. It is this difference that makes the strain energy of a mode within an element different from one mass-matrix approach to another.

From equation \[(4.10)\] it can be seen that the whole strain energy of a mode within the structure is, in value, equal to the half of the eigenvalue if the eigenvector of the mode is normalised to mass. From equation \[(4.36)\] the whole strain energy of a mode within the FE model, no matter which mass-matrix approach is used, is, in value, equal to the half of the eigenvalue if the eigenvector of the mode is normalised to the mass matrix. Thus, the discrepancy of the estimated strain energy of a mode from an FE model of the bar in extension is, in value, half of the discrepancy of the estimated eigenvalue of the mode.

**Summary of the discrepancies**

From the above discussion, the discretisation errors in an FE model of a bar in extension will affect the predicted dynamic properties. If a model has only discretisation errors, the discrepancies between the predicted natural frequencies and the natural frequencies of the structure depend on the mesh size and the mode number, as well as the mass-matrix approach used in the model calculation. The predicted eigenvectors should have high \( MAC \) values when compared with the mode shapes of the structure, although the amplitude of a predicted eigenvector will be equal to that of the mode shape of the structure only when the lumped mass-matrix approach is used in the FE model calculation.
4.2.1.2 Discretisation errors of FE models with general types of elements

The above analysis of discretisation errors is for structures and FE models of a bar in extension. From the practice of model updating, it is well known that the discretisation errors in an FE model with elements of any kind have their effects on the predicted natural frequencies as the function of the mesh size of the model. The more elements the model has, or the finer the mesh is, the closer the predicted natural frequencies will be to the natural frequencies of the structure if there is no other error in the model.

In the process of model validation, one of the purposes of the convergence check is to find the convergent frequency range or the convergent modes predicted. Although it is impossible to give a formula for calculating the natural frequency discrepancies due to discretisation errors of FE models with elements of the type other than a bar in extension, the idea of assessing the discretisation errors in FE models of a bar in extension can be extended to those models.

4.2.1.2.1 Determining the convergence of modes with two mass-matrix approaches

As in the case of a bar in extension, for a general FE model the mass-matrix approach used in model calculation is a factor that will affect the model predictions. Most FE model programs, such as NASTRAN and ANSYS, have two mass-matrix approaches for most types of element – the lumped mass-matrix approach and the coupled mass-matrix approach. The lumped mass matrix for an element of any kind has the mass of the element divided and distributed around all the grid points of the element so that the off-diagonal elements of the mass matrix are all zero. The formulae for forming coupled mass matrices are different for different types of element. The coupled mass-matrix approach in NASTRAN uses the same formulae of the consistent mass-matrix approach for elements of the types other than a bar in extension. All the mass matrices formed with the coupled mass-matrix approach have some off-diagonal elements with non-zero values.

If an FE model for a structure has no parameter errors or configuration errors but only discretisation errors, the lumped mass-matrix approach will usually produce lower natural frequencies than the natural frequencies of the structure and the consistent mass-matrix approach will produce higher natural frequencies than the natural frequencies of the structure. For convenience, the difference between the natural frequency of a mode predicted by an FE model (analytical model) and that of the structure is called the discrepancy, and the difference between the natural frequencies of a mode predicted by the same FE model but with different mass-matrix approaches is called the difference in this section.

Figure 4-1 shows the natural frequencies of some modes of a bar in extension and those predicted by an FE model with the consistent mass-matrix approach and the
lumped mass-matrix approach respectively. The meaning of the terms of “discrepancy” and "difference" that will be used in this chapter is also explained in the figure.

From an FE model, the normal mode extraction with two different mass-matrix approaches respectively can produce two sets of modal data. The natural frequency difference of a mode in these two data sets is usually larger than and approximately proportional to the natural frequency discrepancy of the same mode due to discretisation errors of the model with any one of these mass-matrix approaches. As the number of elements in the model approaches infinity, the natural frequency difference of the mode will approach zero, and the natural frequencies predicted for the mode by the model with two different mass-matrix approaches will be the same and equal to the natural frequency of the mode of the structure.

The feature that the natural frequency difference of a mode predicted by a model with two different mass-matrix approaches is related to the discrepancy of the mode due to the discretisation errors of the model can be used for a convergence check. Once a threshold on the natural frequency discrepancy is set for the convergence check for a model, the threshold on the natural frequency difference can be set as double the value of the threshold on the discrepancy. The frequency range, over which the difference between the natural frequencies of the same mode from these two predictions is smaller than the threshold on the natural frequency difference, can be considered as the convergent range of the model.

Usually, the coupled mass-matrix approach results in smaller discrepancies on the predicted natural frequencies than the lumped mass-matrix approach does. The method of convergence check by the threshold on the natural frequency difference will give a conservative result for the coupled mass-matrix approach – some of the predicted
modes out of the considered-as-convergent frequency range will have the natural frequency discrepancies less than the threshold on the discrepancy. In contrast, in the considered-as-convergent frequency range by the method of convergence check with the threshold on the natural frequency difference, a few modes predicted with the lumped mass-matrix approach will have the discrepancies a little greater than the threshold on the discrepancy.

Case study on FE models of a rectangular plate

Consider the analytical model and the reference model for a rectangular plate that are used in Case 2 of the previous chapter. Performing the normal mode extraction with two different mass-matrix approaches respectively on the analytical model that has only discretisation errors, two sets of predicted natural frequencies can be obtained. The discrepancies between the natural frequencies of these two sets and those from the reference model are plotted in Figure 4-2. The natural frequencies differences of the modes between these two predicted modal data sets from the same analytical model but with two different mass-matrix approaches are plotted in Figure 4-3.

![Figure 4-2 Natural frequency discrepancies](image1)

![Figure 4-3 Natural frequency differences](image2)
In Figure 4-2, it can be seen that the natural frequency discrepancies between the reference model and the analytical model with two different mass-matrix approaches are on the two sides of the line of zero. As the mode number increases, the natural frequency discrepancies with any mass-matrix approach will increase.

Figure 4-3 shows that the natural frequency difference of the same mode between these two sets has a trend of increasing as the mode number increases. This is consistent with the general concept of discrepancies of the natural frequencies predicted by an FE model with discretisation errors.

From Figures 4-2 and 4-3, it can be seen that if a criterion for the natural frequency discrepancy in Figure 4-2 is set for checking the convergent frequency range or the convergent modes, a criterion for the natural frequency differences in Figure 4-3 can also be set for the same purpose. For example, if the criterion for the natural frequency discrepancy is set to 5%, the number of convergent modes from the analytical model with the lumped mass-matrix approach is about 10 and the number of convergent modes with the coupled mass-matrix approach is 14. If the criterion for the natural frequency differences in Figure 4-3 is set to 10%, the number of the convergent modes is 13.

In this case study, because the reference model is known, the discrepancies of the natural frequencies due to the discretisation errors in the analytical model can be obtained by comparing the natural frequencies predicted by the analytical model and those predicted by the reference model. The relationship between the natural frequency discrepancies and the differences of model predictions with different mass-matrix approaches shows that the natural frequency difference of a mode can give a rough idea of the amplitude of the natural frequency discrepancy of the same mode due to discretisation errors of an FE model.

Case study on the FE model of the Rear Bearing Housing (RBH) component structure

The FE model of the RBH structure has been used as a case study of model updating in the previous chapter. Although the discrepancies of the natural frequencies due to the discretisation errors of the model cannot be distinguished from the discrepancies due to all other errors of the model, the natural frequency differences of modes predicted by the model with two different mass-matrix approaches can still be used to check the convergent frequency range of the model.

Figure 4-4 shows the natural frequency differences of the modes predicted by the initial RBH model with two different mass-matrix approaches. The words beside the curve are brief descriptions of the eigenvectors. From the figure, it can be seen that if the threshold on the natural frequency difference is set to 5%, the first 21 modes predicted from the model can be considered as convergent modes. However, from the figure it can also be seen that the first four out-of-phase modes have greater natural
frequency differences than the in-phase modes of the same nodal diametral orders. The different effects of the discretisation errors of the model on the predicted natural frequencies of the out-of-phase modes and the in-phase modes are revealed clearly in the figure.

Compared with the natural frequency discrepancies of the model predictions, which are listed in Table 3-11 in the previous chapter, the natural frequency differences plotted in Figure 4-4 show that the discretisation errors of the model are not the main reason for the discrepancies of the natural frequencies. Refining the model (with the same configuration and same parameter values) can only improve the natural frequency discrepancies by up to 2 percentage points for the lower order modes and up to 5 percentage points for the 5D out-of-phase modes.

4.2.1.2.2 Examining the convergence of modes with the number of elements in one wavelength

In equation (4.52), the natural frequency discrepancy of a mode predicted by a model of the bar in extension is expressed as a function of the parameter $L$ – the number of elements in a wavelength. For the dynamic properties predicted by a general FE model with other types of element, it is difficult to estimate accurately the natural frequency discrepancy for each mode. However, by examining the mode shape represented by the predicted eigenvector data, it is possible to get an approximate value of the parameter $L$ for a mode in certain axis. This parameter can help to identify the effects of the discretisation errors of the model on the predicted dynamic properties.

Case study on FE models of a rectangular plate

Consider the analytical model and the reference model that are used in Case 2 of the case studies of model updating in the previous chapter. Figure 4-5 shows the mesh of the analytical model. There are 15 elements along the $y$ axis and 10 elements along the $x$ axis. The correlation between the modal data from the models is listed in Table 4-5. From the table, it can be seen that the discrepancy of the natural frequency for Mode 3, which is a bending mode along the $x$ axis, is almost double the discrepancy of
the natural frequency of Mode 2, which is a bending mode along the $y$ axis. For these two bending modes, the numbers of elements per wavelength are all about twice the number of elements of the model along the bending axes. However, for Mode 2 – bending along the $y$ axis – the parameter $L$ is approximately equal to 30, while for Mode 3 – bending along the $x$ axis – it is approximately equal to 20. The difference of the values of the parameter $L$ is the reason why the modes with similar format will have significantly different discrepancies from the structural dynamic properties.

![Figure 4-5 FE model for a rectangular plate](image)

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>NFD (%)</th>
<th>MAC %</th>
<th>Nodal lines</th>
<th>Mode No.</th>
<th>NFD (%)</th>
<th>MAC %</th>
<th>Nodal lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.247</td>
<td>100.0</td>
<td></td>
<td>5</td>
<td>-2.851</td>
<td>99.55</td>
<td></td>
</tr>
<tr>
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<td>100.0</td>
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<td>6</td>
<td>-2.679</td>
<td>99.80</td>
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</tr>
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<td>4</td>
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<td>100.0</td>
<td></td>
<td>8</td>
<td>-4.404</td>
<td>99.99</td>
<td></td>
</tr>
</tbody>
</table>

Table 4-5 Correlation between models with different mesh sizes

From the above analysis, it might be considered that we should refine the mesh size along the $x$ axis in order to decrease the discrepancies of the natural frequencies. Another FE model, the refined model, for the same plate was built, which had 15 elements along both the $x$ and $y$ axes. The physical parameter values of the model were the same as those of the reference model in order to make the model with only discretisation errors. The natural frequency correlation between the refined model and the reference model is listed in Table 4-6. The improvement of the correlation of the natural frequencies is also listed in the table.
<table>
<thead>
<tr>
<th>Mode No.</th>
<th>NFD (%)</th>
<th>NFD decreased (percentage)</th>
<th>Mode No.</th>
<th>NFD (%)</th>
<th>NFD decreased (percentage)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.770</td>
<td>0.477</td>
<td>5</td>
<td>-2.171</td>
<td>0.680</td>
</tr>
<tr>
<td>2</td>
<td>-1.357</td>
<td>0.037</td>
<td>6</td>
<td>-1.677</td>
<td>1.002</td>
</tr>
<tr>
<td>3</td>
<td>-1.278</td>
<td>1.593</td>
<td>7</td>
<td>-2.966</td>
<td>0.481</td>
</tr>
<tr>
<td>4</td>
<td>-1.740</td>
<td>0.458</td>
<td>8</td>
<td>-2.784</td>
<td>1.610</td>
</tr>
</tbody>
</table>

Table 4-6 Natural frequency correlation and its improvement with the refined model

From this table, it can be seen that increasing the number of elements along the x axis improves significantly the NFDs for Modes 3 and 8, which have two nodal lines along the x axis, by about 1.6 percentage points, while the NFD for Mode 2, which is the bending mode along the y axis, is improved by only 0.037 percentage points. Thus, the refined model has a balanced distribution of the effects of the discretisation errors on Modes 2 and 3.

Case study of an FE model of RBH component structure

The FE model of the RBH structure was used previously as a case study for a convergence check. There are significant differences between the natural frequency differences of the in-phase modes and those of the out-of-phase modes (Figure 4-4 on Page 116). Examining the initial FE model (Figure 3-9 on Page 75), it can be seen that in the main body of the outer case there are only 6 shell elements in a line along the axis of the case. Comparing this number with the number of elements along a circumference, which is 48, the mesh size of the model is not balanced. The wavelength of the in-phase modes along the axis of the structure is much longer than that of the out-of-phase modes. Thus, the in-phase modes of the structure have more elements per wavelength than the out-of-phase modes have. This is one of the reasons for the poor correlation of the out-of-phase modes of the model.

From the above case studies, a conclusion can be drawn that the number of elements per wavelength of a mode along each direction can help to assess the effects of discretisation errors of the model and can help to refine FE models in order to balance the effects of discretisation errors on the modes of interest.

4.2.2 Estimation of the compensation for discretisation errors

As discussed in the previous chapter, when an FE model with discretisation errors is subjected to an updating procedure, the obtained updating parameter values are not the same as the correction to parameter errors but a combination of the correction to the parameter errors and compensation of the parameters for the discretisation errors. If the effects of the discretisation errors on the structural properties are significant, the compensation of the updating parameters for the discretisation errors may distort the
physical meaning of the updating parameter values. Thus, in order to assess the results of an updating procedure, the compensation of updating parameters for discretisation errors of the initial model needs to be estimated.

From the analysis in section 4.2.1 it is known that the normal eigenvalue solution of an FE model with two different mass-matrix approaches will obtain two sets of predictions. The natural frequency differences between the predictions of the same mode are related to the natural frequency discrepancies of the mode due to discretisation errors of the model. This feature can be used in estimating compensation of the updating parameters for the discretisation errors. An estimation method based on this feature is described as follows and shown in Figure 4-6.

**Figure 4-6 Estimation of compensations for discretisation errors**

If the lumped mass-matrix approach is used when a model is subjected to a model updating session, the predictions of the model with the coupled mass-matrix approach can be taken as a special set of reference properties. (If the coupled mass-matrix approach is used in the formal-updating session, the lumped mass-matrix approach should be used to create the special set of reference properties.) Two model-
updating sessions, with the same settings for model updating calculations but with two different reference properties, can be performed on the initial model. One session is with this special set of reference properties and is called the pre-updating session in order to distinguish from the formal-updating session that is with the actual reference properties.

The obtained updating parameter values from the pre-updating session are the modification to the initial model such that the predictions from this modified model with the lumped mass-matrix approach are close to the predictions of the initial model with the coupled mass-matrix approach. Thus, the updating parameter values from the pre-updating session can be considered as an estimate of compensations of the updating parameters for discretisation errors when the same settings of model updating are used in the formal-updating session with the actual reference properties. The estimated compensations of updating parameter values can be used in the assessment of the updated results from the formal updating session.

For most types of element, because the natural frequency difference of a mode predicted by a model is greater than the natural frequency discrepancy of the same mode, the estimated compensation is a conservative estimate.

Take the analytical model and the reference model for a rectangular plate that were used previously in this chapter as an example. The analytical model has no other errors but discretisation errors when compared with the reference model. In Case 2 of the previous chapter, a model updating procedure of four sessions was undertaken to the analytical model with the lumped mass-matrix approach. Thus, the "true" compensation for the discretisation errors of the analytical model is the updating parameter values obtained in the sessions. For estimating the compensation, the coupled mass-matrix approach was used to create a special set of reference data. Three updating parameters the same as those in Case 2 of the previous chapter were selected. Four pre-updating sessions were carried out with the special set of reference data. The results from these sessions and those from the sessions with the data from the reference model (in Case 2 of the previous chapter) are listed in Table 4-7.

From the table, it can be seen that the estimated compensations in Sessions 1, 2 and 4 are reasonably close to the "true" compensation. Although the distances between the estimated compensation and the "true" compensation are different from a session to other sessions, each estimated value is with the same order and in the same direction as the "true" compensation.

For Session 3, in which only eigenvectors were used in updating, the estimated compensation is quite different from the "true" compensation. It is known from the analysis on the discretisation errors that the discretisation errors of a model have little effect on predicted eigenvectors. Therefore, the differences of the eigenvectors
predicted by the *analytical model* with two different mass-matrix approaches may have small amplitudes and are different in direction from the discrepancies between the eigenvectors predicted by the *analytical model* and the *reference model*. This makes the estimation distance away from the "true" compensation. However, in the practice of model updating, it is rare that only eigenvectors are used in an updating session.

<table>
<thead>
<tr>
<th>Session</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_k$</td>
<td>1.0</td>
<td>$\lambda_i^{-1}$</td>
<td>0</td>
<td>$\lambda_i^{-1}$</td>
</tr>
<tr>
<td>$W_{\phi}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$</td>
</tr>
<tr>
<td>&quot;True&quot; compensation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta p_1$</td>
<td>0.3507</td>
<td>0.0977</td>
<td>0.0317</td>
<td>0.0537</td>
</tr>
<tr>
<td>$\delta p_2$</td>
<td>-0.2788</td>
<td>-0.0307</td>
<td>0.0097</td>
<td></td>
</tr>
<tr>
<td>$\delta p_3$</td>
<td>0.1390</td>
<td>0.0576</td>
<td>0.0603</td>
<td>0.0854</td>
</tr>
<tr>
<td>Estimated compensation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta p_1$</td>
<td>0.3312</td>
<td>0.1561</td>
<td>-0.0167</td>
<td>0.1017</td>
</tr>
<tr>
<td>$\delta p_2$</td>
<td>-0.1768</td>
<td>-0.0219</td>
<td>0.0326</td>
<td></td>
</tr>
<tr>
<td>$\delta p_3$</td>
<td>0.1583</td>
<td>0.1072</td>
<td>-0.0189</td>
<td>0.0978</td>
</tr>
</tbody>
</table>

Table 4-7 Parameter compensation for discretisation errors

In this case study, the method for estimating the compensation of updating parameters for the discretisation errors has been used successfully. When eigenvalues, which are easily affected by the discretisation errors, of any form are used in updating, the estimated compensation for the errors can show the order and the direction of the "true" compensation.

### 4.3 Model verification for configuration check

When an FE model is constructed for predicting the dynamic properties of a structure, there are usually some simplifications that represent complicated parts in the structure by standard types of element provided by an FE model program. Although in this construction process the simplifications are made according to the experience of the modeller, the effects of the simplifications on the dynamic properties of the model are in general unknown, or at least not clear for some of them.

If a simplification made to the model has the capability to provide all the “key” features of the corresponding part of the structure for predicting the required dynamic properties, even if the predicted properties are not accurate, this simplification does not cause configuration errors. However, if a simplification results in a loss of some key features and makes the model unable to predict the required dynamic properties accurately, even by modifying parameter values in the model, this simplification results in the model having configuration errors. In general, if some key features of a structure are missing in an FE model for the structure, and this makes the model unable to predict
the dynamic properties in a specific frequency range with the required precision, the discrepancies of the predicted properties cannot be reduced significantly even by modifying parameters in the model, and the model is said to have configuration errors.

In the previous chapter, a case study of updating the FE model of the Rear Bearing Housing component of an aero-engine did not result in an updated model. A model updating session undertaken on the model resulted in an unacceptable modification for the stiffness data of the elements for the vanes. The discussion on the updating results showed that there are possible configuration errors in the model which are connected with the simplification of the vanes of the structure by a layer of shell elements. This simplification in the model makes the model not capable of predicting some dynamic properties accurately and, even more, not capable of being updated with parameter values modified in an acceptable adjustment range.

As a task in the procedure of model verification, the aims of the configuration error study in this section are (i) to propose a method for determining whether there is any configuration error in an FE model that will affect the ability of the model to predict the required dynamic properties, and (ii) to find a way that can detect the unsuitable simplifications in FE models that may introduce configuration errors into the models and affect significantly the predicted dynamic properties in the frequency range of interest.

4.3.1 Projection of a vector onto a space and onto a subspace

According to the theory of linear algebra [104], projection of a vector, $u \in \mathbb{R}^n$, onto a vector space, $V \in \mathbb{R}^n$, denoted as $\text{proj}_V u$, can be expressed as a linear combination of the vectors in a matrix, $B$, which are linearly independent and constitute a basis:

$$ \text{proj}_V u = [B][c] $$

where, $\det(B) \neq 0$, and $\{c\} = [B]^{-1}\{u\}$.

If the vectors in the matrix, $[B]$, span only a subspace, $H$, of the vector space $V \in \mathbb{R}^n$, there must exist another subspace, $H^\perp$, which is the orthogonal complement of $H$. A matrix, $[A]$, consisting of another set of linearly independent vectors in the subspace $H^\perp$ can be found that each vector in $[A]$ is orthogonal to every vector in $[B]$ and all vectors in the matrix $[B, A]$ span the vector space $V \in \mathbb{R}^n$.

$$ \{b_j\}^T \{a_j\} = 0 $$

$$ \det([B, A]) \neq 0 $$

(4.54)

In this situation, the projection of a vector $\{u\}$ onto the space, $V$, can be written as:
\[ \text{proj}_v u = [B, A][c] \]
\[ = [B][c_b] + [A][c_a] \]
\[ = \text{proj}_v u + \text{proj}_w u \] (4.55)

When \([B]^T [A] = 0\) and \([A]^T [B] = 0\), it can be proved that:
\[ [B, A]^{-1} = \begin{bmatrix} ([B]^T [B])^{-1} [B]^T \\ ([A]^T [A])^{-1} [A]^T \end{bmatrix} \] (4.56)

Therefore,
\[ \{c\} = [B, A]^{-1} \{u\} \]
\[ = \begin{bmatrix} ([B]^T [B])^{-1} [B]^T \\ ([A]^T [A])^{-1} [A]^T \end{bmatrix} \{u\} \]
\[ = \begin{bmatrix} c_b \\ c_a \end{bmatrix} \]
\[ \{c_b\} = ([B]^T [B])^{-1} [B]^T \{u\} \]
\[ \{c_a\} = ([A]^T [A])^{-1} [A]^T \{u\} \] (4.57)

If the vector, \(\{u\}\), is in the same subspace as that spanned by the vectors in \([B]\), it can be expressed as a linear combination of the vectors in \([B]\):
\[ \{u\} = [B]\{b\} \] (4.59)

Replacing \(\{u\}\) in equation (4.58) with the above equation, it can be seen that:
\[ \{c_b\} = ([B]^T [B])^{-1} [B]^T \{u\} \]
\[ = ([B]^T [B])^{-1} [B]^T [B]\{b\} \]
\[ = \{b\} \]
\[ \{c_a\} = ([A]^T [A])^{-1} [A]^T \{u\} \]
\[ = ([A]^T [A])^{-1} [A]^T [B]\{b\} \]
\[ = ([A]^T [A])^{-1} [0]\{b\} \]
\[ = \{0\} \] (4.60)

The vector, \(\{u\}\), is also linearly independent from the vectors in the subspace \(H^\perp\). Therefore,
\[ \text{proj}_v u = [B]\{c_b\} = [B]\{b\} = \{u\} \]
\[ \text{proj}_w u = [B]\{c_a\} = [B]\{0\} = \{0\} \] (4.61)

The projection of the vector, \(\{u\}\), onto the subspace, \(H\), is equal to the vector itself. The projection of the vector, \(\{u\}\), onto the subspace, \(H^\perp\), is a null vector.

From the above mathematical manipulations, two statements can be made:

(1) When a vector, \(v \in \mathbb{R}^n\), is in the same space (or subspace) spanned by a matrix, \([B]\), consisting of a set of vectors, \(b_i \in \mathbb{R}^n\), the projection of the vector onto the
space (subspace) will equal to the vector itself. There is no information lost in the process of projection.

(2) When the vector, \( \mathbf{v} \in \mathbb{R}^n \), is in a vector space, \( \mathcal{V} \), while the matrix, \( \mathbf{B} \), spans a subspace of \( \mathcal{V} \), the projection of the vector \( \mathbf{v} \) onto the subspace spanned by the matrix \( \mathbf{B} \) may result in a different vector. Some information of the vector \( \mathbf{v} \) may be lost in the process of projection.

### 4.3.2 Projection of reference eigenvectors onto analytical eigenvector matrix

The dynamic properties of a structure in a specific frequency range can be described by the eigenvalues and eigenvectors in that range (plus the residual effects of the modes outside the range). Suppose that these dynamic properties are the same as those of an FE model, which is called reference model in this section, the structural dynamic properties can be represented by the eigenvalues, \( \lambda_X \), and eigenvectors, \( \phi_X \), of this reference model. The FE model that is to be validated for predicting the dynamic properties of the structure is called the analytical model in order to distinguish it from the reference model. The eigenvalues and the eigenvectors of this analytical model are \( \lambda_A \) and \( \phi_A \).

Suppose that both the reference and analytical models have \( n \) degrees-of-freedom. All reference eigenvectors and analytical eigenvectors are in a vector space \( \mathbb{V} \in \mathbb{R}^n \). The elements in each reference eigenvector and analytical eigenvector are arranged in such a way that the DOF corresponding to the element at a row of each reference eigenvector is the same as that at the same row of each analytical eigenvector. This is the same as the requirement when performing correlation calculations.

The projection of a reference eigenvector, \( \{ \phi_X \}_i \), onto the analytical eigenvector matrix, \( \{ \phi_A \} \), can be expressed as:

\[
\{ \hat{\phi}_X \}_i = \{ \phi_A \}_i \{ c_i \} \quad (4.62)
\]

According to the discussion in the previous section, the constant vector, \( \{ c_i \} \), can be obtained by:

\[
\{ c_i \} = ([\phi_A]^T [\phi_A])^{-1} [\phi_A]^T [\phi_X]_i \quad (4.63)
\]

If the analytical eigenvector matrix, \( \{ \phi_A \} \), is of full rank, it can be taken as a basis of the vector space \( \mathbb{V} \in \mathbb{R}^n \). Then, each reference eigenvector can be spanned by the basis. Then, the constant vector, \( \{ c_i \} \), can be obtained as:

\[
\{ c_i \} = [\phi_A]^{-1} [\phi_X]_i \quad (4.64)
\]

In this situation, the projection of the reference eigenvector, \( \{ \phi_X \}_i \), onto the analytical eigenvector matrix, \( \{ \phi_A \} \), can be obtained as:

\[
\{ \hat{\phi}_X \}_i = [\phi_A] \{ c_i \} = [\phi_A] [\phi_A]^{-1} [\phi_X]_i = [\phi_X]_i \quad (4.65)
\]
It can be seen that when the analytical eigenvector matrix contains all $n$ modes and the matrix is of full rank, the projection of any reference eigenvector of the same $n$ degrees-of-freedom onto the matrix is equal to the reference eigenvector itself.

If there are $m$ ($<n$) modes (columns) in the analytical eigenvector matrix, or some DOFs in the analytical model are constrained in a way different from that in the reference model, the analytical eigenvectors in the matrix span a subspace $H$ of $R^n$. A reference eigenvector in the vector space $V \in R^n$ (it may belong to a subspace of the vector space $V \in R^n$) can be represented as:

$$
\{\phi_x\}_i = \{\bar{\phi}_x\}_i|_H + \{\bar{\phi}_x\}_i|_{H^\perp}
$$

(4.66)

where, $\{\bar{\phi}_x\}_i|_H$ is the projection of the reference eigenvector onto the subspace $H$ which is spanned by the analytical eigenvectors, and $\{\bar{\phi}_x\}_i|_{H^\perp}$ is the projection of the reference eigenvector onto a subspace which is the orthogonal complement of the subspace $H$. The projection, $\{\bar{\phi}_x\}_i|_H$, can be calculated by the equation of:

$$
\{\bar{\phi}_x\}_i|_H = \{\phi_x\}_i|_H
$$

(4.67)

In this case, the constant vector, $\{c_i\}$, can be obtained by equation [4.63]. If the reference eigenvector is in the same subspace as the analytical eigenvector matrix, the projection of the reference eigenvector can still be equal to the vector itself (equation [4.61]). If the reference eigenvector is in a subspace different from $H$, some (or all) information of the reference eigenvector will be lost in the process of projection. Therefore, the projection of the reference eigenvector would not equal to the eigenvector itself.

4.3.3 Projection of reference eigenvectors and configuration errors

If there are neither discretisation errors nor configuration errors but only parameter errors present in the analytical model, both the eigenvectors of the reference model and the eigenvectors of the analytical model have the same number of DOFs and the same DOFs. Although the parameter errors of the analytical model will make its eigenvectors different from those of the reference model, both reference eigenvectors and analytical eigenvectors are in the same vector space when the parameter errors are not too large. Therefore, the projection of each reference eigenvector onto the space spanned by the analytical eigenvectors might not lose any information and the results of the projection will be the same as the initial reference eigenvectors.

$$
[\bar{\phi}_x] = [\phi_x]
$$

(4.68)

When there are no errors other than discretisation errors in the analytical model and the effects of the errors to the dynamic properties of the model in the frequency range of interest are not significant, which can be checked by a convergence check using the method proposed in section 4.2 of the thesis, the analytical eigenvectors
in the frequency range of interest will be only slightly affected by these errors. Therefore, projecting each reference eigenvector in the frequency range onto the set of analytical eigenvectors in the same frequency range will get a projection that is equal to the initial reference eigenvector.

However, if there are configuration errors in the analytical model, some features of the structure that are important to the dynamic properties of interest will be missing from the analytical model. This is equivalent to situations where some DOFs in the reference model are either missing from the analytical model or replaced by other kinds of DOF. For example, if a joint in the actual structure (or a spring element in the reference model) is represented by a rigid bar (or MPC) element in the analytical model, the DOF at one end of the joint will be constrained to have the same eigenvector data as the DOF at the other end of the joint. This constraint introduces a configuration error into the analytical model and makes the analytical eigenvectors limited in a subspace of \( n \)-dimension vector space, \( V \in \mathbb{R}^n \), (\( n \) here is the number of DOFs in the reference model). If a mode is affected significantly by this configuration error, the reference eigenvector of the mode will be in a subspace that is different from the subspace that is spanned by the analytical eigenvectors. When projecting the reference eigenvector of the mode onto the subspace spanned by the analytical eigenvectors, the projection of the reference eigenvector will exhibit significant differences from the reference eigenvector itself.

From the above discussion, it can be seen that the comparison between each reference eigenvector and its projection onto a subspace spanned by the analytical eigenvectors predicted by an analytical model might be able to distinguish configuration errors from other errors in the analytical model.

### 4.3.4 Configuration error check

In the previous section, all DOFs in the reference model are used to represent reference eigenvectors. In the practice of model validation (and in the procedure of model verification) only some DOFs in the actual structure (or in the reference model) can be measured. From the initial analytical model, not all modes but only some of them will be extracted to examine its dynamic properties in the frequency range of interest. Thus, the practice of a configuration error check for the process of model validation can be carried out in a method described below.

For reference eigenvectors, \([\phi_X]\), of all modes in a given frequency range, the analytical eigenvectors, \([\phi_A]\), predicted by an FE model in a little wider frequency range than the experiment frequency range are selected (the reason for selecting analytical eigenvectors in this way will be discussed in the section 4.3.6 – Study of the configuration check method). Calculating a matrix \([C]\) by the equation:
\[ [C] = ([\phi_A]^T[\phi_A])^{-1}[\phi_A][\phi_X] \]  \hspace{1cm} (4.69)

Then, the projection of all \textit{reference eigenvectors} onto the subspace spanned by the \textit{analytical eigenvectors} can be calculated by the equation:

\[ [\tilde{\phi}_X] = [\phi_A][C] \]  \hspace{1cm} (4.70)

The matrix \([C]\) is called the "projecting matrix" because it serves to project the \textit{reference eigenvectors} onto the \textit{analytical eigenvectors}. The comparison between an initial \textit{reference eigenvector} and its projection can be made with the MAC parameter, as follows:

\[
(MAC_{\phi_X})_{ii} = \frac{([\tilde{\phi}_X]^T[\phi_X])^2}{([\phi_X]^T[\tilde{\phi}_X]) ([\phi_X]^T[\phi_X])}
\]  \hspace{1cm} (4.71)

The values of the diagonal elements of the MAC matrix indicate the degree of similarity between the \textit{reference eigenvectors} and their projection. By examining both the MAC matrix and projecting matrix, \([C]\), it is possible to detect if there are configuration errors in the \textit{analytical model}.

First, consider the situation where there is no noise on the \textit{reference eigenvectors}. If the value of the \(i^{th}\) diagonal element of the MAC matrix is close to 1.0, and there are only a few elements in the \(i^{th}\) column of \([C]\) that have amplitudes significantly greater than those of other elements in the same column, it can be said that the \textit{reference eigenvector} of the \(i^{th}\) mode can be expressed as a linear combination with a few of the selected \textit{analytical eigenvectors} and, thus, the analytical model, by which these \textit{analytical eigenvectors} are predicted, has no configuration errors or possible configuration errors in the model do not affect the ability of the model to predict this mode with high accuracy.

If the value of the \(i^{th}\) diagonal element of the MAC matrix is close to 0.0, that means that the information of the \(i^{th}\) \textit{reference eigenvector} is lost in the process of projection of the eigenvector onto the subspace spanned by the selected \textit{analytical eigenvectors}. Thus, the model is unable to predict this mode, even by modifying parameters in the model, and the model is considered to have configuration errors.

If the value of the \(i^{th}\) diagonal element of the MAC matrix is somewhere between 0.0 and 1.0, the value reflects the degree to which the \(i^{th}\) \textit{reference eigenvector} is similar to its projection onto the subspace spanned by the selected \textit{analytical eigenvectors}. Usually, a MAC value lower than 0.8 for a \textit{reference eigenvector} means that even if many \textit{analytical eigenvectors} predicted by the model are included in projecting this \textit{reference eigenvector}, the projection of the \textit{reference eigenvector} is still not the same as, or close to, the initial one. This means that the initial reference eigenvector is in a subspace significantly different from the subspace spanned by the selected analytical eigenvectors. Therefore, it can be concluded that the model has configuration errors that
make the model unable to predict this mode.

The projecting matrix $[C]$ plays an important role in the projecting process. Each element in the projecting matrix is a constant multiplied to an analytical eigenvector for expressing a reference eigenvector with a linear combination of the analytical eigenvectors. If an FE model can predict the reference properties accurately with only slight modification to the parameter values of the model, the difference between a reference eigenvector and its corresponding analytical eigenvector will be small. This means that the reference eigenvector can be expressed as a linear combination of the corresponding analytical eigenvector together with relatively small contributions from other analytical eigenvectors. Thus, only one element in the column of the projecting matrix for this reference eigenvector has a relatively large amplitude and the others in the same column have relatively small amplitudes. If the reference eigenvector is one of a pair of double modes, the projecting matrix may have two elements in the corresponding column with large amplitudes. Only under this condition can good correlation between the initial reference eigenvector of a mode and its projection indicate that the model is capable of predicting this reference eigenvector after the model is modified when necessary.

If, in a column of the projecting matrix, there are several elements with relatively large amplitudes of the same order, several analytical eigenvectors are needed for expressing the reference eigenvector corresponding to this column. This indicates that the analytical model may have configuration errors that make the model difficult to be updated for predicting this reference eigenvector.

When there is noise on the experimental eigenvectors, the values of the diagonal elements in the MAC matrix will drop to some extent, and the number of elements in columns of the matrix $[C]$ that have relative large amplitudes will increase. Thus, when using this method in practical cases, these effects should be taken into account.

### 4.3.5 Case study of the method with plate FE models

In Case 3 of the previous chapter, two FE models, one with configuration errors and one without, were subjected to model updating procedures. The results showed that the model with configuration errors was not suitable for model updating. The same FE models will be used in this section for a case study of the proposed configuration check method.

The mesh of these two analytical models and the reference model is shown in Figure 3-7 in Page 71. The physical parameters of these three models are listed in Table 3-8 in Page 71. In the case studies with these models, all translation DOFs of the model will be included in eigenvectors in order to simplify the application condition.
4.3.5.1 Configuration check on the model without configuration errors

*Analytical model A* has no configuration errors or discretisation errors but does include parameter errors. With the all translation DOFs of the model represented in the eigenvectors, the correlation between the modal data of the first 30 modes from *analytical model A* and the first 20 modes from the *reference model* is shown in Figure 4-7. Figure 4-8 shows the *MAC* matrix of the eigenvector correlation. From these figures, it is difficult to give an opinion whether the model has configuration errors or not.

![Figure 4-7 Correlation between analytical model A and the reference model](image)

Projecting each of the 20 *reference eigenvectors* onto the subspace spanned by the *analytical eigenvectors* of the first 30 modes, a set of the projections of the *reference eigenvectors* can be obtained. Comparing the initial *reference eigenvectors* and the their projections, the *MAC* value between each initial *reference eigenvector* and the their projections is plotted in Figure 4-9 and the *MAC* matrix is shown in Figure 4-10. From these two figures it can be seen that the *MAC* values for all pairs of the initial *reference eigenvectors* and their projections are all higher than 98%. That means that all *reference eigenvectors* are almost equal to their projection onto the subspace spanned by the *analytical eigenvectors* of the first 30 modes predicted by *analytical model A*.

The projecting matrix in the above projection process, \( C \), is plotted in Figure 4-11. For examining the matrix easily, the normalised projecting matrix is plotted in Figure 4-12. Each column of the projecting matrix is normalised in such a way that the element with the largest amplitude in the column is set to +1.0 or –1.0.
Figure 4-8 MAC matrix between analytical model A and the reference model

Figure 4-9 MAC values of the initial modes and their projections with 30 modes

Figure 4-10 MAC between the reference eigenvectors and the their projections
From Figure 4-12, it can be seen that most modes in the initial *reference eigenvector* set can be expressed as a linear combination of one or two eigenvectors in the *analytical* modal data set. For some modes, one or two projecting constants have relatively large amplitudes and up to three other projecting constants have the absolute normalised values between 0.2 and 0.6. This means that the initial *reference eigenvector* of each of these modes is quite different from any of the *analytical eigenvectors*. However, by linear combinations of several *analytical eigenvectors*, these initial *reference eigenvectors* can be well expressed.
The above projection results indicate that analytical model A has no configuration errors, or that the configuration errors the model may have will not affect the ability of the model to predict the first 20 modes provided that some parameters of the model are modified to some degree.

4.3.5.2 Configuration check for the model with configuration errors

The same projection process was carried out on analytical model B, which has configuration errors when compared with the reference model. The correlation between the modal data from the reference model and analytical model B is shown in Figure 4-13. In the figure, there are natural frequency differences and the MAC values of the correlated mode pairs. The MAC matrix between the reference model and analytical model B is also shown in the figure.

Figure 4-13 Correlation between the reference model and analytical model B

From this figure, it is difficult to say whether the analytical model has configuration errors or not. The eigenvector correlation for the first ten modes is almost the same as those for analytical model A. The eigenvalue discrepancies of these modes from analytical model B are a little greater than those from analytical model A. For the last ten modes, the main difference between the correlation results is the number of the uncorrelated modes. This number is 3 with analytical model A and is 5 for analytical model B.

Projecting the reference eigenvectors onto the subspace spanned by the analytical eigenvectors of the first 40 modes from analytical model B, the correlation between the initial reference eigenvectors and their projections is shown as the MAC matrix in Figure 4-14. From the figure, it can be seen that the first ten modes in Figure 4-14 have MAC values greater than 80%. This means that configuration errors of the
model, if they exist, hardly affect the ability of the model to predict the first ten reference modes. However, there are six modes with MAC values lower than 80% and two of them lower than 40%.

The amplitudes of the elements of the projecting matrix are shown as the lengths of the columns in Figure 4-15. The normalised projecting matrix is shown in Figure 4-16. From these pictures, it can be seen that most reference eigenvectors need only one or two analytical eigenvectors when expressing them as linear combinations of the analytical eigenvectors, although some of these reference eigenvectors are very similar to their projections while others are not so similar.

Figure 4-14 MAC of the initial eigenvectors and the projections with analytical model B

Figure 4-15 projecting matrix for the model with configuration errors
Figure 4-16 Relative values of the projecting matrix - with configuration errors

For the 13th reference mode, which has the lowest MAC value in Figure 4-14, six analytical eigenvectors have their contributions in expressing the reference eigenvector of the mode. Two of these have relative constants greater than 80%, and all other four have relative constants between 40% and 80%. In the projecting process, the first 40 eigenvectors from the analytical model are included for the first 20 eigenvectors from the reference model. According to engineering experience, it is clear that no mode beyond mode 40 from the analytical model can be expected to have an eigenvector which is the same as, or close to, any of the first 20 eigenvectors from the reference model. Thus, combining the analysis on the projecting matrix, it can be concluded that the reason for the low MAC value for mode 13 in Figure 4-14 is that the analytical model has configuration errors which affect the ability of the model to predict the reference eigenvector of the mode even after modifying the parameter values of the model.

One of the aims of a configuration error check is to find the locations of possible configuration errors in a model. This can be accomplished by comparing the initial reference eigenvectors and their projections that have low MAC values between them. Figure 4-17 shows the initial reference model shape for Mode 13 of the reference model and the projection of the reference model shape for the same mode. The initial mode shape shows relative movement between the two ends of each of the connecting elements. Because the connecting elements in analytical model B are MPC elements, there is no relative movement between the two ends of each MPC element. The projection of the reference model shape for the mode cannot reproduce the initial reference model shape of the mode. Thus, the difference between the initial reference model shape and its projection shows the locations of the configuration errors of the analytical model.
4.3.6 Study of the configuration check method

The configuration check method proposed here verifies FE models with the help of reference eigenvectors. Before performing the method, the number and the positions of the DOFs by which the eigenvectors are represented need to be determined. The number of modes of the analytical data set used in the projection process is another factor that will affect the verification results. In the case study of the method with the plate models, the eigenvectors are represented by all the translation DOFs of the model. However, in the practice of model validation, the number of DOFs in the reference data set (usually it is an experimental data set) can be only a small part of the total DOFs of the analytical model. In this section, the selection of DOFs and modes for a configuration error check will be studied.

4.3.6.1 Number of analytical modes

In equation (4.69), the projecting matrix \( C \) is calculated with the pseudo-inverse of the analytical eigenvector matrix. Thus, a condition of using the configuration check method is to keep the analytical eigenvector matrix non-singular – the number of the DOFs in the eigenvector matrix should be equal to or greater than the number of the analytical modes and the rank of the analytical eigenvector matrix should be equal to the number of modes.

The number of modes in the reference data set is determined by the purpose of model validation. Because there are some uncertainties in the initial FE model, the analytical eigenvectors predicted by the model will not the same as the reference eigenvectors. Thus, in the verification procedure, the number of the analytical modes from the model should be more than the number of the reference modes in order to project all reference modes onto the subspace spanned by the analytical eigenvectors. This is the lower limit for the number of analytical modes for the verification procedure.

Consider an extreme condition. Suppose that the DOFs in analytical eigenvectors are selected by a test-planning technique so that the eigenvectors are linearly independent of each other, and suppose that the number of the analytical modes is equal to the number of the DOFs. In this case, each reference eigenvector can be expressed as a linear combination of the analytical eigenvectors. No matter what kind of
error the model may have, the projections of the reference eigenvectors will be the exactly the same as the reference eigenvectors themselves. Possible configuration errors in the model cannot be detected from the projection results. Thus, the number of the analytical modes should be less than the number of DOFs by which the eigenvectors are represented. This is the upper limit for the number of analytical modes for the verification procedure.

From the practice of model validation, it is known that by slightly modifying parameter values of an FE model, the natural frequencies of the predicted modes will not be changed too much. Applying this idea to configuration error check gives that analytical modes used in the projection process should be in a frequency range that is a little wider than that of the reference modes.

In the case study of the configuration check method with the plate model without configuration errors, the first 30 analytical eigenvectors took part in the projection process. The results showed that all of the first 20 reference eigenvectors were well expressed by a linear combination of the 30 analytical eigenvectors. From Figure 4-12 it can be seen that a few of modes above Mode 20 in the eigenvector data set from analytical model A have relatively large amplitudes in the projecting matrix. This means that these modes have contributions for expressing some reference modes. If the analytical eigenvectors of Modes 21 to 30 are not included in the projecting process, the projections of the reference eigenvectors will not be so similar to the initial reference eigenvectors. Figure 4-18 shows the MAC matrix between the initial reference eigenvectors and their projections onto a subspace spanned by the first 20 analytical eigenvectors from analytical model A. It can be seen that the last two modes have low MAC values.

![Figure 4-18 MAC of the projections onto the first 20 analytical eigenvectors](image)
4.3.6.2 Number and positions of the DOFs

In general, the number of DOFs in reference eigenvectors will be much less than that in the analytical model. The current test planning methods for selecting measurement positions are generally for the purpose of distinguishing the measured eigenvectors from each other. The methods are also based on the initial FE model of the structure. When there are some uncertainties in the model, the predicted eigenvectors will be different from those of the actual structure. Thus, the number of DOFs selected by the methods is always more than the number of modes in order to compensate for the inaccuracy of the predicted eigenvectors.

In the verification procedure, reference eigenvectors will be used to check the configuration of analytical models. According to the formulae for projecting reference eigenvectors onto analytical eigenvectors, the analytical eigenvector matrix should be non-singular. Thus, the number of DOFs in both the reference eigenvectors and the analytical eigenvectors should be at least greater than the number of modes in the analytical eigenvector matrix and the selected DOFs should make the rank of the analytical eigenvector matrix equal to the number of modes in the matrix.

Besides the above requirement which can be fulfilled by purely mathematical manipulation on the analytical eigenvectors predicted by the initial FE model, engineering experience and judgement are also important for selecting DOFs of eigenvectors that will be used in the verification procedure, especially when there may be some configuration errors in the model.

Take the case study on the plate model with configuration errors as an example. In this case study, all translation DOFs in the analytical model were included in the eigenvectors. The verification results showed that some reference eigenvectors couldn't be well expressed by the analytical eigenvectors of the first 40 modes. When performing a test-planning technique on the eigenvectors from analytical model B, one of the two DOFs of the same direction at the two ends of an MPC element might be removed from the measurement DOF candidates. Thus, the selected DOFs may not be the best selection to reflect the effects of the configuration errors of the model. As an example with analytical model B, the number of DOFs is limited to 74 and reference eigenvectors are represented by the DOFs selected with an Effective Independence test-planning method. The verification procedure results in good correlation between the initial reference eigenvectors and their projections as shown in Figure 4-19.

This case study indicates that some DOFs should be selected on the locations of possible configuration errors in order to avoid biased information from a verification procedure. This can only be achieved by the use of test-planning methods together with engineering experience and the understanding of the model and the structure.
4.3.6.3 Information from the projecting matrix

Beside the information from the MAC matrix between the initial reference eigenvectors and the projections, the projecting matrix \([C]\) is another source of information that helps to verify FE models. Each element in the projecting matrix is a constant applied to an analytical eigenvector for expressing a reference eigenvector as a linear combination of the analytical eigenvectors.

In the above case study, the normalised projecting matrix is shown in Figure 4-20. From the figure, it can be seen that in some columns several analytical modes have normalised amplitudes greater than 20%. Particularly in columns 13 and 19, the number of elements with normalised amplitudes greater than 20% is more than 10. Figure 4-21 shows the values of the elements in the 13\(^{th}\) column of the projecting matrix.
The picture shows that 16 modes in the analytical data set have their contributions for expressing the reference eigenvector. That means that the high MAC value between the initial reference eigenvector and the projection for Mode 13 is obtained by a linear combination of the analytical eigenvectors that is a purely mathematical manipulation. There is no physical meaning for the good eigenvector correlation.

From the above analysis of the configuration check method and the analysis of the projection results with different sets of the analytical eigenvectors from the plate models, the conclusion is that the method can be successfully applied in configuration checks provided that some conditions are met on selection of DOFs in eigenvectors and on the number of modes in the analytical eigenvector matrix. The results from a configuration error check should be assessed not only on the correlation between the initial reference eigenvectors and their projections but also on the projecting matrix.

4.3.7 Case study on an industrial model

An FE model of the Rear Bearing Housing (RBH) component structure has been subjected to a model updating procedure in the previous chapter. The results of model updating showed that the model was difficult, if not impossible, to update with improved correlation to the reference data. The comparison of the model and the structure showed that the model might have configuration errors located on the vanes of the structure. Although the reference data used for model updating may not be the best for verification, the configuration check method proposed in this chapter was applied to this case in order to check the applicability of the method to industrial models.

The reference eigenvectors were measured at 24 grid points and in three orthogonal directions at each point. All the grid points were located in the outer case of the structure. The Auto-MAC of the reference eigenvectors and the MAC matrix between the initial reference eigenvectors and the analytical eigenvectors from the FE model are shown in Figure 4-22.
Figure 4-22 Auto-MAC of the reference eigenvectors (left) and MAC (right) for the RBH structure and the model

From the Auto-MAC, some reference eigenvectors are seen to have MAC values greater than 20% with respect to other reference eigenvectors. This means that the eigenvectors represented by the measurement DOFs are not totally linearly independent. The MAC between the reference eigenvectors and the analytical eigenvectors shows that some reference modes are correlated with analytical modes and others are not. It is not possible to give the reasons for the poor correlation for the uncorrelated modes by just examining the MAC matrix. Performing a configuration check by the proposed method, the correlation between the initial reference eigenvectors and the projections is plotted in Figure 4-23. From the figure, it can be seen that most of the diagonal elements of the MAC matrix are greater than 80% and one diagonal element, Mode 9, has MAC value between 60% and 80%. Thus, the analytical eigenvectors can express all reference eigenvectors except for Mode 9.

Examine the projecting matrix, of which the normalised element values are shown in Figure 4-24. For some reference eigenvectors, many analytical modes have normalised amplitude greater than 20%. For example, for the reference eigenvector of Mode 8, seven analytical modes have their contributions with relatively large amplitudes. Although the MAC value for reference mode 8 in Figure 4-23 is greater than 80%, the fact that seven analytical eigenvectors are needed to express it indicates a significant difference between the actual structure and the analytical model. For reference mode 9, only two analytical eigenvectors take part in the projection process. However, the projection of the reference eigenvector of this mode is a distance from the initial reference eigenvector. This shows that the model cannot predict this reference eigenvector even when parameter values of the model are modified.

All the above analysis on the projection results leads to a conclusion that the analytical model for the structure has configuration errors which make the model incapable of predicting the structural dynamic properties represented by the reference eigenvectors, even after modifying parameter values of the model.
4.4 Concluding remarks on model verification study

There are several papers published in last ten years dealing with discretisation errors in FE models and some methods have been proposed to reduce the effects of discretisation errors on the required dynamic properties. The method in literature for convergence check is to refine the mesh until the solutions converge. The method proposed in this thesis for convergence check is for a pre-updating check. An FE model subjected to the check will not be changed at all in the procedure. The results of the convergence check are the convergence range of model predictions and an estimate of updating parameter compensations for discretisation errors. These results can give the vibration analyst help in the process of model validation. Through the case studies in this chapter, the convergence check method demonstrates itself as being easy to use and effective for the purpose of the method. However, no test has been presented in this thesis for checking the possibility of discretisation errors caused by stiffness effects. Therefore a convergence check by the method proposed here might not be sufficient.

Configuration errors in FE models have been discussed in this chapter. The discussion has led to the proposal of a method to detect their existence in FE models. The MAC parameter is used in the proposed method to compare each initial reference eigenvector and the corresponding projection that is actually a linear combination of analytical eigenvectors with a projecting matrix. By assessing the projection results and the projecting matrix, configuration errors of an FE model might be distinguished from parameter errors in the same model. Although the method cannot correct configuration errors of an FE model, it is possible to locate the errors by comparing visually the reference mode shapes and their projections.

Model verification is one procedure in the model validation process. It is also possible to say that this is a new procedure introduced into the process especially for the step of configuration check. There are many practical cases where an FE model is
considered to be unsuitable for being updated with a normal updating program because
the model is too coarse or the dynamic properties predicted by the initial model are too
far away from the reference properties. The final conclusion on this kind of model is
usually obtained after several trials of performing model updating on the model and
much time has been spent on performing model updating calculation and on
understanding the model and the structure. If an FE model has been subjected to the
verification procedure with a convergence check and a configuration check, the model
can be said to be suitable for model updating, or not, with high confidence.
Chapter 5

Correlation for Model Validation

5.1 Introduction

Correlation is a numerical manipulation in which two sets of dynamic properties are compared quantitatively. The results from the correlation calculation quantify the similarity and the difference between these two data sets. Many numerical correlation methods have been proposed in the last two decades and some of these have been widely used in the area of structural dynamics, especially in comparing experimental data and analytical predictions.

This chapter describes the role of correlation in the process of model validation in general. Furthermore, a study of the problem of performing correlation on quasi-axisymmetric structures will lead to the proposal of a method of performing numerical correlation for such structures.

5.2 Role of correlation in model validation

The process of model validation, as discussed in the first chapter of this thesis, is to create an FE model by modifying an initial model so that the dynamic predictions from the validated model are similar to those of the actual structure under consideration. In this process, the correlation procedure measures the similarity and difference between the dynamic properties predicted by the initial or a modified FE model and the reference dynamic properties that are usually obtained from an experiment performed on the actual structure. The information from the correlation calculation will be used in several procedures of the FE model validation process.

*A quasi-axisymmetric structure means a structure which is approximately axisymmetric, either by virtue of a cyclic periodicity (such as in a bladed wheel) or of imperfections such as blade to blade variation, or irregular holes or seams in an otherwise axisymmetric profile*
5.2.1 Modal data correlation

The most widely used methods for modal data correlation are the Modal Assurance Criterion (MAC) and the Natural Frequency Difference (NFD).

The MAC correlates two sets of eigenvectors (mode shapes), \( \{ \phi_X \} \) and \( \{ \phi_A \} \), and is based on the formula:

\[
MAC(X, A) = \frac{\left( \{ \phi_X \}^H \{ \phi_A \} \right)^2}{\left( \{ \phi_X \}^H \{ \phi_X \} \right) \left( \{ \phi_A \}^H \{ \phi_A \} \right)}
\] (5.1)

where the superscript \( H \) means the conjugate transpose of a complex vector.

Two eigenvector matrices can be correlated to construct a MAC matrix, which is formed of the individual MAC parameters. When there are \( m_1 \) modes in one eigenvector matrix and \( m_2 \) modes in the other, the MAC matrix calculated on these two eigenvector matrices is of dimension \( (m_1 \times m_2) \). If the eigenvector matrices to be correlated are in fact the same matrix, the resulting MAC matrix is an Auto-MAC matrix, which means the correlation of an eigenvector matrix with itself.

A Natural Frequency Difference (NFD) correlation coefficient gives an assessment of the difference between the natural frequencies of a pair of modes in two modal data sets. For two modal data sets with \( m_1 \) modes and \( m_2 \) modes respectively, the NFD coefficients form a matrix of dimension \( (m_1 \times m_2) \). Each element in the matrix represents the difference between the natural frequency of a mode in first data set and that of a mode in the other data set:

\[
NFD(X, A) = \frac{|(\omega_X)_i - (\omega_A)_j|}{\min\left((\omega_X)_i, (\omega_A)_j\right)} \times 100\% \quad i = 1, \ldots, m_1; j = 1, \ldots, m_2.
\] (5.2)

In the formula, \( (\omega_X)_i \) is the natural frequency of the \( i \)th mode in a data set, and \( (\omega_A)_j \) is the natural frequency of the \( j \)th mode in the other data set. Usually, the correlation is made between an experimental data set and an analytical data set. In this case, \( X \) and \( A \) in the equation denote the experimental data set and the analytical data set, respectively.

5.2.2 Correlation in the model validation process

In the process of model validation, correlation is involved not only in the procedures for assessing the initial model and the validated model but also in the procedures of model verification and model updating. When the structural dynamic properties are represented by modal data, the main roles of correlation in the process of model validation can be categorised in two groups: (i) assessing the similarity of two dynamic property data sets, and (ii) determining the Correlated Mode Pairs (CMPs) for further analysis.
5.2.2.1 Assessing the similarity of two dynamic property data sets

Before an FE model is subjected to a model updating procedure, the model has to be assessed in order to determine if it is suitable for model updating. The correlation between the analytical modal data and the experimental modal data shows how close the predictions from the initial FE model are to the measured structural dynamic properties. The correlation results, together with the results from a convergence check and a configuration check, determine whether the model can usefully be subjected further to the model updating procedure. When some modes in the experimental data set are correlated (in both eigenvalues and eigenvectors) with modes in the analytical data set, the initial FE model may be good enough for model updating. If none of the modes in the experimental data set can be correlated, there must be some problems either in the initial FE model or in the experimental data. Until several modes can be correlated between the experimental data set and the analytical data set, the procedure of model updating cannot be undertaken.

When an FE model is subjected to a model updating procedure with the inverse eigen-sensitivity method, the results of correlation between the reference modal data and the modal data predicted from the modified FE model in each iteration of updating calculations are used for monitoring the performance of model updating. Usually, the evolution curves of the correlation parameters as functions of iteration number will be plotted after each iteration when running a model updating program.

After completing a model updating procedure, the modified model should also be assessed to check whether predictions from the modified model are improved with respect to the reference dynamic properties and whether the model can fulfil the requirements proposed on the process of model validation. There are several ways of making this assessment: using modal data correlation is one of them.

5.2.2.2 Correlated Mode Pairs (CMPs)

For a model updating procedure with modal data, the results from correlating the modal data are not only used for assessing the initial FE model and the modified models but are also used for identifying the CMP information which will be used within the updating procedure. In a typical updating equation (see Chapter 3 of the thesis), the sensitivities of dynamic properties to the updating parameters are calculated for the eigenvalues and eigenvectors of the correlated modes in the analytical data set. On the right hand side of the updating equation, there are the differences of the eigenvalues and the eigenvectors of the correlated mode pairs. Thus, in order to build up an updating equation with the inverse eigen-sensitivity method, the information of the correlated mode pairs between the reference dynamic properties and the predictions from models needs to be determined.

The determination of CMPs is made by examining the MAC matrix and the NFD
values. Two thresholds can be proposed on \( MAC \) and \( NFD \) values, respectively. The pairs of modes in two data sets with \( MAC \) values greater than the \( MAC \) threshold and with \( NFD \) values smaller than the \( NFD \) threshold can be considered as correlated mode pairs.

The thresholds on \( MAC \) and \( NFD \) values may be different in different situations. For example, the thresholds for determining CMPs within an updating procedure should be different from those for measuring the similarity of an FE model to a test structure. In order to include more information in the updating procedure, more modes should be considered as correlated modes. That means that the \( MAC \) threshold within the model updating procedure may need to be lower than that for assessing the initial model and modified models and the \( NFD \) threshold within the model updating procedure should be higher than that for assessing the initial model and the modified models.

From the above analysis, it can be seen that correlation plays an important role in the whole process of model validation. The improvement of correlation in the model updating procedure measures the performance of that procedure. The information of CMPs between a reference data set and model predictions can only be determined from the correlation results with the preset thresholds and this information will be used within the model updating procedure.

5.3 **Eigenvector correlation for axisymmetric structures**

Axisymmetric structures and quasi-axisymmetric structures are widely used in the aero engine industry and other industries. FE models for this kind of structure are usually constructed as axisymmetric although the actual physical or “real” structures are often quasi-axisymmetric due to inherent geometry or imperfections.

5.3.1 **Correlation of eigenvectors for axisymmetric structures**

When calculating the \( MAC \) matrix on eigenvectors of a quasi-axisymmetric structure and eigenvectors predicted by an FE model for the structure, the \( MAC \) values obtained do not, sometimes, represent the “true” correlation between pairs of modes from the experiment data set and the analytical data set respectively. For a pair of modes that seem very closely correlated by visual comparison of the mode shapes, the \( MAC \) value might not even be close to 100%.

Consider a typical structure such as a cylinder. When an FE model is constructed

* The term of “a pair of modes” can be used in two situations. (1) Two orthogonal modes that are in the same data set and have the same or close natural frequencies and the same nodal diametral pattern but orthogonal eigenvectors - they will be called “an orthogonal mode-pair” or “double modes” in the thesis. (2) A pair of correlated or considered-to-be-correlated modes that are from two different modal data sets. In the thesis, “a pair of modes” is used to represent the modes in the second situation.
to analyse the cylinder, the structure is usually considered to be axisymmetric. Figure 5-1 shows such a model. However, there might be some effects that make the actual structure not exactly axisymmetric. To simulate an experimental data set for the structure, a quasi-axisymmetric FE model is constructed which is based on the axisymmetric FE model but with parameter values for a few of elements different from those for the other elements. The modal data set predicted by this model is taken to represent an experimental data set.

![Figure 5-1 FE model of the cylinder as an axisymmetric structure](image)

The nodal diameter modes of the structure can be grouped into orthogonal mode-pairs, – the 2ND mode-pair, 3ND mode-pair, etc. Each such pair has two modes with the same or almost the same natural frequencies but different mode shapes. Figure 5-2 shows the mode shapes of two 2ND mode-pairs for the cylinder: A1 and A2 are predicted by the exactly-axisymmetric FE model; B1 and B2 are the “experimental” eigenvectors which are actually obtained from the quasi-axisymmetric FE model.

![Figure 5-2 Orthogonal 2ND modes of the cylinder](image)

By visually comparing the mode shapes, these two pairs of orthogonal modes can readily be considered as two pairs of closely-correlated modes. Each of them clearly relates to a 2ND pattern of deformation. However, when using the MAC parameter to evaluate the correlation between these modes, it is found that the MAC value between
any pair of modes from two different data sets is far away from 100% (see Table 5-1). When 20 modes from each data set are put into a correlation calculation, the MAC values for many considered-to-be-correlated mode pairs are smaller than 80% (see Figure 5-3).

<table>
<thead>
<tr>
<th></th>
<th>Mode A1</th>
<th>Mode A2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode B1</td>
<td>0.494</td>
<td>0.503</td>
</tr>
<tr>
<td>Mode B2</td>
<td>0.504</td>
<td>0.495</td>
</tr>
</tbody>
</table>

Table 5-1 MAC values between two mode-pairs

The MAC values between the analytical data and the experimental data are used to correlate the mode pairs. Differences of the natural frequencies and the eigenvectors between the modes of correlated mode pairs provide the reason for having the FE model updated and are used to form the residual of updating equations. However, it might be misleading when determining correlated mode pairs by examining the MAC values of axisymmetric structures such as those in the MAC matrix of Figure 5-3.

From Figure 5-2, it can be seen that the shape of each 2ND mode looks like an oval. The angles of the principal axes of the ovals are different for the different modes. If the principal axis for Mode A1 rotates anticlockwise by a certain angle about the axisymmetric axis, the rotated shape of the mode will be almost the same as the shape of Mode B1. Rotating Mode A2 by the same angle about the axisymmetric axis, this mode becomes similar to Mode B2. The angle of the principal axis of the oval shape for a 2ND mode is referred to as the spatial phase angle of the mode. For other nodal diameter modes, the lines connecting the points with the maximum deformation of the modes and the axisymmetric axis of the structure can also be referred to as the principal axes of the modes. Thus, each nodal diameter mode can be expressed with the order number of the nodal diameter mode family, the amplitude of the deformation, and the spatial phase angle of the principal axis of the mode.

When an FE model is axisymmetric, the mode shapes representing the predicted
eigenvectors will have spatial phase angles determined by a principle that is of the benefit for calculation. For an orthogonal mode pair predicted by the FE model, we can consider the spatial phase angle of one mode being arbitrary determined while the other mode being orthogonal to this mode. However, the actual structure must have some imperfections or other factors that make the structure not exactly axisymmetric. Even when it is axisymmetric, the mode shapes obtained from an experiment have spatial phase angles that are determined by the test settings. These spatial phase angles of the experimental mode shapes might be different from those predicted by the FE model and the differences of the spatial phase angles will cause low $MAC$ values between pairs of considered-to-be-correlated modes. The differences of mode shapes due to the different spatial phase angles will also make updating results uncertain if eigenvectors are used directly in an updating equation.

In order to ensure that $MAC$ correlation results can be used on quasi-axisymmetric structures to correlate the analytical eigenvectors and the experimental eigenvectors, the spatial phase angles of the predicted nodal diameter modes may need to be shifted.

### 5.3.2 Harmonic components and $MAC$ values

When the mode shapes of an axisymmetric structure are displayed in a cylindrical coordinate system, they can be considered as combinations of trigonometric functions. For the modes shown in Figure 5-2, the mode shape data in the radial direction can be plotted along the $\theta$ axis of the cylindrical coordinate system as in Figures 5.4 and 5.5.

![Figure 5-4 Predicted mode shapes in the cylindrical coordinate System](image1)

![Figure 5-5 "Experimental" mode shapes in the cylindrical coordinate System](image2)
Because all of these modes are 2ND modes, the basic harmonic component has two cycles around a circumference. In Figure 5-4, the orthogonal mode pair have different phase angles – Mode A2 can be considered as delayed by $\pi/2$ with respect to Mode A1 in the phase angle of the basic harmonic component. In Figure 5-5, Mode B2 lags $\pi/2$ behind Mode B1.

From these two figures, it can be also seen that the phase angle of Mode B1 is different from that of Mode A1 and the phase angle of Mode B2 is different from that of Mode A2. When Modes A1 and A2 are shifted to make this phase angle difference zero, one can obtain two pairs of highly correlated modes.

In general, there may be a number of harmonic components in one mode. If trigonometric functions are used to describe the eigenvector data in the radial direction, the eigenvector data of the $i^{th}$ mode at the $k^{th}$ grid point can be expressed as:

$$\phi^{(R)}_{ki} = \sum_{m=2}^{N_c} R_{i0} \cos(m \times \theta_k + m \theta_{i0}).$$

(5.3)

where $R_{i0}$ is the amplitude of the $m^{th}$ component of the $i^{th}$ mode in the radial direction; $\theta_k$ is the phase angle coordinate of the $k^{th}$ grid point; $m \theta_{i0}$ is the spatial phase angle of the $m^{th}$ harmonic component of the $i^{th}$ mode; $N_c$, the upper limit of the summation, is determined by $N_p$, the number of grid points around a circumference. When all grid points or measurement points are equally spaced around each circumference,

$$N_c = \text{int}(\frac{1}{2} \times N_p - 1).$$

The MAC value for two modes (suppose that they are all real modes) can be expressed as:

$$MAC_{ij} = \frac{\left(\{\phi_{ij}\}^T \{\phi_{ij}\}\right)^2}{\left(\{\phi_{ij}\}^T \{\phi_{ij}\}\right) \left(\{\phi_{ij}\}^T \{\phi_{ij}\}\right)} = \frac{\left(\sum_{k} \phi_{ik} \phi_{jk}\right)^2}{\left(\sum_{k} \phi_{ik}^2\right) \left(\sum_{k} \phi_{jk}^2\right)}.$$

(5.4)

Consider the eigenvector data in the radial direction only and suppose that the grid points are equally spaced around each circumference:

$$\sum_{k} \phi^{(R)}_{ki} \phi^{(R)}_{kj} = \sum_{k} \left(\sum_{m} R_{i0} \cos(m \times \theta_k + m \theta_{i0})\right) \left(\sum_{n} R_{j0} \cos(n \times \theta_k + n \theta_{j0})\right)$$

$$= \sum_{m} R_{i0}^2 \sum_{n} R_{j0}^2 \sum_{k} \cos(m \times \theta_k + m \theta_{i0}) \times \cos(n \times \theta_k + n \theta_{j0})$$

Because:

$$\sum_{k} \cos(m \times \theta_k + m \theta_{i0}) \times \cos(n \times \theta_k + n \theta_{j0}) = \begin{cases} 0 & \text{when } m \neq n \\ \frac{N_p}{2} \cos(m \theta_{i0} - n \theta_{j0}) & \text{when } m = n \end{cases}$$
so,
\[ \sum_{k} \phi_{ki}^{(R)} \phi_{kj}^{(R)} = \sum_{m} R_{m0} \sum_{n} R_{n0} \sum_{k} \cos(m \times \theta_{k} + m \theta_{j0}) \times \cos(n \times \theta_{k} + n \theta_{j0}) \]
\[ = \frac{1}{2} N_{p} \sum_{m} R_{m0} R_{n0} \cos(m \theta_{j0} - n \theta_{j0}) \]  \hspace{1cm} (5.5)

Consider the DOFs in all directions,
\[ \phi_{ki}^{(R)} = \sum_{m=2}^{N_{c}} R_{m0} \cos(m \times \theta_{k} + m \theta_{j0}) \]
\[ \phi_{ki}^{(T)} = -\sum_{m=2}^{N_{c}} T_{m0} \sin(m \times \theta_{k} + m \theta_{j0}) \]
\[ \phi_{ki}^{(Z)} = \sum_{m=2}^{N_{c}} Z_{m0} \cos(m \times \theta_{k} + m \theta_{j0}) \]

where \( mR_{i0}, mT_{i0}, \) and \( mZ_{i0} \) are the \( m^{th} \) component amplitudes of the \( i^{th} \) mode in the radial, tangential and axial directions respectively. Putting the above expression and equation (5.5) into equation (5.4), it can be shown that:
\[ MAC_{ij} = \frac{\left( \sum_{m} (mR_{i0} \cdot mR_{j0} + mT_{i0} \cdot mT_{j0} + mZ_{i0} \cdot mZ_{j0}) \cos(m \theta_{i0} - n \theta_{j0}) \right)^{2}}{\sum_{m} (mR_{i0}^{2} + mT_{i0}^{2} + mZ_{i0}^{2}) \cdot \sum_{m} (mR_{j0}^{2} + mT_{j0}^{2} + mZ_{j0}^{2})} \]  \hspace{1cm} (5.6)

If for all components of two modes, \( mR_{i0} = mT_{i0} = mZ_{i0} = C \), these two modes can be considered to be correlated. However, by putting the above relation into equation (5.6), it is found that:
\[ MAC_{ij} = \frac{\left( \sum_{m} (mR_{i0}^{2} + mT_{i0}^{2} + mZ_{i0}^{2}) \cos(m \theta_{i0} - n \theta_{j0}) \right)^{2}}{\left( \sum_{m} (mR_{i0}^{2} + mT_{i0}^{2} + mZ_{i0}^{2}) \right)^{2}} \]

It is obvious that the \( MAC \) value is influenced directly by the spatial phase angle difference. If for a pair of modes, the amplitudes of each harmonic component are equal between the modes but \( m \theta_{i0} \neq n \theta_{j0} \), such as the modes shown in Figure 5-2, the \( MAC \) parameter of the pair will not be equal or even close to 100%.

When the spatial phase angle difference between a mode pair is \( \pm \pi/2 \) for each harmonic component, such as in the case for an orthogonal mode pair, the \( MAC \) value for the mode pair is equal to zero, because \( \cos(m \theta_{i0} - n \theta_{j0}) = 0 \).

Consider the orthogonal mode pairs, A1 and A2, B1 and B2, in Figure 5-2. Because all of them belong to the 2ND mode family, only one harmonic component dominates the mode shapes. Due to the difference of the spatial phase angles, none of the four \( MAC \) values between these two pairs of modes is equal to 100%. However, because Mode B1 is orthogonal to Mode B2, \( \theta_{B1,0} = \theta_{B2,0} \pm \pi/2 \), thus,
\[ MAC_{11} + MAC_{12} = \cos^2(\theta_{A1,0} - \theta_{B1,0}) + \cos^2(\theta_{A1,0} - \theta_{B2,0}) = \cos^2(\theta_{A1,0} - \theta_{B1,0}) + \cos^2(\theta_{A1,0} - \theta_{B1,0} \pm \pi/2) = \cos^2(\theta_{A1,0} - \theta_{B1,0}) + \sin^2(\theta_{A1,0} - \theta_{B1,0}) = 1 \]

From the above equation, it can be seen that the sum of the MAC values of a orthogonal mode pair with respect to one mode with the same nodal diameter pattern should be equal or close to 100%. The same result can be obtained for \((MAC_{21} + MAC_{22})\). This relationship between MAC values for orthogonal mode pairs can be proved by the data listed in Table 5-1.

The feature described above is helpful for identifying the problem of low MAC values due to spatial phase angle differences. For a quasi-axisymmetric structure, if an analytical nodal diameter orthogonal mode pair have both MAC values with respect to one experimental nodal diameter mode not equal or close to 100%, but the sum of the MAC values equal or close to 100%, then this orthogonal mode pair probably belong to the same nodal diameter mode family as the experimental mode. However, they have different spatial phase angles which cause neither of the MAC values equal or close to 100%.

5.3.3 Shifting spatial phase angle

In order to obtain a MAC value that can represent the “true” correlation between a considered-to-be-correlated pair of an analytical mode and an experimental mode for a quasi-axisymmetric structure, it is necessary to create a rotated analytical mode by shifting the spatial phase angle of each harmonic component of the analytical mode. The difference between the spatial phase angle of this rotated analytical mode and that of the 'initial' analytical mode is referred to as the shifting spatial phase angle. It is determined in such a way that the rotated analytical mode has the maximum MAC value with respect to the experimental mode. To keep the mode shape of the 'initial' analytical mode intact with only spatial phase angles of the harmonic components shifted, the rotated analytical mode is composed of the same harmonic components of the analytical mode with a shifting spatial phase angle the same for each component. For an orthogonal mode pair, the shifting spatial phase angles for the modes should be the same in order to keep two rotated analytical modes orthogonal to each other.

Suppose an analytical mode with an eigenvector \(\{\phi_A\}_{i}\) is to be rotated by a shifting phase angle \(\tilde{\Theta}\) in order to maximise the MAC value between this mode and an experimental mode with an eigenvector \(\{\phi_X\}_{j}\). The rotated analytical mode has an eigenvector, \(\{\phi_{A'}\}_{i}\), referred to as the rotated eigenvector in this chapter of the thesis. At the \(k^{th}\) grid point, the rotated eigenvector in the three orthogonal directions can be
expressed as:

\[
\tilde{\phi}_{ij}^{(R)} = \sum_{m=2}^{N} R_{io} \cos(m \times \theta_k + m \theta_{i0} + m \times \tilde{\theta})
\]

\[
\tilde{\phi}_{ij}^{(T)} = -\sum_{m=2}^{N} T_{io} \sin(m \times \theta_k + m \theta_{i0} + m \times \tilde{\theta})
\]

\[
\tilde{\phi}_{ij}^{(Z)} = \sum_{m=2}^{N} Z_{io} \cos(m \times \theta_k + m \theta_{i0} + m \times \tilde{\theta})
\]

(5.7)

5.3.3.1 Shifting spatial phase angle with equally-spaced grid points

In most of modal tests undertaken on axisymmetric or quasi-axisymmetric structures, the measurement DOFs are equally spaced around circumferences. In order to identify each nodal diameter mode in the experimental frequency range, the number of grid points in each circumference is usually more than twice the highest order number of the nodal diameter mode family. Under these conditions, it can be shown that:

\[
\sum_{k} \tilde{\phi}_{ki} \phi_{ki} = \sum_{k} (\phi_A)_{ki} (\phi_A)_{ki}
\]

Thus, the MAC value between the rotated eigenvector \{\tilde{\phi}_i\} and the experimental eigenvector \{\phi_X\}_j can be expressed as:

\[
MAC_{ij} = \frac{\left(\sum_{k} \tilde{\phi}_{ki} (\phi_X)_{kj}\right)^2}{\left(\sum_{k} (\phi_X)_{kj} (\phi_X)_{kj}\right) \left(\sum_{k} (\phi_A)_{ki} (\phi_A)_{ki}\right)}.
\]

The shifting spatial phase angle \(\theta\) that maximises the MAC value can be obtained by letting the partial derivative of the MAC value with respect to \(\tilde{\theta}\), thus the partial derivative of the numerator of the right hand side of the above equation with respect to \(\tilde{\theta}\), to be zero.

\[
\frac{\partial}{\partial \tilde{\theta}} \left[\left(\sum_{k} \tilde{\phi}_{ki} (\phi_X)_{kj}\right)^2\right] = 2 \times \left(\sum_{k} \tilde{\phi}_{ki} (\phi_X)_{kj}\right) \times \frac{\partial}{\partial \tilde{\theta}} \left(\sum_{k} \tilde{\phi}_{ki} (\phi_X)_{kj}\right) = 0.
\]

(5.8)

When both data sets are real modes, the above equation is equivalent to:

\[
\frac{\partial}{\partial \tilde{\theta}} \left(\sum_{k} \tilde{\phi}_{ki} (\phi_X)_{kj}\right) = 0,
\]

which, after performing algebraic manipulation, becomes:

\[
\sum_{k} \sum_{m} m \times \left(\begin{array}{c}
(\phi_X)_{kj}^{(R)} \cdot m R_{io} \cdot \sin(m \times \theta_k + m \theta_{i0} + m \times \tilde{\theta}) - \\
(\phi_X)_{kj}^{(T)} \cdot m T_{io} \cdot \cos(m \times \theta_k + m \theta_{i0} + m \times \tilde{\theta}) + \\
(\phi_X)_{kj}^{(Z)} \cdot m Z_{io} \cdot \sin(m \times \theta_k + m \theta_{i0} + m \times \tilde{\theta})
\end{array}\right) = 0
\]

(5.9)
Supposing that for each \( m \), the value of \( \hat{\theta}_m \) satisfies the equation:

\[
\sum_{k=1}^{N_c} \left( \varphi_{Xk}^{(R)} \cdot R_{i0} \cdot \sin(m \times \theta_k + m \times \hat{\theta}_m) - \right)
\left( \varphi_{Xk}^{(T)} \cdot T_{i0} \cdot \cos(m \times \theta_k + m \times \hat{\theta}_m) + \right)
\left( \varphi_{Xk}^{(Z)} \cdot Z_{i0} \cdot \sin(m \times \theta_k + m \times \hat{\theta}_m) \right) = 0
\]

then

\[
\tan \left( m \times \hat{\theta}_m \right) = \frac{-\left( \varphi_{Xk}^{(R)} \cdot R_{i0} \cdot \sin(m \times \theta_k + m \times \hat{\theta}_m) - \right)
\left( \varphi_{Xk}^{(T)} \cdot T_{i0} \cdot \cos(m \times \theta_k + m \times \hat{\theta}_m) - \right)
\left( \varphi_{Xk}^{(Z)} \cdot Z_{i0} \cdot \sin(m \times \theta_k + m \times \hat{\theta}_m) \right)}{\sum_{k=1}^{N_c} \left( \varphi_{Xk}^{(R)} \cdot R_{i0} \cdot \sin(m \times \theta_k + m \times \hat{\theta}_m) - \right)
\left( \varphi_{Xk}^{(T)} \cdot T_{i0} \cdot \cos(m \times \theta_k + m \times \hat{\theta}_m) + \right)
\left( \varphi_{Xk}^{(Z)} \cdot Z_{i0} \cdot \sin(m \times \theta_k + m \times \hat{\theta}_m) \right)} \tag{5.10}
\]

If the value of \( \hat{\theta}_m \) obtained from the above equation is the same for \( m=1, 2, \cdots, N_c \), then \( \hat{\theta} = \hat{\theta}_m \) is the solution for equation \( 5.9 \). If the value of \( \hat{\theta}_m \) obtained from the above equation is not the same for different \( m \), the approximate solution of \( \theta \) for equation \( 5.9 \) can be obtained by averaging the \( \hat{\theta}_m \) values under weighting of the moduli of the harmonic component amplitudes:

\[
\hat{\theta} \equiv \frac{\sum_{m=1}^{N_c} \left( R_{m}^2 + T_{m}^2 + Z_{m}^2 \right)^{1/2} \times \hat{\theta}_m}{\sum_{m=1}^{N_c} \left( R_{m}^2 + T_{m}^2 + Z_{m}^2 \right)^{1/2}} \tag{5.11}
\]

In most applications, FE models predict real modes while experimental modes are complex. Supposing:

\[
\sum_k \varphi_{k} \left( \varphi_{Xk} \right)_k = a + i \cdot b \quad \text{and} \quad \frac{\partial}{\partial \theta} \left( \sum_k \varphi_{k} \left( \varphi_{Xk} \right)_k \right) = c + i \cdot d
\]

where \( i = \sqrt{-1} \), equation \( 5.8 \) becomes:

\[
(a + i \cdot b) \times (c + i \cdot d) = (ac - bd) + i \cdot (ad + bc) = 0 .
\]

Setting either the real or imaginary part of the above equation equal to zero and after performing algebraic manipulation, the angle \( \hat{\theta} \) that maximises the MAC value can be obtained.

### 5.3.3.2 Shifting spatial phase angle with unequally-spaced grid points

When the Effective Independence test planning method (see Chapter 2 of the thesis) is used for the modal test of an axisymmetric or quasi-axisymmetric structure, measurement DOFs are usually unequally spaced around each circumference. In this
case, the MAC parameter between the rotated eigenvector and the experimental eigenvector is expressed as:

$$MAC_{ij} = \frac{\left( \sum_{k} \theta_{kij}(\phi_{X})_{ij} \right)^2}{\left( \sum_{k} (\phi_{X})_{ij} \right) \left( \sum_{k} \theta_{kij} \right)}$$

The shifting spatial phase angle \( \tilde{\theta} \) that maximises the MAC value can be obtained by letting the partial derivative of the MAC value to \( \tilde{\theta} \) go to zero.

$$\frac{\partial MAC_{ij}}{\partial \tilde{\theta}} = 0$$

When using the expression in equation (5.7) to replace \( \tilde{\phi}_{kij} \), the above equation will be very complicated. However, when the FE model is axisymmetric, the analytical eigenvector for a nodal diameter mode usually has only one harmonic component. Under this assumption, the above equation can be simplified as shown below.

Supposing that there is only the \( m^{th} \) harmonic component in the analytical eigenvector, and supposing that the experimental eigenvector, \( \{ \phi_{X} \} \), is a real vector and is represented by the DOFs in the radial direction, then the above equation is equivalent to:

$$(a_2 \cdot a_3 - a_1 \cdot a_5) \cos^3(m \times \tilde{\theta}) + (a_2 \cdot a_5 - a_1 \cdot a_4) \cos^2(m \times \tilde{\theta}) \sin(m \times \tilde{\theta}) + (a_2 \cdot a_5 \cdot a_3 - a_1 \cdot a_4) \cos(m \times \tilde{\theta}) \sin^2(m \times \tilde{\theta}) + (a_2 \cdot a_5 - a_1 \cdot a_4) \sin^3(m \times \tilde{\theta}) = 0 \tag{5.12}$$

where, $a_1 = \sum_{k} (\phi_{X})_{ij} m R_{io} \sin(m \times \theta_k + m \theta_{i0})$

$$a_2 = \sum_{k} (\phi_{X})_{ij} m R_{io} \cos(m \times \theta_k + m \theta_{i0})$$

$$a_3 = \sum_{k} m R_{io}^2 \sin^2(m \times \theta_k + m \theta_{i0})$$

$$a_4 = \sum_{k} m R_{io}^2 \cos^2(m \times \theta_k + m \theta_{i0})$$

$$a_5 = \sum_{k} m R_{io}^2 \sin(m \times \theta_k + m \theta_{i0}) \cos(m \times \theta_k + m \theta_{i0})$$

$\ m R_{io}$ and $\ m \theta_{i0}$ are the amplitude in the radial direction and the initial spatial phase angle, respectively, of the harmonic component in the analytical eigenvector; \( \theta_k \) is the angle coordinate of the \( k^{th} \) grid point in the cylindrical coordinate system, and \( N_p \) is the
number of the measurement points in the radial direction. Equation (5.12) is a cubic equation with real constants. One of the solutions of the equation, the one that must be real (as opposed to complex), is:

\[
\tan\left(m \times \tilde{\theta}\right) = \frac{a_2 \cdot a_3 - a_1 \cdot a_5}{a_2 \cdot a_5 - a_1 \cdot a_4}
\]

(5.13)

When there are measurement DOFs in the tangential and/or axial directions, the five parameters, \(a_1\) to \(a_5\), should include contributions from these DOFs. Then, the shifting spatial phase angle, \(\tilde{\theta}\), can be calculated with the above equation.

From the above analysis, the analytical eigenvector of each nodal diameter mode predicted by an axisymmetric or quasi-axisymmetric FE model should be rotated by an angle such that the MAC values between the considered-to-be-correlated pairs of analytical and experimental modes are maximised and the improved MAC values will then reflect the "true" correlation between the pairs of modes. When the measurement DOFs are equally spaced around circumferences, the shifting spatial phase angles of analytical modes can be calculated using equations (5.10) and (5.11). If the measurement DOFs are unequally spaced around circumferences, the main harmonic component of each analytical eigenvector should be determined before equation (5.13) can be used to calculate the shifting spatial phase angle of the analytical eigenvector.

5.3.4 Two case studies

The method proposed in this chapter has been applied in two case studies: a simple cylinder structure and an engine casing structure.

The cylinder structure shown in Figure 5-1 at Page 147 was used as the first case study for the method. Two FE models were built. One was axisymmetric with all 384 elements having the same parameter values. The second model was based on the first one, but the thickness of 16 elements was changed in order to make the model not exactly axisymmetric and to represent an “experimental” version. This model is called the quasi-axisymmetric model in order to distinguish it from the axisymmetric model. 20 modes were predicted from both models. The MAC matrix between the two data sets was shown in Figure 5-3 at Page 148.

When using the method to rotate the mode shapes predicted by the axisymmetric model, the amplitudes of the harmonic components of the mode shape data in each circumference are identified separately, while the spatial phase angle of each harmonic component of one mode is assumed to be the same for all circumferences. The shifting spatial phase angle for one mode is also confined to be the same for all circumferences and all harmonic components. The mode shapes of one particular mode from three data sets – (i) predicted by the axisymmetric model, (ii) predicted by the quasi-axisymmetric model, and (iii) the rotated one based on the axisymmetric model prediction – are
shown in Figure 5-6. In order to show clearly the difference between these three eigenvectors, different scaling values are used for plotting the mode shapes in the radial direction and those in the tangential direction. The dotted lines in the plots show the mode shape predicted by the axisymmetric model. It is clear that this mode shape has a different spatial phase angle from the others.

Figure 5-6 Rotating mode shape for the axisymmetric structure

Although the rotated analytical mode shape has the spatial phase angle shifted in order to make the eigenvector as close as possible to that predicted by the quasi-axisymmetric model, these two mode shapes are not exactly the same. This is because the quasi-axisymmetric model is not exactly axisymmetric and the amplitudes of the harmonic components of the mode predicted by this model are different from those predicted by the axisymmetric model. It is the differences between the rotated eigenvectors and those predicted by the quasi-axisymmetric mode instead of the differences between the eigenvectors predicted by the axisymmetric model and those predicted by the quasi-axisymmetric model that should be used in updating equations.

After shifting the spatial phase angle for each mode in the axisymmetric model predictions, the rotated analytical eigenvectors were compared with the eigenvectors predicted by the quasi-axisymmetric model. Figure 5-7 shows the MAC matrix between these two eigenvector data sets. This MAC matrix is referred to as the improved MAC matrix in order to distinguish from the initial MAC matrix that is between the initial eigenvectors from the axisymmetric model and those from the quasi-axisymmetric model. For comparison, the initial MAC matrix is also shown in Figure 5-7. From this figure, it can be clearly seen that the MAC values of all considered-to-be-correlated mode pairs in the improved MAC matrix are greater than the MAC values of those pairs in the initial MAC matrix. The mode pairs with uncertain correlation by examining the values in the initial MAC matrix (for example, Modes 1 and 2 in both data sets) can be certainly identified to be or not to be closely correlated by the values in the improved MAC matrix.
The second case study is on the fan casing of an aero engine, of which the FE model and the experimental mesh are shown in Figure 5-8. The structure of the fan casing is not exactly axisymmetric. It is composed of 12 components around the axis and all these components are the exactly same as each other. Therefore, it can be roughly considered as an axisymmetric structure.

Figure 5-9 shows the eigenvector correlation between predictions from the FE model and the experimental results from a modal test on the casing. Some pairs of modes from these two data sets are correlated by the \( MAC \) values in the figure. However, some modes in one data set seem to be correlated with two modes in the other data set. For example, Mode 9 in the experimental data set is correlated to both Modes 13 and 14 in the analytical data set with \( MAC \) values between 40% and 60%. It is not clear one mode in one set is correlated to which mode in the other data set. The method of rotating eigenvectors was applied to this case for those considered-to-be-correlated mode pairs.

The \( MAC \) matrix between the rotated analytical eigenvector data set and the experimental data set is shown in Figure 5-10. The \( MAC \) values for those correlated mode pairs in both the initial and improved \( MAC \) matrices are plotted in Figure 5-11.
Figure 5-9 Initial $MAC$ matrix between FE model predictions and the experimental data

Figure 5-10 $MAC$ matrix between the rotated mode shapes and the experimental data

Figure 5-11 $MAC$ values for the correlated mode pairs
From these two figures, it can be seen that the \( MAC \) values of those considered-as-correlated mode pairs from the initial \( MAC \) matrix are all increased by some degree. For Mode 10 in the experimental data set, the \( MAC \) value of the mode with respect to Mode 13 in the initial analytical data set is about 50% while the value becomes over 90% with respect to the same mode in the rotated analytical data set. It is also very clear that Mode 9 in the experimental data set is now closely correlated to Mode 13 in the rotated analytical data set.

From Figure 5-9, it can be seen the number of correlated mode pairs with \( MAC \) values greater than 80% is only 5 – the analytical model cannot be considered as a good model for predicting the dynamic properties of the casing structure. However, from Figure 5-10 the number of correlated mode pairs with \( MAC \) values greater than 80% is 15 for the first 25 modes in the experimental data set. From these improved correlation results, the analytical model can be considered as not too bad, and the improved correlation results provide, at least, a good start point for subjecting the model to model verification and model updating procure.

### 5.4 Concluding remarks on correlation study

Correlation is a tool that is used widely for numerically comparing analytical predictions and experimental data. In this chapter, the role of modal data correlation in the model validation process has been discussed.

FE models for quasi-axisymmetric structures are usually axisymmetric. Sometimes, some eigenvectors predicted by this kind of FE model are found to have low \( MAC \) values when with respect to experimental eigenvectors although these modes can be considered as closely correlated by visual comparison of the mode shapes. A study of the problem in this chapter has led to the proposal of a method to improve the \( MAC \) parameter values for these modes.

By decomposing the harmonic components and shifting the spatial phase angles of the components, the \( MAC \) values between analytical eigenvectors and experimental eigenvectors of axisymmetric or quasi-axisymmetric structures can be maximised. From improved \( MAC \) values, correlated mode pairs can be clearly identified, especially for those nodal diameter modes. Two case studies, on a simple cylinder structure and on a complicated casing structure respectively, were used to demonstrate the method. The results from both cases show the effectiveness of the method.
Chapter 6

Tests for Model Validation

6.1 Introduction

An FE model validation process constructs an FE model which has acceptably similar dynamic behaviour to that of a structure under consideration. The dynamic behaviour of the structure is usually represented by the experimental data that are obtained from tests undertaken on an actual test structure. There are several forms of experimental data that can be used in the process of model validation and the most widely used are: frequency response functions (FRFs) and modal data (eigenvalues and eigenvectors).

In this chapter, the requirements placed on experimental data for model validation will be discussed. The applications of the Effective Independence (EI) test planning method and the Complex Mode Indicator Function (CMIF) parameter for quasi-axisymmetric structures are explained for the purpose of ensuring that experimental data from a modal test undertaken on a given structure can fulfil the requirements.

6.2 Requirement on experimental data for model validation

In model validation, experimental data are used in several individual procedures of the overall process. The role of experimental data in the process can be categorised into two groups: (i) providing a reference for assessing the initial model and modified/updated models; and (ii) providing information of the structure that will be used in the verification and updating procedures.

6.2.1 Experimental data for assessing models

For assessing the initial model and modified models in the process of model validation, experimental data represent the dynamic properties of the actual structure under the consideration. The frequency range of the experimental data (FRFs or modal
data) must cover the frequency range over which the validated model will be used to predict the dynamic properties of the actual structure. When using modal data in the model validation process, all the modes, or at least most of the modes, in the frequency range of interest should be included in the experimental data set. The DOFs by which the experimental eigenvectors are represented need to be selected in order to distinguish each mode from all others in the experimental data set.

### 6.2.2 Experimental data for verifying and updating models

Providing information of a structure for a model validation process, experimental data are used as the reference dynamic properties in the model verification and model updating procedures of the process.

In the model verification procedure, the experimental data provide information for a configuration error check (see Chapter 4 of the thesis). The selection of the measurement DOFs in the experimental eigenvectors should make the verification procedure as capable as possible of detecting configuration errors in the initial model. That means that the selection of the measurement DOFs should be sensitive to the effects of the possible configuration errors in the model.

For the model updating procedure, the experimental data should be measured in such a way that the modified model resulting from the updating procedure is the most likely to be reckoned as an updated model. That means that the experimental data should provide the information of the structure that is sensitive to the updating parameters and is also sensitive to the boundary conditions of the structure.

Furthermore, for the benefit of solving updating equations, both experimental eigenvalues and eigenvectors will be used in the equations (see Chapter 3 of the thesis). Thus, the experimental eigenvectors will be used not only for determining Correlated Mode Pairs (CMPs) but also for calculating the coefficient matrices and the residual vectors of the updating equations. From this point of view, the more eigenvector data and the more linearly-independent the eigenvector data, the more benefit will be gained for the updating procedure.

### 6.2.3 Comparison of the requirement on modal test data

Many papers and books have been published for modal testing [15] and test planning (see Chapter 2 of the thesis). The principles for a good performance of modal testing in a model validation process should be the same as those for a general modal test. However, the experimental modal data for the process of model validation are not used directly for assessing the dynamic properties of the structure but used for validating an FE model that, then, is going to be used further to predict and to assess the dynamic properties of the structure. Thus, compared with a general modal test, different information and more accurate information are required from a modal test conducted for
the model validation process.

For example, for the purpose of assessing the dynamic properties and/or troubleshooting, the modal test results are required to represent the deformation picture and to identify the natural frequencies of the modes that are of interest to analysts and designers. However, for the purpose of undertaking a model validation process, all modes in the frequency range of interest should be identified from experiments. The selection of DOFs for representing experimental eigenvectors is not for displaying the deformation of a structure but for correlation with analytical eigenvectors and for a configuration check and model updating.

6.2.4 Requirement on experimental data for model validation

From the above analysis, the requirements on experimental data from a modal test for model validation are listed below:

(1) the frequency range over which the experimental data are measured must cover the frequency range over which the validated model will be used;

(2) as many as possible of the modes, or all modes – if possible, in the experimental frequency range should be identified from the experimental data;

(3) the selection of DOFs by which the experimental mode shapes are represented must be capable of distinguishing the mode shapes from each other. For the purpose of model verification, each analytical eigenvector used in the configuration check must be also distinguishable from others when represented by the measurement DOFs; and

(4) the DOFs that can reflect the possible configuration errors in the initial FE model should be measured in the experiment.

6.3 Modal tests on quasi-axisymmetric structures

Many mechanical components and assemblies in the aero engine industry are axisymmetric or quasi-axisymmetric. Because of the special structural dynamic characters of these structures, a special test-planning procedure and special test settings are needed for undertaking a modal test on such structures in order to fulfil the requirements of experimental modal data for the model validation process.

6.3.1 Selecting measurement DOFs

In planning for a modal test, the main issues that should be determined are: excitation DOF(s) and measurement DOFs. For a modal test in a process of model validation, the selection of excitation DOF(s) and measurement DOFs is made to ensure that the experimental modal data can fulfil the requirements listed in the previous sub-chapter.
6.3.1.1 Auto-MAC of eigenvectors on the selected DOFs

The Effective Independence (EI) test-planning method that performs numerical manipulations on analytical eigenvectors predicted from the initial FE model can give advice on the selection of measurement DOFs for general structures. The main criterion for the EI test planning method (see Chapter 2 of the thesis) is to maximize the linear independence between the experimental mode shapes (eigenvectors). In this section, the application of the EI method on quasi-axisymmetric structures will be discussed.

The initial FE model for an axisymmetric or quasi-axisymmetric structure is usually constructed as axisymmetric. As discussed in the previous chapter, there are differences between the spatial phase angles of the eigenvectors predicted by the FE model and those of the eigenvectors of the actual structure, and these spatial phase angle differences are different from one mode to another.

Suppose that the analytical eigenvectors represented by the selected measurement DOFs are in matrix $\phi_\lambda$ and the experimental eigenvector matrix is $\phi_\Lambda$. The analytical eigenvectors need to be rotated in order to maximize the MAC values with respect to the experimental eigenvectors. Suppose that the matrix of the rotated analytical eigenvectors is $\hat{\phi}$.

Because each rotated analytical eigenvector has the maximum MAC value with respect to an experimental eigenvector, the linear independence of the experimental eigenvectors is approximately equivalent to the linear independence of the rotated analytical eigenvectors. In order to apply the EI method to a quasi-axisymmetric structure, the experimental eigenvectors and thus the rotated analytical eigenvectors on the selected measurement DOFs should be as linearly independent as possible to each other.

The MAC value between two rotated analytical eigenvectors is represented as:

$$MAC_{ij} = \frac{\left(\{\hat{\phi}_i\}^T \{\hat{\phi}_j\}\right)^2}{\left(\{\hat{\phi}_i\}^T \{\hat{\phi}_j\}\right)\left(\{\hat{\phi}_j\}^T \{\hat{\phi}_i\}\right)}$$  \hspace{1cm} (6.1)

From the analysis in the previous chapter, it is known that nodal diameter modes of a quasi-axisymmetric FE model need their eigenvectors to be rotated for the purpose of correlating with experimental eigenvectors. In the analytical model data set, each $n$-nodal diameter mode is one mode of a pair of orthogonal modes. The elements of the rotated analytical eigenvectors of such a pair of modes at the $k^{th}$ grid point and in the radial direction can be expressed as:
\[
\begin{align*}
\tilde{\phi}_{ik}^R &= \sum_n R_{i0} \cos \left( m \theta_k + m \theta_0 + m \theta \right) \\
\tilde{\phi}_{i+1,k}^R &= -\sum_n R_{i0} \sin \left( m \theta_k + m \theta_0 + m \theta \right)
\end{align*}
\] (6.2)

Usually, only one harmonic component dominates the analytical eigenvector of a nodal diameter mode. Suppose for a pair of orthogonal modes, modes \(i\) and \(i+1\), the \(m^{th}\) harmonic component is the main component. After numerical manipulations of the above equation, it can be shown that:

\[
\begin{align*}
\tilde{\phi}_{ik}^R &\approx m R_{i0} \cos \left( m \theta_k + m \theta_0 \right) \cos(m \theta) - m R_{i0} \sin \left( m \theta_k + m \theta_0 \right) \sin(m \theta) \\
&= (\phi_A)_{ik}^R \cos(m \theta) + (\phi_A)_{i+1,k}^R \sin(m \theta)
\end{align*}
\] (6.3)

\[
\begin{align*}
\tilde{\phi}_{i+1,k}^R &\approx -m R_{i0} \sin \left( m \theta_k + m \theta_0 \right) \cos(m \theta) - m R_{i0} \cos \left( m \theta_k + m \theta_0 \right) \sin(m \theta) \\
&= - (\phi_A)_{ik}^R \sin(m \theta) + (\phi_A)_{i+1,k}^R \cos(m \theta)
\end{align*}
\]

Thus, each rotated analytical eigenvector can be taken approximately as a linear combination of the analytical eigenvectors of a pair of orthogonal modes:

\[
\begin{align*}
\{\tilde{\phi}_i\} &\approx c_i \{\phi_A\}_i + c_{i+1} \{\phi_A\}_{i+1} \\
\{\tilde{\phi}_{i+1}\} &\approx -c_{i+1} \{\phi_A\}_i + c_i \{\phi_A\}_{i+1}
\end{align*}
\] (6.4)

where

\[
\begin{align*}
c_i &= \cos(m \theta) \\
c_{i+1} &= \sin(m \theta)
\end{align*}
\]

and their amplitudes are smaller than 1.0. \(\{\phi_A\}_i\) and \(\{\phi_A\}_{i+1}\) are the initial analytical eigenvectors of the orthogonal mode pair. Replacing the rotated analytical eigenvectors in equation (6.1) with the expression in equation (6.4) and after algebraic manipulations, the MAC value between two rotated analytical eigenvectors can be obtained as:

\[
MAC_{ij} = \frac{c_j c_i \{\phi_A\}_i^T \{\phi_A\}_j + c_i c_{j+1} \{\phi_A\}_i^T \{\phi_A\}_{j+1} + c_{i+1} c_j \{\phi_A\}_{i+1}^T \{\phi_A\}_j + c_{i+1} c_{j+1} \{\phi_A\}_{i+1}^T \{\phi_A\}_{j+1}}{\left(c_j^2 \{\phi_A\}_j^T \{\phi_A\}_j + c_i^2 \{\phi_A\}_i^T \{\phi_A\}_i + c_{i+1}^2 \{\phi_A\}_{i+1}^T \{\phi_A\}_{i+1} + 2 c_i c_{i+1} \{\phi_A\}_i^T \{\phi_A\}_{i+1}\right) \cdot \left(c_j^2 \{\phi_A\}_j^T \{\phi_A\}_j + c_{j+1}^2 \{\phi_A\}_{j+1}^T \{\phi_A\}_{j+1} + 2 c_j c_{j+1} \{\phi_A\}_j^T \{\phi_A\}_{j+1}\right)}
\] (6.5)

Because the measurement DOFs are selected by applying the EI method to the initial analytical eigenvectors, the off-diagonal elements of the Auto-MAC matrix of \(\{\phi_A\}_j\) should have amplitudes which are much smaller than 1.0. That means that \(\{\phi_A\}_j^T \{\phi_A\}_j = 0\) when \(i \neq j\). Putting this condition into the equation (6.5) and taking
into account that the amplitude of each constant, $c$, is smaller than 1.0, it can be shown that the numerator of the right hand side of equation (6.5) is approximately equal to 0.0, and thus the MAC value between two rotated analytical eigenvectors is approximately equal to 0.0.

From the above analysis, it can be concluded that when the test planning method based on the Effective Independence is used on a quasi-axisymmetric structure, the rotated analytical eigenvectors on the selected measurement DOFs will be as linearly independent as the 'initial' analytical eigenvectors. If these rotated analytical eigenvectors are a good prediction for the eigenvectors of the structure, the experimental eigenvectors on the selected measurement DOFs should be capable of being distinguished from each other. Therefore, the conditions for using the EI test planning method on quasi-axisymmetric structures are the same as the method is used on general structures.

6.3.1.2 Case studies in test planning on quasi-axisymmetric structures

The first case study is based on two FE models: one is axisymmetric and another is quasi-axisymmetric. Figure 6-1 shows the mesh of the models. The test planning method based on Effective Independence Distribution (see section 2.2.1.4.1 of the thesis) was applied to the eigenvectors of the first 20 modes from the axisymmetric model. As an extreme condition for test planning, only 21 DOFs were selected using the method. The spots in the figure show the positions of the selected DOFs, and the numbers besides the spots are the identity numbers for the grid points. The selected DOFs are all in the radial direction.

Figure 6-1 The FE model mesh and the measurement locations

The Auto-MAC matrix of the analytical eigenvectors from the axisymmetric model on the selected DOFs is shown in Figure 6-2. Most off-diagonal elements in the Auto-MAC matrix have values smaller than 20%: only two have values between 20% and 40%.

From the quasi-axisymmetric model, the eigenvectors of the first twenty modes were extracted and were represented on the selected DOFs as those indicated in Figure
The $MAC$ matrix between the eigenvectors from the axisymmetric model and those from the quasi-axisymmetric model, as shown in Figure 6-3, indicates that there are spatial phase angle differences between the eigenvectors of the “considered-to-be-correlated” pairs of modes. However, the $Auto-MAC$ of the eigenvectors from the quasi-axisymmetric model, Figure 6-4, shows that the eigenvectors on the selected measurement DOFs can be distinguished from each other. There are only two off-diagonal elements in the $Auto-MAC$ matrix with values between 20% and 40%. The values of other off-diagonal elements in the matrix are all smaller than 20%.

Figure 6-2 $Auto-MAC$ of the eigenvectors from the axisymmetric model

Figure 6-3 $MAC$ of the eigenvectors from the axisymmetric and quasi-axisymmetric models on the selected DOFs

Figure 6-4 $Auto-MAC$ of the eigenvectors from the quasi-axisymmetric model
The second case study is on the Turbine Case of an aero engine. The initial FE model of the structure is shown in Figure 6-5. The test planning method based on Effective Independence Distribution was applied to the eigenvectors of the first 18 modes predicted by the FE model, and 19 grid points were selected, as shown in Figure 6-6 for the experiment mesh. The vibration in the radial and axial directions on the selected grid points was measured in a modal test undertaken on the casing structure.

The Auto-MAC matrices of the analytical eigenvectors and the experimental eigenvectors, respectively, are shown in Figure 6-7. Both the analytical and experimental eigenvectors are on the measurement DOFs only. Although there are some off-diagonal elements in the matrices, with values between 20% and 40%, the first 18 analytical modes can easily be distinguished from each other, as well as can the first 11 experimental modes. The correlation between the analytical and experimental eigenvectors is shown in Figure 6-8. It can be seen that Modes 8 and 9 in the experimental data set may have different spatial phase angles from Modes 11 and 12 in the analytical data set. However, this difference has little effect on the Auto-MAC of the experimental eigenvectors.

![Figure 6-5 FE model of the Turbine Case](image1)

![Figure 6-6 Exp. mesh of the Turbine Case](image2)

![Figure 6-7 Auto-MAC matrices of the analytical eigenvectors (left) and experimental eigenvectors (right)](image3)
From the above case studies, it can be concluded that the possible differences between the spatial phase angles of the analytical eigenvectors and those of the experimental eigenvectors of a quasi-axisymmetric structure will not change significantly the Effective Independence of the eigenvectors represented on the selected DOFs. Therefore, the EI test planning method can be used on quasi-axisymmetric structures.

6.3.2 Identifying all modes in modal test with CMIF parameter

When a modal test for a model validation process is undertaken on a quasi-axisymmetric structure, it is required to identify all modes from the experimental FRFs in the experimental frequency range. However, because of the closeness of the natural frequencies of a pair of orthogonal modes, one mode of the pair may be identified while the other is missing in many cases. In this section, the Complex Mode Indicator Function (CMIF) will be applied to modal tests on quasi-axisymmetric structures in the aim of providing a helpful tool for effective modal test planning.

6.3.2.1 Complex Mode Indicator Function (CMIF)

Mode indicator functions have been developed for multiple-input-sine-dwell modal testing in order to isolate normal modes [75]. The functions can also be used to process the Frequency Response Functions (FRFs) of a structure for indicating the existence of the modes of the structure.

For a linear structure, the FRF matrix, \([H(\omega)]\), can be considered as the relationship between inputs to the structure and outputs from the structure.

\[
\{x\} = [H(\omega)]\{f\} \tag{6.6}
\]

According to standard vibration theory, this FRF matrix can be written as:

\[
[H(\omega)] = [\phi_o] \left[ \frac{Q_r}{j\omega - \nu} \right] [\phi_i]^H \tag{6.7}
\]

where \([\phi_o]\) and \([\phi_i]\) are, respectively, the eigenvector matrix of dimension \(N_o \times N_m\) on
the output DOFs and the eigenvector matrix of dimension $N_I \times N_m$ on the input DOFs. $N_o$ and $N_I$ are the numbers of output DOFs and of input DOFs, respectively, and $N_m$ is the number of modes. $\begin{bmatrix} Q_r \\ j\omega - \lambda_r \end{bmatrix}$ is a diagonal matrix of dimension $(N_m \times N_m)$. $Q_r$ is the scaling factor of the $r^{th}$ mode, and $\lambda_r$ is the complex natural frequency of the $r^{th}$ mode. $\left[ \right]^H$ means the conjugate transpose of a matrix.

From a modal test, if the experimental FRFs are measured at $N_f$ frequency points, the FRFs can be considered as $N_f$ matrices. Performing a Singular Value Decomposition (SVD) on the matrix at a frequency point $\omega_k$, it is shown that:

$$[H(\omega_k)] = [U_k] [\Sigma_k] [V_k]^H = [U_k] \begin{bmatrix} \sigma_{k1} \\ \sigma_{k2} \\ \vdots \\ \sigma_{kn} \end{bmatrix} [V_k]^H$$

where the singular values, $\sigma_{ki}$, are in the descending order.

$$\sigma_{k1} \geq \sigma_{k2} \geq \cdots \geq \sigma_{kn} \quad \text{and} \quad n = \min(N_I, N_o) \quad (6.9)$$

When SVDs are performed on all the FRF matrices along the frequency axis, the singular values $\sigma_{ki} (k = 1, 2, \cdots N_f)$ in all frequency points can be taken as a function of frequency. The Complex Mode Indicator Function (CMIF) is defined as the square of each singular value at a spectral line:

$$CMIF_i(\omega_k) = \sigma_{ki}^2 \quad i = 1, 2, \cdots, n \quad \text{and} \quad k = 1, 2, \cdots N_f \quad (6.10)$$

Plotting $CMIF$ values along the frequency axis, $CMIF_1(\omega_k)$ is the primary $CMIF$ curve, $CMIF_2(\omega_k)$ is the secondary $CMIF$ curve. For all singular values, $n$ $CMIF$ curves can be plotted [86].

Although equation [6.8] has the similar format as equation [6.7] – both matrices $\begin{bmatrix} Q_r \\ j\omega - \lambda_r \end{bmatrix}$ and $[\Sigma_k]$ are diagonal, the matrices in the equations are different. The matrices $[U_k]$ and $[V_k]$ in equation [6.8] are unitary matrices, while the matrices $[\phi_o]$ and $[\phi_r]$ in equation [6.1] may not be orthogonal – their orthogonality is affected by the DOFs by which the eigenvectors are represented. The dimension of the singular value matrix, $[\Sigma]$, and that of the matrix $\begin{bmatrix} Q_r \\ j\omega - \lambda_r \end{bmatrix}$ are also different. For a continuous structure, the number of modes in the experimental FRFs can be infinite, while in a modal test, the number of the modes in the experiment frequency range is usually
greater than the number of the input DOFs. Thus, the dimension of the singular value matrix, $[\Sigma]$, is much smaller than that of the matrix $\frac{Q_r}{j\omega - \lambda_r}$.

Because the matrices $[U_k]$ and $[V_k]$ are unitary matrices, each vector in the matrices $[\phi_o]$ and $[\phi_r]$ can be taken as a linear combination of the vectors in the matrices $[U_k]$ and $[V_k]$.

$$[\phi_o] = [U_k][c_k] \quad \text{and} \quad [\phi_r] = [V_k][d_k]$$

(6.11)

where $[c_k]$ and $[d_k]$ are constant matrices applied to the matrices $[U_k]$ and $[V_k]$, respectively. Putting the above equation into equation (6.7) it can be shown that:

$$[H(\omega_k)] = [U_k][c_k]\left[\frac{Q_r}{j\omega_k - \lambda_r}\right][d_k]^H[V_k]^H$$

(6.12)

Comparing this equation with equation (6.8) it can be seen that:

$$\begin{bmatrix}
\sigma_{k1} \\
\sigma_{k2} \\
\vdots \\
\sigma_{kN} \\
\end{bmatrix} = [c_k]\left[\frac{Q_r}{j\omega_k - \lambda_r}\right][d_k]^H$$

(6.13)

From the above equation, each singular value contains contributions from all the modes in the FRFs. At different frequency points, the values of $\frac{Q_r}{j\omega_k - \lambda_r}$ will be different. In order to make the left hand side of the above equation to be diagonal, the constant matrices, $[c_k]$ and $[d_k]$, should be different at different frequency points. Thus, the unitary matrices $[U_k]$ and $[V_k]$ must vary at different frequency points.

From the above analysis, it can be concluded that the singular values from an SVD on an FRF matrix cannot be necessarily considered as proportional to the scaling factors of the modes of a structure, and the singular vectors from the SVD cannot be taken in general as proportional to the eigenvectors of the structure.

6.3.2.2 Peaks in CMIF curves and the existence of modes

Suppose that at a frequency point, say $\omega_k$, there is one mode, mode $r$, whose natural frequency is the closest to $\omega_k$ among all the modes and the absolute value of $\frac{Q_r}{j\omega_k - \lambda_r}$ is much greater than the other elements in the diagonal matrix of equation (6.7). At this frequency point, the eigenvector and the value of $\frac{Q_r}{j\omega_k - \lambda_r}$ of this mode have the greatest contribution to the FRF matrix, $[H(\omega_k)]$. In other words, the information of a principal component of the FRF matrix at $\omega_k$ is dominantly composed
of the information of this mode. With an SVD on the FRF matrix, \( [H(\omega)] \), the vectors in the matrices \( [U_k] \) and \( [V_k] \) that correspond to the singular value, \( \sigma_{k1} \), will be similar to the eigenvectors of the mode on the input DOFs and the output DOFs, respectively. At the same frequency point, other modes have small amplitudes. Their contributions to the FRF matrix are contained in the other singular values and other vectors in the matrices \( [U_k] \) and \( [V_k] \).

Because all vectors in the matrices \( [U_k] \) and \( [V_k] \) are normalized such that \( [U_k]^T[U_k] = [I] \) and \( [V_k]^T[V_k] = [I] \), the singular value \( \sigma_{k1} \) reflects the amplitude of mode \( r \) and other singular values have values much smaller than \( \sigma_{k1} \). As the frequency increases from a point lower than the natural frequency of mode \( r \) to a point higher than the natural frequency, the primary CMIF curve will have a peak at a frequency point equal to or near the natural frequency of the mode. Thus, a peak of the primary CMIF curve indicates the existence of a mode.

Consider that at a given frequency point there are a pair of double modes, modes \( r \) and \( r+1 \), whose natural frequencies are the closest to the given frequency. The eigenvectors and the values of \( \frac{Q_r}{j\omega_k - \lambda_r} \) and \( \frac{Q_{r+1}}{j\omega_k - \lambda_{r+1}} \) of both modes have relatively large contribution to the FRF matrix at the frequency point. Although the eigenvectors of the modes may not be orthogonal when represented by the input DOFs and the output DOFs, two principal components of the FRF matrix can be found so that the corresponding singular vectors are approximately equal to linear combinations of the eigenvectors of these two modes. Therefore, not only the first singular value but also the second singular value will have larger amplitudes than other singular values. As the frequency increases from a point lower than the natural frequency of mode \( r \) to a point higher than the natural frequency, both the primary and secondary CMIF curves will have a peak at that frequency point.

6.3.2.3 Use of CMIF on modal test

From the above analysis on the relationship between peaks in CMIF curves and the existence of modes of a structure, it can be seen that the information from CMIF curves can be used for a modal test on the structure.

For selecting excitation point(s) for the modal test, there are several test planning techniques based on numerical manipulations on analytical eigenvectors predicted by the initial FE model of the structure. All these techniques will give several options for the test engineer to choose. Usually a "preliminary" test will be undertaken in order to determine the excitation DOF(s). From this preliminary test, the experimental FRFs can form a FRF matrix with several input DOFs and several output DOFs. Performing an SVD on the FRF matrix at each experimental frequency point and
plotting the CMIF curves, as described in the previous sub-section, some useful information of the structure can be provided.

The peaks in the primary CMIF curve show the existence of the modes of the structure. The peaks in the secondary CMIF curves, whose frequencies are the same as or close to some peak frequencies of the primary CMIF curve, indicate the existence of double modes or close modes of the structure. When more than two excitation DOFs and several measurement DOFs are used in the preliminary test, all of the structural modes (distinguished from local modes) of the structure in the test frequency range should be excited and be measured in general. Thus, the peaks in the CMIF curves show the existence of all structural modes of the structure in the experimental frequency range.

Information from CMIF curves can also be used to determine excitation DOF(s) for the modal test on a quasi-axisymmetric structure. When a preliminary modal test with several input DOFs and several output DOFs is undertaken, the double modes of the structure can be picked up from the peaks of the primary and secondary CMIF curves. Suppose two CMIF curves are obtained by performing an SVD on FRF matrices with two input DOFs. If the secondary CMIF curve cannot show a double mode of the structure, none of the input DOFs is a good choice as the excitation DOF for the "formal" modal test in the light of identifying this double mode from experimental FRFs. If the primary and the secondary CMIF curves show a pair of double modes, there are three possible situations: (i) each input DOF can excite both the double modes; (ii) only one of the input DOFs can excite both the double modes; (iii) one input DOF can excite one of the double modes and the other input DOF can excite the other of the double modes. In this case, performing a preliminary modal analysis on the FRFs with each input DOF from the preliminary test, the situation can be identified and the excitation DOF(s) for the "formal" modal test can be determined.

6.3.2.4 Case study of selecting excitation DOFs for modal test on a quasi-axisymmetric structure

Figure 6-9 shows the FE model and the experimental mesh of the Fancase component of an aero engine. The main part of the component is axisymmetric. Some seams make the whole component quasi-axisymmetric. A preliminary modal test was carried out using hammer excitation. There were three input DOFs, all of them in the radial direction. They were located at Points 2, 3, and 50 as marked on the experimental grid. In the preliminary modal test, the response of the vibration of the structure in the radial direction was measured at 12 points, as marked from 1 to 12 on the experimental mesh. From the preliminary modal test, a total of 36 FRFs were obtained.

Four groups of CMIF curves, computed for different combinations of the experimental FRFs, are plotted in Figure 6-10.
Figure 6-9 The FE model mesh and the experimental mesh of a casing structure

![Figure 6-9 The FE model mesh and the experimental mesh of a casing structure](image)

Figure 6-10 $CMIF$s for the FRFs from the preliminary modal test

(1) input DOFs: 1R, 2R, 50R
(2) input DOFs: 2R, 3R
(3) input DOFs: 2R, 50R
(4) input DOFs: 3R, 50R

From each of these figures, six modes can be indicated from the peaks of the $CMIF$ curves. In Figure 6-10(1), there are three $CMIF$ curves from all 36 FRFs. The primary and the secondary $CMIF$ curves have peaks showing the existence of the six modes. There are several peaks in the third $CMIF$ curve. However, by comparing the amplitudes of the peaks in the third $CMIF$ curve with those in the primary and secondary $CMIF$ curves, it can be found that the amplitudes of the peaks in the third
CMIF curve are more than one hundred times smaller than the amplitudes of the peaks in the primary and the secondary CMIF curves. Thus it can be confirmed that the peaks in the third CMIF curve do not indicate any modes of the structure.

In Figure 6-10 (2) to (4), all CMIF curves are obtained from the FRFs with two input DOFs. Each of these figures indicates six modes in the test frequency range. This means that these six modes of the structure can be excited by hammer excitation at one or two input DOFs of the preliminary test. Table 6-1 lists the natural frequencies identified from modal analysis on the experimental FRFs from the preliminary test and it can be seen from the table that using 2R or 3R as the excitation DOF can only reveal five of the six modes. Only one mode in the pair of double modes around 103Hz will be excited with input at either 2R or 3R. However, all six modes can be identified from the FRFs with 50R as the excitation DOF. Thus, if only one shaker will be used in the formal test, neither 2R nor 3R but 50R should be selected as the excitation DOF.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Input: 2R</th>
<th>Input: 3R</th>
<th>Input: 50R</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>45.77</td>
<td>45.77</td>
<td>45.75</td>
</tr>
<tr>
<td>2</td>
<td>75.66</td>
<td>75.66</td>
<td>75.63</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>102.99</td>
<td>102.98</td>
</tr>
<tr>
<td>4</td>
<td>103.32</td>
<td></td>
<td>103.29</td>
</tr>
<tr>
<td>5</td>
<td>177.70</td>
<td>177.69</td>
<td>177.62</td>
</tr>
<tr>
<td>6</td>
<td>177.86</td>
<td>177.90</td>
<td>177.80</td>
</tr>
</tbody>
</table>

Table 6-1 Natural Frequencies (Hz) identified from the preliminary modal test

In this case study, the CMIF has been successfully applied in a modal test on a quasi-axisymmetric structure. Information contained in the CMIF curves has been used to determine the number of modes in the experimental frequency range. Particularly, the CMIF curves for FRFs with more than one input DOFs can show the existence of double modes of the structure. Information in the CMIF curves is also useful for selecting excitation DOF(s) so that all modes in the experimental frequency range can be identified from experimental FRFs.

6.4 Concluding remarks on experiments for model validation

Modal testing is an important procedure in the process of model validation. At the moment, all reference information about a structure under the consideration comes from modal tests. When using modal data in the procedures of model updating and model verification, the reference data have to be provided from modal tests.

The requirements of the modal test procedure in the model validation process arise from other procedures of the process. In this chapter, the differences between the
requirements of modal test for the purpose of model validation and those for the purpose of assessing the structural dynamic properties have been discussed. The particular preparation for modal tests on quasi-axisymmetric structures has been studied in the light of selecting the measurement DOFs and the excitation DOF(s).

The application of the EI test planning method on quasi-axisymmetric structures has been studied in this chapter. The study result has proved that the method is suitable of being used for quasi-axisymmetric structures – shifting the spatial phase angles of nodal diameter modes will not change significantly the linear independence of the eigenvectors of the modes.

The $CMIF$ has also been studied in this chapter. An SVD on a FRF matrix at a frequency point results in singular values and $CMIF$ parameters. Although the left and the right vectors from the SVD are not the same as the eigenvectors of the structure on the input DOFs and the output DOFs, the primary and the secondary $CMIF$ parameters reflect the amplitudes of modes that dominate the FRF matrix at the frequency point.

Case studies of preparing modal tests on quasi-axisymmetric structures are included in the chapter to demonstrate the effectiveness of the methods.
Chapter 7

Case Study of Model Validation

7.1 Introduction

The model updating procedure plays a central role in the model validation process. Other procedures in the process can be considered as pre-processing or post-processing for the model updating procedure. The last step in the model validation process is to assess the updated model(s) from the updating procedure. In the procedure of model updating, the modified model from an updating session is assessed by correlation with the reference data for model updating. When the correlation results are improved and the correlation parameters (for example MAC and NFD data) satisfy the requirements demanded of the model, the modified model is an updated model. However, this updated model needs to be further assessed in a situation different from where the reference data were obtained. Only when an updated model satisfies both the requirements within the model updating procedure and in an assessment procedure with other independent information about the structure, can it be said a validated or “valid” model.

In this chapter, the process of validating the FE models of two industrial structures will be presented. One structure is the Combustion Chamber Outer Casing (CCOC) component of an aero engine casing, and the other is the Turbine Casing component of the same engine. To assess the updated models for both structures, the updated models will be assembled together and the dynamic properties of the assembled model will be compared with experimental data from a test undertaken on the physical assembly of these two structures.

Some procedures for validating the FE models of the CCOC have been discussed in previous chapters of the thesis. The procedures for validating the FE model of the Turbine Casing, from convergence check to assessing the modified model, will be discussed in this chapter.
7.2 Updating the FE model of the Turbine Casing

7.2.1 Initial model and convergence check

The initial FE model of the Turbine Casing structure is shown in Figure 7-1. In this model, there are 864 grid points and the number of DOFs is 5184. There are shell elements and beam elements and the total number of elements is 1824. Performing a normal mode extraction, 18 modes were obtained in the frequency range of 10 – 700Hz. Table 7-1 lists the natural frequencies predicted by the initial FE model of Turbine Casing component with the lumped-mass and the coupled-mass matrix approaches, respectively. Figure 7-2 shows the differences between the natural frequencies from these two approaches.

![Figure 7-1 Initial FE model of the Turbine Casing component](image)

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Freq. Lump _m</th>
<th>Freq. Couple _m</th>
<th>Mode Shape</th>
<th>Mode No.</th>
<th>Freq. Lump _m</th>
<th>Freq. Couple _m</th>
<th>Mode Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>44.80</td>
<td>44.85</td>
<td>2ND in</td>
<td>11</td>
<td>368.20</td>
<td>370.87</td>
<td>5ND in</td>
</tr>
<tr>
<td>2</td>
<td>44.80</td>
<td>44.85</td>
<td>2ND in</td>
<td>12</td>
<td>368.21</td>
<td>370.88</td>
<td>5ND in</td>
</tr>
<tr>
<td>3</td>
<td>70.85</td>
<td>71.14</td>
<td>2ND out</td>
<td>13</td>
<td>402.86</td>
<td>408.51</td>
<td>4ND out</td>
</tr>
<tr>
<td>4</td>
<td>70.85</td>
<td>71.14</td>
<td>2ND out</td>
<td>14</td>
<td>402.86</td>
<td>408.52</td>
<td>4ND out</td>
</tr>
<tr>
<td>5</td>
<td>125.25</td>
<td>125.56</td>
<td>3ND in</td>
<td>15</td>
<td>524.45</td>
<td>529.99</td>
<td>6ND in</td>
</tr>
<tr>
<td>6</td>
<td>125.25</td>
<td>125.56</td>
<td>3ND in</td>
<td>16</td>
<td>524.46</td>
<td>530.00</td>
<td>6ND in</td>
</tr>
<tr>
<td>7</td>
<td>218.56</td>
<td>220.69</td>
<td>3ND out</td>
<td>17</td>
<td>608.91</td>
<td>619.80</td>
<td>5ND out</td>
</tr>
<tr>
<td>8</td>
<td>218.56</td>
<td>220.69</td>
<td>3ND out</td>
<td>18</td>
<td>608.91</td>
<td>619.80</td>
<td>5ND out</td>
</tr>
<tr>
<td>9</td>
<td>234.26</td>
<td>235.33</td>
<td>4ND in</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>234.27</td>
<td>235.34</td>
<td>4ND in</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7-1 Natural frequencies from the initial model
From Table 7-1 and Figure 7-2 it can be seen that, for the same order of the nodal diametral mode family, the natural frequency differences of the out-of-phase modes are greater than those of the in-phase modes. However, all 18 modes have the natural frequency differences smaller than 2%. Therefore, the initial model can be considered convergent on these modes.

### 7.2.2 Experiment and correlation

With the EI test-planning method applied on the analytical eigenvectors, 19 grid points were selected as the measurement points. In a modal test on the component, the behaviour of the component was measured in the radial and axial directions at each of these grid points. Below 450Hz, 11 modes (in addition to the rigid body modes) were identified from the experimental FRFs. Figure 7-3 shows the experiment mesh. From the figure it can be seen that the measurement points are unequally spaced around the circumferences on the two end flanges of the component. In Chapter 6 of the thesis, the Auto-MAC of the experimental mode shapes at the measurement DOFs of this component has been discussed and it was demonstrated that they could be distinguished from each other.

The correlation between the analytical data from the initial FE model and the experimental data from the test is shown in Figure 7-4 and the natural frequency correlation is listed in Table 7-2. Modes 2, 4 and 5 in the analytical data set are not correlated with modes in the experimental data set. This is because the experimental data set has missed one mode in each of the first three pairs of orthogonal modes. From the MAC matrix in Figure 7-4, Modes 8 and 9 in the experimental data set have MAC values between 40% and 80% with respect to Modes 11 and 12 in the analytical data set. These low MAC values indicate that there may be spatial phase angle differences between the experimental eigenvectors and the analytical eigenvectors of these modes because the component is axisymmetric. Using the method proposed in Chapter 5 of the thesis to rotate the analytical eigenvectors, the MAC matrix between the experimental eigenvectors and the rotated analytical eigenvectors is plotted in Figure 7-5. The improved eigenvector correlation parameters are also listed in Table 7-2.
Figure 7-3 Experimental mesh of the Turbine Casing component

Figure 7-4 Correlation between the experimental and initial analytical data

Figure 7-5 Correlation after shifting spatial phase angles
<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Freq. Exp.</th>
<th>Freq. Ana_init</th>
<th>MAC (%)</th>
<th>NFD (%)</th>
<th>Mode No.</th>
<th>Freq. Exp.</th>
<th>Freq. Ana_init</th>
<th>MAC (%)</th>
<th>NFD (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>44.10</td>
<td>44.85</td>
<td>93.5</td>
<td>1.88</td>
<td>8</td>
<td>220.59</td>
<td>220.69</td>
<td>93.4</td>
<td>0.59</td>
</tr>
<tr>
<td>2</td>
<td>44.85</td>
<td>9</td>
<td>238.85</td>
<td>235.34</td>
<td>9</td>
<td>239.49</td>
<td>235.33</td>
<td>97.9</td>
<td>-1.37</td>
</tr>
<tr>
<td>3</td>
<td>67.94</td>
<td>71.14</td>
<td>97.6</td>
<td>5.32</td>
<td>10</td>
<td>380.64</td>
<td>370.87</td>
<td>98.1</td>
<td>-2.49</td>
</tr>
<tr>
<td>4</td>
<td>71.14</td>
<td>11</td>
<td>382.16</td>
<td>370.88</td>
<td>12</td>
<td>408.19</td>
<td>408.52</td>
<td>98.6</td>
<td>-2.89</td>
</tr>
<tr>
<td>5</td>
<td>125.56</td>
<td>12</td>
<td>382.16</td>
<td>370.88</td>
<td>13</td>
<td>414.88</td>
<td>408.51</td>
<td>96.6</td>
<td>-0.94</td>
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<tr>
<td>6</td>
<td>125.82</td>
<td>125.56</td>
<td>95.8</td>
<td>-0.05</td>
<td>14</td>
<td>414.88</td>
<td>408.51</td>
<td>96.6</td>
<td>-0.94</td>
</tr>
<tr>
<td>7</td>
<td>215.68</td>
<td>220.69</td>
<td>93.4</td>
<td>2.95</td>
<td>15</td>
<td>414.88</td>
<td>408.51</td>
<td>96.6</td>
<td>-0.94</td>
</tr>
</tbody>
</table>

Table 7-2 Correlation between the experimental and the initial analytical (rotated) data

From Table 7-2 and Figures 7-4 and 7-5, it can be seen that the modal data set predicted by the initial FE model of the Turbine Casing component has a very good correlation with the experimental data set. However, there is one mode – Mode 3 – in the analytical data set that has a natural frequency about 5% higher than the natural frequency of the counterpart experimental mode in the correlated mode pair. In order to make the model suitable for being used further, the model needs to be updated and validated. Because all 11 experimental eigenvectors are highly correlated with analytical eigenvectors, there is no need to perform a configuration check on the initial model.

7.2.3 Sensitivity and updating parameter selection

In order to keep the model axisymmetric, each candidate updating parameter is the Young’s modulus or mass density of all the elements within one circumference ring. Figure 7-6 shows the candidates for the sensitivity analysis. Figure 7-7 shows the eigenvalue sensitivities to the stiffness parameters and those to the mass parameters. In the sensitivity figures, the mode numbers are the correlated mode pair numbers and the absolute values of the sensitivities to mass parameters are plotted for convenience.

Figure 7-6 Shell elements (upper) and beam elements (lower) for sensitivity analysis
From Figure 7-7, it can be seen that the eigenvalues of the modes included are more sensitive to the mass properties of candidate elements than to the stiffness properties of the same elements. The sensitivities of the eigenvalues to the mass densities of shell elements are greater than those to the mass densities of beam elements, and it is the same for the sensitivities of eigenvalues to the stiffness parameters. Thus, the updating parameters should be selected mainly from shell elements in this case.

7.2.4 Model updating on the Turbine Casing component

According to the correlation and sensitivity analyses on the model of the Turbine Casing component, a procedure for updating the initial FE model was undertaken by the following settings.

(1) The first 11 modes from the experimental modal data set were selected as the reference data for model updating.

(2) Two sets of updating parameters were selected for two updating sessions. These are listed in Table 7-3. In this table, "K" and "M" refer to stiffness and mass properties, respectively, of the candidate elements. In the both sessions, parameter 1 was selected for improving the correlation of in-phase modes. Parameter 2 was selected for improving the correlation of out-of-phase modes. In addition, parameter 3 in session 2 was selected for balancing the improvements on in-phase modes and out-of-phase modes.

<table>
<thead>
<tr>
<th>Parameter No.</th>
<th>Session 1</th>
<th>Session 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S1-3, S9-11 (K)</td>
<td>S1-3 (M)</td>
</tr>
<tr>
<td>2</td>
<td>S6-8, S17-19 (K)</td>
<td>S6-8 (M)</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>S4, 5, 13-16 (M)</td>
</tr>
</tbody>
</table>

Table 7-3 Updating parameter sets for the turbine case model
The inverse eigen-sensitivity method was used for the updating procedure. The weighting factor for each eigenvalue was assigned to be ten times of the inverse of the eigenvalue, and the weighting factor for each eigenvector was assigned to be the inverse of the norm of that eigenvector.

Figure 7-8 shows eight pictures, four for each updating session. The $MAC$ matrix picture in the figure for each session (the upper left hand corner of a group of four pictures) shows the correlations between the reference eigenvectors and the analytical eigenvectors from the modified model in the final iteration of the updating session. The other three pictures for each session show the evolution curves of updating parameter values (upper right hand corner), the $MAC$ values of correlation mode pairs (lower left hand corner), and the natural frequency differences of the correlated mode pairs (lower right hand corner). Table 7-4 lists the updating parameters and their final values in the updating sessions. Each value in the table is the ratio between the increment of a property and the initial value of the property. Table 7-5 lists the natural frequencies of the experimental data set and those predicted by the modified FE models from these two updating sessions.

<table>
<thead>
<tr>
<th>Updating Parameter</th>
<th>Session 1</th>
<th>Final Value</th>
<th>Session 2</th>
<th>Final Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1-3, S9-11 (K)</td>
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<td></td>
<td>S1-3 (M)</td>
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</tr>
<tr>
<td>S6-8, S17-19 (K)</td>
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<td></td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>S4, 5,13-16 (M)</td>
<td>0.1773</td>
</tr>
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</table>

Table 7-4 Final Values of the Updating Parameters
<table>
<thead>
<tr>
<th>Model No.</th>
<th>Exp. Natural Frequency (Hz)</th>
<th>NFD(%)</th>
<th>Session 1</th>
<th>Session 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Freq. (Hz)</td>
<td>NFD (%)</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>Freq. (Hz)</td>
<td>NFD (%)</td>
</tr>
<tr>
<td>1</td>
<td>44.10</td>
<td>1.88</td>
<td>45.23</td>
<td>2.56</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>45.23</td>
<td>2.59</td>
</tr>
<tr>
<td>3</td>
<td>67.94</td>
<td>5.32</td>
<td>70.71</td>
<td>4.08</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>70.71</td>
<td>2.52</td>
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<tr>
<td>5</td>
<td></td>
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<td>126.82</td>
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</tr>
<tr>
<td>6</td>
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<td>126.82</td>
<td>0.79</td>
</tr>
<tr>
<td>7</td>
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<td>2.95</td>
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<td>1.34</td>
</tr>
<tr>
<td>8</td>
<td>220.59</td>
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<td>218.58</td>
<td>-0.91</td>
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<tr>
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<tr>
<td>11</td>
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<td>376.35</td>
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</tr>
<tr>
<td>12</td>
<td>382.16</td>
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<td>376.35</td>
<td>-1.52</td>
</tr>
<tr>
<td>13</td>
<td>408.19</td>
<td>0.63</td>
<td>403.47</td>
<td>-1.16</td>
</tr>
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<td>14</td>
<td>414.88</td>
<td>-0.94</td>
<td>403.47</td>
<td>-2.75</td>
</tr>
</tbody>
</table>

Table 7-5 Eigenvalue correlation with the modified models

The curves in Figure 7-8 and the data in Tables 7-4 and 7-5 show that both sessions of updating the initial FE mode of the Turbine Casing component have improved the eigenvalue correlation with the experimental data. The eigenvector correlation is largely unchanged or becomes a little worse in both sessions. Considering the improvement on the eigenvalue correlation, the modified model from session 2 is better than that from session 1, and thus is taken as an updated model.

### 7.3 Assessment of the updated models of the components

In order to assess an updated model of a structure, one effective way is to modify the structure and to use the model to predict the effects of the modification. Assembling two components together can be considered as a modification to both components. If the model assembled from the updated models of the components can represent the dynamic properties of the assembly in a frequency range, the updated models of the individual components can be considered to be capable of representing their dynamic properties and to be validated models of the components in the frequency range.

The Turbine Casing component and the Combustion Chamber Outer Casing component are two adjacent substructures of an aero engine. The components are
assembled physically using more than one hundred nuts and bolts in their end flanges.

In order to assess the validity of the updated models of these two components, two FE models for the assembly were constructed. One was assembled from the initial FE models of the components, and another was assembled from the updated models of the components. Figure 7-9 shows the FE model mesh of the assembly. The connection between these two components was approximated by rigid bars elements.

A modal test was undertaken on the assembly. Figure 7-10 shows the experimental mesh. The vibration was measured at a total 60 DOFs on 48 grid points in the test. In the frequency range of 10-450Hz, 15 modes were identified from the experimental FRFs.

Correlation between the experimental modal data and the corresponding data predicted by the assembled FE models is shown in Figures 7-11 and 7-12. The natural frequency correlation is listed in Table 7-6.

From Figure 7-11 and Table 7-6, it can be seen that the FE model assembled from the initial models of the components cannot get good correlation with the first three modes of the experimental data. The natural frequency differences of some correlated mode pairs are greater than 5%, and one of them is greater than 8%. In the experimental frequency range, there are a total of 6 modes in the experimental data set that are not correlated with any of the modes in the analytical data set.

From Figure 7-12 and Table 7-6, it can be seen that the model assembled from the updated models of the components achieves a good correlation with the first ten experimental modes. Another two experimental modes – Modes 12 and 15 – are also correlated with analytical modes, although their MAC values are smaller than those of the same mode pairs between the experimental mode shapes and the analytical eigenvectors from the model assembled from the initial models. The natural frequency
differences for most of the correlated mode pairs in Figure 7-12 are smaller than 2%. Only two pairs of the correlated modes have the natural frequency differences greater than 3% and smaller than 4%, and one pair of correlated modes have natural frequency difference about 8%.

Comparing the correlation results shown in Figures 7-11 and 7-12 it can be clearly seen that the model assembled from the updated models of the components can better represent the dynamic properties of the assembly than does the model assembled from the initial models of the components. Thus, the updated models of the components can not only predict the modal data of the components more accurately than the initial models, but they can also represent the physical properties of the components with an improved accuracy. Therefore, the updated models of the Turbine Casing and CCOC components can be considered as validated models for the components in the frequency range over which the correlation between the FE model and the structure of the assembly is satisfactory.

Figure 7-11 Correlation - Assembled from the initial models

Figure 7-12 Correlation - Assembled from the updated models
### 7.4 Conclusion

In this chapter, the FE model of an industrial structure has been subjected to the full model validation process. Most of the procedures in the process have been performed on the model. Although the dynamic properties of the initial model are quite similar to those of the structure, the model updating procedure has improved the correlation of the dynamic properties with respect to the experimental modal data.

The assessment of the updated models of two adjacent components of an aero engine has been undertaken by comparing the dynamic properties of the model assembled from the updated models with those of the physical assembly. Because both components will function in situations where they are assembled together, the experimental situation for the assembly is closer to the working environment than the experimental situations for the individual components. Assessment of the updated models of the components in this way can give a high confidence to the validated models in the sense that the models will be used to predict the dynamic properties of the components in the working conditions.
Chapter 8

Discussion on Model Validation

Strategy

8.1 Introduction

In the first chapter of the thesis, a strategy for model validation was proposed with a block diagram (Figure 1-1). That strategy includes all the important procedures for model validation. With the variations that exist for the main steps in the process, multiple approaches for validating an FE model for structural dynamics can be adopted. In this chapter, the advantages and the disadvantages of several possible specific strategies for model validation will be discussed.

8.2 Strategies with and without verification

Model verification is one procedure in the overall process of model validation. In most model validation (updating) case studies published in the literature, the initial models are subjected to model updating procedures without being verified first. The strategy of model validation adopted in these cases can be illustrated by Figure 8-1. In this block diagram, once the reference data for model updating are obtained from a test on the structure, the initial FE model is subjected to the model updating procedure. Sometimes, as in most of the cases published in papers, the initial model has dynamic properties similar to those of the actual structure, and the updating procedure is seen to improve the correlation on either/both eigenvalues or/and eigenvectors. It is a straightforward strategy for model validation. When the modelling engineer has a high confidence in the initial model – for example in a theoretical research case study – the
modified model from the updating procedure could be a validated model when considering the similarity of the dynamic properties of the model to those of the structure in a specified frequency range.

![Diagram of model validation strategy]

Figure 8-1 Strategy of model validation without verification

However, in some industrial cases, the initial model may have configuration and/or parameter errors as discussed in Chapter 4 of the thesis. When such a model is subjected to a model updating procedure without being verified, it will be found, usually after time-consuming calculations have been made, that the model is unsuitable for updating with any model-updating program. Thus, it is essential that such a situation is detected and avoided before costly calculations are made and/or erroneous conclusions are drawn.

With the verification methods proposed in Chapter 4 of the thesis, a model verification procedure can be introduced into the process of model validation. Figure 8-2 shows a block diagram of the strategy with two steps of verification.

The first step of verification – a convergence check – is performed immediately after the initial model is acquired. In this step, discretisation errors in the model will be assessed in order to minimise the effects of the errors to the dynamic properties of interest. If discretisation errors of a model have significant effects on the dynamic properties in the frequency range of interest and the model is subjected to an updating procedure, the final updating parameter values may distort the physical meaning of the model parameters. When a model cannot pass the convergence check, it means that the model is too coarse and the whole model validation process should stop or a new model with a finer mesh is required.
After a (modal) test is undertaken on the structure, the reference data for model verification and model updating are available. Then, the second step of verification – a configuration check – should be performed. In this step, with the help from the experimental data, the similarity and the differences between the dynamic properties of the initial model and those of the structure are assessed and the model is checked for the possible configuration errors (structural errors). Comparison of the dynamic properties can be carried out with the correlation tools, such as the $MAC$ and $NFD$ parameters, and/or FRF comparison. The configuration check, as discussed in Chapter 4 of the thesis, is used to detect the effects of the possible configuration errors of the model to the dynamic properties in the frequency range of interest. The method developed in Chapter 4 of the thesis proposes a necessary condition for a model to be configuration error-free. If a model cannot pass the check, then it has configuration errors and these may significantly affect the ability of the model to predict the required dynamic properties. This model is not suitable for being subjected to a model updating procedure. If a model does pass the configuration check, it cannot be guaranteed that the model is configuration error-free, but it is possible that any configuration error in the model will not affect significantly the ability of the model to predict the required dynamic properties in the frequency range of interest, and the model is suitable for subjecting to a model updating procedure.

In this strategy for model validation, there are two more steps than in the strategy in Figure 8-1. Although performing each of these steps will take a certain amount of time, the results will save much more time if the model is discovered to be
unsuitable for model updating. Besides, when a model passes the checks, a higher confidence can be attributed to the model and this can help the engineer undertaking the following procedures in the model validation process.

8.3 Strategy with reference data from "super model"

Reference data are used in the model verification and the model updating procedures in a model validation process. Different kinds of reference data will require different strategies of model validation.

Usually, the reference data for the model validation process are experimental data, such as experimental FRFs or modal properties. The strategies discussed in the previous section use experimental modal data as the reference data. The experimental data are obtained from the tests on the actual structure under consideration. When the test results are of a good quality, the experimental data can be considered to be a "true" reflection of the dynamic properties of the actual structure. However, it is not always easy to obtain experimental data of a good quality. First of all, the tests have to be undertaken on actual structures. In the design loop of a new product, it is prototypes of the design that will generally be used in the tests. Because of different manufacturing methods for prototypes and for products, the dynamic properties of the prototypes will be, more or less, different from those of the products. Secondly, in a process of troubleshooting, the time window for undertaking a test on the structure is very limited. Furthermore, the performance of test planning and experience of the test engineer are also critical for obtaining test results with a good quality.

Even in the best situation for a test on a structure, noise on test data cannot be avoided. Some boundary conditions that are required to apply to the structure cannot be exactly applied. All these factors will result in the experimental data being an approximation to the true dynamic properties of the structure.

With the growing power of computers, another source for reference data of a structure – an FE model with a very fine mesh – is emerging. The FE model for a general industrial structure for the purpose of design, denoted as the design model, usually has the number of DOFs limited by the efficiency for the final use of the model. With the development of the CAD technique and the fast growth in the power of both hardware and software of computers, it is possible to construct another model with much finer mesh and much less approximation in the details of the structure. This model, denoted here as the "super model", not only has a mesh which is fine enough to avoid the discretisation errors but also has elements to represent all detailed structural design features so that it contains very few configuration errors. The only uncertainties the model may have are the values assigned to model parameters. The non-linearity of some materials in the structure will also make the properties of the materials different
from those from material laboratory. However, the dynamic properties of the "super model" must be more similar to those of the structure than the dynamic properties of the "design model".

Figure 8-3 Model validation strategy with reference data from a "super model"

Figure 8-3 shows a validation strategy for a "design model" using information from a "super model" instead of experimental data as the reference properties in the procedures of model verification, model updating, and the final assessment of the updated models.

There are two main advantages to this strategy. The first is the great amount of information that can be made use of in the process of model validation. The amount of information from a super model is definitely much more than that from experimental data. By taking the advantage of this, some limitations to the application of the configuration check method and to the selection of updating methods will be automatically eliminated. For example, the updating methods based on direct matrix modification require the reference eigenvector data represented by all the DOFs of the model being updated. This requirement can be fulfilled in the above strategy. The second major advantage of this strategy is the reduction in time for obtaining a validated design model. With the help of CAD and CAE techniques, it takes much less time to construct an FE model than to produce a prototype and to undertake a modal test. Although the dynamic properties predicted from the "super model" may not exactly be
the same as those of the structure, the validated model from the above strategy can still be used in the first few iterations of the design-examine-redesign loop, especially in the early stage of a new design or modification.

The key issue for using the above strategy is to make the "super model" of a good quality. For practical industrial structures, the complexity in some physical features, such as joints and contacts, will make the model not the same as the actual structure. Thus, in Figure 8-3, four blocks with broken lines are introduced into the above strategy. These procedures can only be undertaken when prototypes are available. The experimental data from the test procedure will be fed back to modify/update the "super model" in order to correct possible errors in the model. Then the design model, or the validated design model with reference to the initial "super model", needs to be further validated with reference to the new updated "super model".

### 8.4 Strategy within design loop

Usually, a new design of a structure in industry has some similarities to some example structures, and the modification on an existing structure keeps the basic features of the original design. From this point of view, the models of the example structures and/or of the original design and the experience from the validation process on these models are useful in the process of constructing a model for a new design and in the process of validating the new model. Thus, embedding the model validation process into the industrial design-modify-production loop will benefit industry by reducing the time of the design-examine-redesign loop. Figure 8-4 shows such a design loop.

The blocks in the right hand column of Figure 8-4 are the procedures in a typical design-examine-redesign loop. There are two returning points in the analysis and test procedures. With the model validation process introduced into the loop, the model used for analysis (prediction) will be updated/validated when the test data are available. The updated/validated model will, then, be used in the next iteration of the loop to improve the quality of the prediction from the model. From the model validation process, the structure and the model can be better understood. Knowledge of the structure and the model is further used in the modelling process, especially when the initial model contains configuration errors.

In the left hand column of Figure 8-4, there is a validation process on the model of an example structure. The experience of constructing and validating the model can be used to develop and update a modelling method or modelling standard for similar structures. This standard will be used in the modelling process for a new design. Furthermore, the validated model of an existing structure can be used directly for predicting dynamic properties of a modified structure. The example structure and/or the
existing structure are available for test when a new structure or a modified structure is in the design stage. As in the model validation strategy with the "super model", the model for the new structure constructed with the experience of the existing models can give predictions for the new structure when its prototype is still a long way from being available, although these predictions may not be accurate.

The main problem in this design loop is how to develop the modelling procedure or the modelling standard for similar structures. Each modelling engineer has his own experience and his style for constructing FE models of certain types of structure. With the development of intelligent CAE techniques, it become possible to let computer programs learn by experience and build up and update the method and standard for constructing FE models of similar structures.

![Figure 8-4 Modern design loop with embedded model validation processes](image)

8.5 Concluding remarks on model validation strategy

Several variations of model validation strategy have been presented in this chapter. Which strategy should be adopted in a practice of model validation depends on the source of the information about the structure under consideration and the quality of the initial model.

The strategy that is adopted the most at the moment is a straightforward strategy. Introducing model verification into the model validation process is to avoid costly
calculations in the case where the initial model is not suitable of being subjected to model updating.

Two strategies have been proposed which make the use of the development of computers. However, these strategies have not been adopted in the case studies in this thesis. With the fast growth of computer power in both hardware and software, it is believed that these strategies and more complicated and sophisticated strategies will be developed and adopted in the near future for the model validation practice.
Chapter 9

Conclusions and Future Work

9.1 Conclusions of the thesis

- The concept of model validation has been defined in the thesis. A strategy for conducting model validation on representative industrial structures has been proposed as a process composed of several procedures, or steps.

- State-of-art methods for the main steps in the model validation process have been studied through a literature review. The methodology of model validation/updating has been developed for about two decades and is still being developed. It has been shown that some essential issues for model validation need to be explored further.

- Any FE model used for structural dynamics analysis may be affected by any of three kinds of error: discretisation errors, configuration errors, and parameter errors. Model updating can only minimise the discrepancies caused by parameter errors in the model. It cannot reduce the discrepancies caused by discretisation or configuration errors without losing physical meaning of the updating parameters.

- With the inverse eigen-sensitivity method, the information contained in both eigenvalues and eigenvectors is important in the model updating procedure. Weighting factors can be applied to updating equations to balance the different effects of eigenvalues and eigenvectors on the solution of the equations.

- The concept of model verification, one of the steps in the model validation process, has been defined in the thesis. In this step, the dynamic properties of the initial FE model are examined with regard to the convergence of model predictions and the possible configuration errors of the model. A verified model is capable of predicting the dynamic properties of the structure in a given
frequency range with a certain accuracy by only modifying parameter values of the model.

- There are a number of approaches for constructing the mass matrix of each element of an FE model. Without configuration errors and parameter errors in an FE model, the dynamic properties predicted by the model will depend on the mesh size of the model and the mass matrix approach for dynamic property calculation. This feature has been used here to derive a method for a model convergence check and for estimating compensations of updating parameters for discretisation errors in a model which is going to be subjected to a model updating procedure.

- Configuration errors in an FE model may cause the model to be incapable of predicting eigenvectors which are the same as the experimental data, even when the parameter values of the model are modified. Using analytical eigenvectors from the model to curve-fit each experimental eigenvector (mode shape) and comparing the curve-fitted eigenvector with the experimental eigenvector, the configuration check method presented here is capable of detecting configuration errors in the model. When an FE model has passed the configuration error check using this method, it is possible for the model to be updated with a numerical updating program. If an FE model does not pass the check, then there are deemed to be configuration errors in the model and it is not suitable for model updating.

- With the standard numerical correlation parameter – the Modal Assurance Criterion (MAC) – the eigenvector correlation between experimental and analytical eigenvectors is, in many cases, not capable of indicating clearly the Correlated Mode Pairs (CMPs) for quasi-axisymmetric structures. For the modes in a believed-to-be-correlated mode pair, the difference between the spatial phase angle of the analytical eigenvector and that of the experimental eigenvector of the mode results in a small MAC value. A method has been presented in this thesis to rotate each analytical eigenvector by shifting its spatial phase angle in order to maximise the MAC value with the counterpart experimental eigenvector. This method can be applied not only to situations where the experimental eigenvectors are measured on equally-spaced DOFs around circumferences but also to situations where the measurement DOFs are unequally-spaced in a quasi-axisymmetric structure.

- The Effective Independence (EI) test-planning method selects the measurement DOFs in such a way that the experimental eigenvectors on the measurement DOFs are as linearly independent as possible. The method has been demonstrated here to be applicable for quasi-axisymmetric structures – the
differences between the spatial phase angles of the analytical eigenvectors used in the method and those of the eigenvectors of the actual structure will not significantly affect the linear independence of the experimental eigenvectors represented by the selected DOFs.

• The peaks in the Complex Mode Indicator Function (CMIF) curves on experimental FRF matrices indicate the existence of the modes of the test structure. By undertaking a preliminary multi-input and multi-output modal test on a quasi-axisymmetric structure, the features of the CMIF curves can be used to indicate the existence of double modes and the modes with close frequencies of the structure. Furthermore, with the information of the double modes from the CMIF curves, the excitation DOF(s) for the formal modal test on the structure can be selected in such a way as to ensure that all modes in the frequency range of interest can be excited.

• From a model updating procedure on the initial FE model of a structure, there is generally no unique solution. The assessment of the modified models from sessions in the updating procedure results in an updated model or several updated models. These updated models need to be further assessed in order to obtain a validated model of the structure. Comparing the dynamic properties of the assembly of components and those of a model assembled from updated models of the components is a practical and efficient way for assessing these updated models.

9.2 Contributions of present work

The main contributions delivered by this research are listed below.

• Performing model updating on FE models with one or two of the three kinds of error has been studied. The results from the study show that only the models with parameter errors only can be updated without losing physical meaning of the model parameters.

• The influence of both eigenvalues and eigenvectors on solutions in model updating has been examined. Using weighting factors to balance the contributions from eigenvalues and eigenvectors was explored in case studies.

• Two methods have been proposed for model verification. One method checks the convergence of an FE model and estimates parameter compensations for discretisation errors of the model. The second method checks the effects of configuration errors in an FE model on the predicted dynamic properties.

• Based on the study of model updating and model verification, the requirement on correlation and testing has been presented in the thesis.
• A method for achieving "true" numerical eigenvector correlation parameter for quasi-axisymmetric structures has been developed. The method can be applied to most situations – no matter whether the measurement DOFs are equally-spaced or unequally-spaced around the circumferences of the structure.

• The Effective Independence test-planning method has been studied for applying the method for quasi-axisymmetric structures. In the area of modal tests, the Complex Mode Indicator Function (CMIF) has been used for determining the excitation DOF(s) when there are double modes or close-frequency modes of the structure undertaken the modal test.

• The concepts of a verified model, a modified model, an updated model and a validated model have been described. The method by comparing the assembly of substructures and the model assembled from updated models of the substructures was used to assess the updated models.

• Several variations of model validation strategies have been proposed and the advantages of each of those strategies have been described. The problems we are facing at the moment when any one of these strategies is adopted have also been discussed.

9.3 Suggestion for future work

Using FE models to predict structural dynamic properties and to replace some costly experiments is becoming more and more active within industry, especially in the aero engine industry. In order to make sure that predictions from FE models have an acceptable accuracy with respect to experimental data, more and more practice of model validation is expected in the future.

The study undertaken in this thesis has covered the major steps in the model validation process. There are some encouraging results for the successful practice of validating industrial models. However, some results from the study indicate that model validation is not, at the moment, a mature tool that every engineer can make the best use of. From the study results in the thesis, some general suggestions for future work in the area of model validation are outlined below.

(1) Construction of "Super Models". A "Super model", as mentioned in the previous chapter, has a very fine mesh to describe the details of the structure. However, the mesh size of a model is not the only factor to affect the accuracy of the predictions by the model. For the practical use of "super models", other issues in the modelling process, such as representing the dynamic properties of some complicated physical features (for example, joints and contacts), needs to be studied in order to make the model to be configuration error-free.

(2) Acquisition of initial FE models of high quality. The quality of an initial
FE model is critical for the success of model validation practice on the model. When the dynamic properties predicted by the initial model is too far away from the structural properties, it is most likely that a model validation practice on the model cannot achieve satisfactory results. The construction of an FE model, at the moment, is mainly based on the analyst's experience. With the information from the "super model" that has been mentioned in the previous chapter, it is expected to develop new methods for constructing a "design model" that is small in model size but can have similar dynamic properties to the "super model" in the frequency range of interest.

(3) Validation strategy study. Several variations of model validation strategy were proposed in the previous chapter. The major procedures in the basic validation strategies (Figures 8-1 and 8-2) have been studied in this thesis. Other two strategies proposed in the previous chapter may be used for speeding up a model validation process. However, some steps in these strategies need to be further studied before the whole strategies can be applied in practical industrial cases. The key issue here is how to make the best use of the development in computer power to benefit the practice of these strategies.

(4) Model verification and updating. Model verification is introduced into the model validation process in this thesis. The method presented in Chapter 4 of the thesis for a configuration check proposes a necessary condition for an FE model to be updatable. There is no sufficient condition that has been proposed at the moment. However, further study of model verification and model updating may help to get a clearer idea about the updatability of FE models.

9.4 Closure

Although many methods have been proposed and developed in the past two decades for each procedure in the model validation process, the strategy for the process is still not mature enough to be applied routinely to industrial cases. The quality of the initial model is an important factor for the success of model validation. Unless a strategy is developed for constructing initial models that are configuration error-free and have negligible discretisation errors, the practice of model validation is likely to continue to be case-dependent.

The objective of the current research has been to develop a practical but rigorous strategy for the model validation process to be applied to industrial structures. This objective is largely achieved by the study of the main procedures of the model validation process, especially by the development of the methods for model verification.
Chapter 10

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