THE RISK OF OPTIMAL, CONTINUOUSLY REBALANCED HEDGING STRATEGIES AND ITS EFFICIENT EVALUATION VIA FOURIER TRANSFORM

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The Risk of Optimal, Continuously Rebalanced Hedging Strategies and Its Efficient Evaluation via Fourier Transform

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Abstract

This paper derives a closed-form formula for the hedging error of optimal and continuously rebalanced hedging strategies in a model with leptokurtic IID returns and, in contrast to the standard Black–Scholes result, shows that continuous hedging is far from riskless even in the absence of transaction costs. Our result can be seen as an extension of the Capital Asset Pricing Model and the Arbitrage Pricing Theory, allowing for intertemporal risk diversification.

The paper provides an efficient implementation of the optimal hedging strategy and of the hedging error formula via fast Fourier transform and demonstrates their speed and accuracy. We compute the size of hedging errors for individual options based on the historical distribution of returns on FT100 equity index as a function of moneyness and time to maturity. The resulting option price bounds are found to be non-trivial, and largely insensitive to model parameters, while the optimal hedging strategy remains virtually identical to the standard Black-Scholes delta hedge. Thus, with leptokurtic returns Black-Scholes price is the right value to hedge towards, but not the right value to price at.

Key words: hedging error, Fourier transform, mean–variance hedging, locally optimal strategy, exponential Lévy process, incomplete market, option pricing, excess kurtosis

JEL classification code: C61, C63, G12, G13

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Introduction

We examine a model where log returns are generated by a Lévy process with stationary increments, which provides a simple but flexible framework to analyze impacts of excess kurtosis on option hedging. It is in fact the only possible representation of a continuous-time model with IID returns. In this model we examine the mean–variance trade-off of (locally) optimal hedging strategy. Our paper makes an important contribution in four directions: i) we give closed-form expressions for the representative agent price, optimal delta and the dynamic Sharpe ratio of the optimal hedging strategy in the limit as the rebalancing interval goes to zero; ii) we compare, in closed form, the performance of locally optimal and dynamically optimal hedging strategies; iii) we demonstrate that the gain in computational speed afforded by the closed-form results, as compared to traditional backward recursion schemes, is highly significant and can be likened to the difference between the Black–Scholes formula and its binomial implementation; iv) we show that a calibrated model of high frequency FT100 returns yields robust and non-negligible option price bounds, while the optimal hedging strategy remains very close to the Black–Scholes delta.

Modelling of excess kurtosis in equity return data has attracted considerable attention since mid 1960s. Building on the geometric Brownian motion of Osborne (1959) and Samuelson (1965) researchers have proposed different parametric distributions for log returns: stable in Mandelbrot (1963); Brownian motion with normally distributed jump sizes at Poisson arrival times in Press (1967); Student’s t in Praetz (1972), obtainable as variance inverse gamma mixture; variance lognormal mixture in Clark (1973), variance gamma mixture in Madan and Seneta (1990), and hyperbolic distribution in Eberlein et al. (1998). All of the above are special cases of the geometric Lévy process.

Option pricing in exponential Lévy models is complicated by the fact that it is no longer possible to construct a dynamic self-financing portfolio that replicates the option. One can visualize this situation in discrete time by imagining a multinomial stock price lattice instead of the standard binomial tree. Since the hedged position is risky (synonymously, market is incomplete), it is necessary to formulate a reward-for-risk measure telling us which option prices are sensible, and which, in contrast, lead to near-arbitrage opportunities. Based on the seminal work of Von Neumann and Morgenstern (1944), it has become customary in economic literature to use utility functions for this purpose. Markowitz (1952) pioneered the use of quadratic utility in static portfolio selection, and his results were extended to dynamic setting and other utility functions by Merton (1969). Hodges and Neuberger (1989) were the first to apply dynamic optimal portfolio selection with a random endowment to option valuation. Since optimization with random endowment is notoriously difficult
to solve further simplifications ensued. Assuming that the option position is small one obtains so-called representative agent price. When log-normal returns in the classical Rubinstein’s (1976) reformulation of Black–Scholes result are replaced with log-infinitely divisible distributions one obtains so-called Esscher risk-neutral measures which permit fast pricing via characteristic function in exponential Lévy models, see Madan and Milne (1991), Gerber and Shiu (1994), Carr and Madan (1999) and Fujiwara and Miyahara (2003). Duffie and Richardson (1991), Cochrane and Saá-Requejo (2000) and Henderson (2002) give closed form solutions for non-trivial option positions in a market with continuous price processes and basis risk for quadratic, truncated quadratic and exponential utility, respectively.

Since utility levels per se are meaningless one prefers to replace them with scale-free measures of investment performance. These reward-for-risk measures are typically related to certainty equivalent rate of return normalized by the coefficient of local risk-aversion. In the case of quadratic utility this yields the classical Sharpe ratio, introduced in Sharpe (1966). Similar measures exist for other utility functions: Cochrane and Saá-Requejo (2000) use truncated quadratic utility, Bernardo and Ledoit (2000) employ Domar–Musgrave utility, Hodges (1998) derives a Generalized Sharpe ratio based on exponential utility and Černý (2003a) extends this definition to power utility and shows that price bounds in realistically calibrated models are robust across utility functions.

Among the various utility functions the quadratic utility is the most tractable. Dynamic option hedging under mean–variance criteria has been examined in great generality since the beginning of 1990s. In its simplest form with discrete time and a finite number of contingencies mean–variance hedging is a discrete Markov decision problem (MDP), with exogenous state variables including stock price, volatility and other factors as the case may be, and one endogenous state variable represented by the value of the self-financing hedging portfolio. The fully general dynamic programming solution of the discrete MDP is described in Černý (2004a). One of the remarkable features of dynamic mean-variance hedging is that the optimal hedge is an (affine) function of the endogenous state variable, which introduces path dependency absent in the Black–Scholes hedge. It is therefore interesting to examine suboptimal but purely exogenous hedging strategies, which has lead to the concept of locally optimal hedging, see Föllmer and Schweizer (1991). With infinitely many states or in continuous time the structure of the problem remains the same but the existence of a solution is no longer automatic – see Schweizer (2001) for a survey of more or less restrictive existence criteria. From the practical point of view one wishes to compute the exogenous components of the solution as functions of the exogenous state variables. In the discrete setup this is done by backward recursion, in continuous time one obtains non-linear partial difference-differential equations which are then solved by numerical methods not dissimilar to the backward recursion, see Bertsimas et al. (2001), Heath
et al. (2001), Lim (2004). The present paper and Hubalek et al. (2004) side-step the need to perform the backward recursion numerically by expressing the option pay-off as a linear combination of exponential affine terms in log stock price, and thereby obtaining all exogenous characteristics of the solution in closed form.

The paper is organized as follows. Section 1 gives a brief introduction to dynamic mean–variance hedging and discusses the properties of the locally optimal and the dynamically optimal strategies. Section 2 explains the link between the objective and the variance-optimal measures and finds an expression for the variance-optimal characteristic function of log returns in terms of the objective characteristic function. Section 3 expresses the mean value process as a Fourier transform à la Carr and Madan (1999). Section 4 derives the delta of dynamically optimal and locally optimal hedging strategies. The main theoretical result is in Section 5 which evaluates the expected squared hedging error to maturity for both strategies. Section 6 is concerned with numerical implementation of the hedging error formula and Section 7 tests the numerical accuracy and speed of proposed algorithms. Section 8 uses calibrated models of high frequency FT 100 returns to examine optimal option hedging strategies and hedging errors as a function of moneyness and time to maturity. Section 9 concludes.

A thorough mathematical treatment of the continuous-time limit is beyond the scope of the present paper, and can be found in a companion paper Černý (2003b). Alternative derivation using only continuous-time stochastic calculus is given in Hubalek et al. (2004).

1 Brief introduction to dynamic mean–variance hedging

This section aims to provide the minimal necessary background to (locally) optimal mean–variance hedging strategies and the resulting hedging errors. More detailed and more general exposition with proofs can be found in Černý (2003b, 2004a, 2004b).

Consider a trading horizon $T$ divided into $N_{\triangle t}$ trading periods of length $\triangle t = T/N_{\triangle t}$. Suppose we have a market with a stock, whose returns $R_t$ are IID, and a risk-free bank account with the same interest rate for borrowing and lending $R_{t\Delta t} = e^{r\Delta t}$. The stock bears no dividends and it generates the information filtration $\mathcal{F}_{j\Delta t}^{N_{\triangle t}}$. To avoid technicalities one can assume that $\ln R_t$ takes finitely many equidistantly spaced values, so that the stock price model is represented by a multinomial lattice.

Consider a contingent claim $H_T$ whose pay-off depends (at most) on the stock
price history up to time $T$. We wish to construct a self-financing hedging strategy with value $V_t$ and number of shares $\theta_t$ purchased at price $S_t$, where $\{\theta_t\}$ is adapted to the stock price information filtration $\mathcal{F}_{j\Delta t}^{N\Delta t}$. The initial value of the self-financing strategy $V_0$ is considered fixed for the purposes of this exposition.

The aim of the mean–variance hedge is to minimize the ex-ante expected squared hedging error to maturity by a suitable choice of hedging strategy $\theta_t$ which is allowed to depend on the stock price history up to and including time $t$:

$$\min_{\{\theta_t\}_{j=0,1,...,N\Delta t-1}} \mathbb{E}_0^P \left[ (V_T - H_T)^2 \right]$$  \hspace{1cm} (1)

$$V_{t+\Delta t} = e^{r\Delta t}V_t + \theta_t S_t X_{t+\Delta t}$$  \hspace{1cm} (2)

$$X_{t+\Delta t} \equiv R_{t+\Delta t} - e^{r\Delta t}.$$  \hspace{1cm} (3)

We commence our account with a description of a suboptimal hedging strategy — so-called locally optimal hedge\(^2\). Locally optimal hedging is important because it uses the same amount of information as the standard Black–Scholes hedge, and it turns out to be almost as good as the fully dynamically optimal hedge. The locally optimal strategy is also easier to explain to someone unfamiliar with dynamic mean–variance hedging. Finally, in models calibrated to historical equity index data the locally optimal delta is numerically very close to the Black–Scholes delta.

### A Locally optimal hedge

Suppose we fix a trading strategy $\{\theta_{j\Delta t}\}_{j=0,1,...,N\Delta t-1}$ such that it depends only on those state variables that determine the value of $H_T$. To be more specific, if $H_T$ is the pay-off of a European call option we take $\theta_t$ that depends only on $t$ and $S_t$, but it is otherwise arbitrary. Note that process $\{S_t\}$ is Markov under measure $P$ by virtue of the IID assumption imposed on log returns.

We wish to break down the total squared hedging error into more manageable portions. By virtue of the law of iterated expectations and (2) we can rewrite the objective function as follows:

\(^2\) The use of ‘locally optimal’ hedging can be traced back to Föllmer and Schweizer (1989). There is one important difference in the usage of this term: in the present paper we think of locally optimal strategy as self-financing, whereas in Föllmer and Schweizer (1989) it is not self-financing.
\[ E_P^0 \left[ (V_T - H_T)^2 \right] \]
\[ = E_P^0 \left[ E_{T-\Delta t}^P \left[ (V_T - H_T)^2 \right] \right] \]
\[ = E_P^0 \left[ E_{T-\Delta t}^P \left[ (e^{r\Delta t}V_{T-1} + \theta_{T-\Delta t}S_{T-\Delta t}X_T - H_T)^2 \right] \right] \]
\[ = E_P^0 \left[ E_{T-\Delta t}^P \left[ (e^{r\Delta t} (V_{T-\Delta t} - H_T^0) \right. \right. \]
\[ + \ e^{r\Delta t} H^\theta_{T-\Delta t} + \theta_{T-\Delta t}S_{T-\Delta t}X_T - H_T)^2 \right] \]

where \( H^\theta_{T-\Delta t} \) is as yet unspecified, but \( \mathcal{F}_{T-\Delta t} \)-measurable. The term

\[ e^{r\Delta t} (V_{T-\Delta t} - H^\theta_{T-\Delta t}) \]

is \( \mathcal{F}_{T-\Delta t} \)-measurable, consequently if we define \( e^{r\Delta t} H^\theta_{T-\Delta t} \) by postulating

\[ E_{T-\Delta t}^P \left[ e^{r\Delta t} H^\theta_{T-\Delta t} + \theta_{T-\Delta t}S_{T-\Delta t}X_T - H_T \right] = 0, \quad (7) \]

then the conditional expectation in (6) simplifies very conveniently to

\[ e^{2r\Delta t} \left( V_{T-\Delta t} - H^\theta_{T-\Delta t} \right)^2 + E_{T-\Delta t}^P \left[ \left( e^{r\Delta t} H^\theta_{T-\Delta t} + \theta_{T-\Delta t}S_{T-\Delta t}X_T - H_T \right)^2 \right]. \]

By its definition (7) and thanks to the Markov property of stock prices under measure \( P \) the value \( H^\theta_{T-\Delta t} \) will only depend on those state variables that govern \( \theta_{T-\Delta t} \); in the case of a call option \( H^\theta_{T-\Delta t} \) will only depend on \( S_{T-\Delta t} \).

One can now take the expression \( (V_{T-\Delta t} - H^\theta_{T-\Delta t})^2 \) and perform the same decomposition as in (4)-(6), obtaining a value of \( H^\theta_{T-2\Delta t} \) as a result. This yields a recursive definition of the mean value process \( H^\theta \):

\[ E_t^P \left[ e^{r\Delta t} H^\theta_t + \theta_tS_tX_{t+\Delta t} - H^\theta_{t+\Delta t} \right] = 0 \]
\[ H^\theta_t \equiv H_T. \quad (8) \]

On defining the one-step-ahead expected squared hedging error \( \gamma \) by setting

\[ \gamma_t(H^\theta_{t+\Delta t}) \equiv E_t^P \left[ \left( e^{r\Delta t} H^\theta_t + \theta_tS_tX_{t+\Delta t} - H^\theta_{t+\Delta t} \right)^2 \right], \quad (9) \]

the total squared hedging error to maturity simplifies to:

\[ E_0^P \left[ (V_T - H_T)^2 \right] = e^{2rT} (V_0 - H_0^\theta)^2 \]
\[ + E_0^P \left[ \sum_{j=0}^{N_{\Delta t} - 1} e^{2r\Delta t(N_{\Delta t} - j)} \gamma_j(H^\theta_{(j+1)\Delta t}) \right]. \quad (10) \]
Crucially, by construction $\gamma_t$ only depends on $S_t$ and $t$. The decomposition of unconditional squared hedging error in (11) is very similar to the analysis in Toft (1996).

Up to now the trading strategy $\theta$ has been arbitrary, apart from the restriction on the amount of information it was permitted to use. With $H^\theta_T$ given by (9) it is now natural to select $\theta_{T-\Delta t}^L$ so as to minimize the one-step ahead squared hedging error $\gamma_{T-\Delta t}(H^\theta_T)$. With the optimal value of $\theta_{T-\Delta t}$ in hand we can find $H^\theta_{T-\Delta t}$ from (8) and then find $\theta_{T-2\Delta t}$ by minimizing $\gamma_{T-2\Delta t}(H^\theta_{T-\Delta t})$, etc. The trading strategy $\{\theta\}$ calculated in this way is called the \textit{locally optimal hedging strategy} and it is denoted by $\{\theta^L\}$. To simplify notation we will denote $H^\theta_T$ simply by $H$.

The minimization of (10) is tantamount to a least squares regression, therefore the locally optimal hedge is computed as a slope coefficient

$$\theta^L_t S_t = \frac{\text{Cov}_t^P(H_{t+\Delta t}, X_{t+\Delta t})}{\text{Var}_t^P(X_{t+\Delta t})}, \quad (12)$$

the intercept is recovered from (8)

$$H_t = E_t^P[H_{t+\Delta t} - \theta^L_t S_t X_{t+\Delta t}] / e^{r\Delta t}, \quad (13)$$

and the minimal variance of the hedging error takes the value

$$\gamma_t(H_{t+\Delta t}) = \text{Var}_t^P(H_{t+\Delta t}) - \frac{(\text{Cov}_t^P(H_{t+\Delta t}, X_{t+\Delta t}))^2}{\text{Var}_t^P(X_{t+\Delta t})}. \quad (14)$$

\textit{B The mean value process as a risk-neutral expectation}

Equation (13) is the familiar Markowitz CAPM asset pricing formula, as can be easily seen by substituting (12) into (13). It can be rephrased equivalently\footnote{The results (15)-(17) can be obtained either by tedious algebra, or much more easily by exploiting an important property of orthogonal projection known in econometrics as the Frisch-Waugh-Lovell theorem (Davidson and MacKinnon 1993, pp. 19-23). The FWL theorem allows one to compute $H_t$ from a modified least squares regression in which $X_{t+\Delta t}$ is removed completely and the intercept and the dependent variable $H_{t+\Delta t}$ are replaced by the respective residuals from their regression on $X_{t+\Delta t}$.
}
using a conditional change of measure $m_{t+\Delta t|t}$:

$$H_t = E_t^P \left[ m_{t+\Delta t|t} H_{t+\Delta t} \right] / R_t, \quad (15)$$

$$m_{t+\Delta t|t} \equiv (1 - a_{\Delta t} X_{t+\Delta t}) / b_{\Delta t}, \quad (16)$$

$$a_{\Delta t} \equiv E_t^P [X_{t+\Delta t}] / E_t^P [X_{t+\Delta t}^2], \quad b_{\Delta t} \equiv 1 - \left( E_t^P [X_{t+\Delta t}] \right)^2 / E_t^P [X_{t+\Delta t}^2]. \quad (17)$$

In view of (15) it is natural to define a risk-neutral measure $Q_{\Delta t}$ by setting

$$E_t^{Q_{\Delta t}} [Z] \equiv E_t^P [m_{t+\Delta t|t} \times \ldots \times m_{T|T-\Delta t} Z], \quad (18)$$

whereby we can write

$$H_t = E_t^{Q_{\Delta t}} [H_{t+\Delta t}] / e^{r_{\Delta t}} = E_t^{Q_{\Delta t}} [H_T] / e^{r(T-t)}. \quad (19)$$

The risk-neutral measure $Q_{\Delta t}$ is known in the literature as the minimal martingale measure and in the special case with IID returns and constant interest rate it coincides with the so-called variance-optimal measure. Note that $Q_{\Delta t}$ is, in general, a signed measure.

### C Dynamically optimal hedging strategy

The locally optimal strategy effectively picks $\theta_t$ assuming that $V_t = H_t$. In an incomplete market this is not always the case and the dynamically optimal strategy $\theta_t^D$ therefore adjusts for the surplus/shortfall $(V_t - H_t)$,

$$S_t \theta_t^D = S_t \theta_t^L - e^{r_{\Delta t}} a_{\Delta t} (V_t^D - H_t), \quad (19)$$

with $a_{\Delta t}$ defined in (17). The decomposition of the dynamically optimal hedging error works similarly as in (11), namely

$$E_0^P \left[ (V_T^D - H_T)^2 \right] = e^{2rT} b_{\Delta t}^{N_{\Delta t}} (V_0 - H_0)^2$$

$$+ E_0^P \left[ \sum_{t=0}^{T-1} \left( b_{\Delta t} e^{2r_{\Delta t}} \right)^{N_{\Delta t} - j} \gamma_{j\Delta t} (H_{(j+1)\Delta t}) \right], \quad (20)$$

where $b_{\Delta t}$ is given in equation (17). Since $b_{\Delta t}$ is very close to 1, comparison of the locally optimal error (11) and the dynamically optimal error (20) reveals

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4 The dynamically optimal strategy is obtained from the least squares regression $\min_{\theta_t} E_t^P \left[ (e^{r_{\Delta t}} V_t^D + \theta_t S_t X_{t+\Delta t} - H_{t+\Delta t})^2 \right]$, this time without intercept. For detailed derivation of (19) and (20) see Černý (2004b), Section 12.4.7.
that one does not lose all that much by following the former (suboptimal) hedging strategy.

2 Distribution of stock returns under objective and risk-neutral measure

A Objective probability

In the continuous-time limit of a model with IID log returns the distribution of log returns must be infinitely divisible. The Lévy–Khintchin representation of the characteristic function of infinitely divisible distributions reads

\[ \phi_T^P(v) \equiv \mathbb{E}^P \left[ e^{iv \ln(S_T/S_0)} \right] = \exp \left( \left( iv \mu_1^P - \frac{\sigma_1^P v^2}{2} \right) + \int_{\mathbb{R}} \left( e^{ivx} - 1 - ivh(x) \right) M^P(dx) \right) T, \tag{21} \]

\[ h(x) \equiv \begin{cases} x \text{ for } |x| \leq 1 \\ 0 \text{ for } |x| > 1 \end{cases}. \]

To visualize equation (21) one should imagine that \( \ln S_t \) follows a Brownian motion process with jumps, where \( \sigma_1^P \) is the volatility of the Brownian motion component, jumps of size \([x, x + \Delta x]\) are compensated by a deterministic drift \(-h(x)dt\) and they keep arriving with intensity \(f_{x+\Delta x} M^P(dx)\) per unit of time. Notice that only small jumps are compensated, and if the intensity of small jumps is small (\(\int_{\mathbb{R}} h(x) M^P(dx) < \infty\)) then the compensation can be left out completely; its contribution will be absorbed in the drift term \(\mu_1^P\).

For example, in the simplest case of a compound Poisson process with \(n\) jump sizes \(x_1, \ldots, x_n\) arriving with intensities \(\lambda_1^P, \ldots, \lambda_n^P\) the characteristic function (21) simplifies to

\[ \phi_T^P(v) = \exp \left( T \left( iv \mu_1^P + \sum_{j=1}^{n} (e^{ivx_j} - 1) \lambda_j^P \right) \right), \tag{22} \]

where \(\mu_1^P\) represents the growth rate of log returns in the absence of jumps.

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5 For a mathematically rigorous treatment of the ‘continuous-time limit’ refer to Černý (2003b).


7 One requires \(\int_{\mathbb{R}} \max(1, x^2) M^P(dx) < \infty\).
We denote the cumulant function for unit time horizon by $\kappa^P$

$$\kappa^P(v) \equiv \mu_1^P v + \frac{\sigma_1^P}{2} v^2 + \int_{\mathbb{R}} (e^{vx} - 1 - vh(x)) M^P(dx),$$

and we will make a standing assumption that the variance of return is finite, requiring $\kappa^P(2) < \infty$. This allows us to define the measure-$P$ variance of the pure jump part of return (per unit of time):

$$\sigma^2_{1P} \equiv \int_{\mathbb{R}} (e^x - 1)^2 M^P(dx),$$

and the total variance including the Brownian motion part:

$$\sigma^2_P \equiv \sigma^2_{1P} + \sigma^2_{2P} = \kappa^P(2) - 2 \kappa^P(1).$$

### B Risk-neutral probability

Combining equations (18) and (21) it is now possible to evaluate the variance-optimal characteristic function of log returns (see Appendix A for details):

$$\phi^Q_T(v) \equiv \lim_{\Delta t \to 0} \mathbb{E}^Q_{\Delta t} \left[ e^{iv \ln(S_T/S_0)} \right]$$

$$= \lim_{\Delta t \to 0} \left( \mathbb{E}^P_t \left[ \left( 1 - a_{\Delta t} \left( \frac{S_{t+\Delta t}}{S_t} - e^{r_{\Delta t}} \right) \right) e^{iv \ln(S_{t+\Delta t}/S_t) / b_{\Delta t}} \right] \right)^{N_{\Delta t}}$$

$$= e^{\kappa^Q (iv) T},$$

(23)

where the risk-neutral cumulant function $\kappa^Q$ is given by

$$\kappa^Q(v) = \kappa^P(v) + \tilde{a} \left( \kappa^P(v) + \kappa^P(1) - \kappa^P(v + 1) \right),$$

(24)

$$\tilde{a} \equiv \lim_{\Delta t \to 0} a_{\Delta t} = \frac{\kappa^P(1) - r}{\kappa^P(2) - 2 \kappa^P(1)} = \frac{\kappa^P(1) - r}{\sigma_P^2}.$$  

(25)

The variance-optimal distribution of log returns is again infinitely divisible $^8$; in terms of its Lévy measure we have:

$^8$ As long as $(1 + \tilde{a}(1 - e^x)) M^P(dx) \geq 0$ the expression $\max(1, x^2) M^Q(dx)$ is a finite measure on $\mathbb{R}$ and it defines a risk-neutral density of log returns. In all other cases $M^Q$ is a signed measure and the corresponding density does not exist. One can, however, in both cases use the expression (23) for the risk-neutral pseudo-characteristic function $\phi^Q_T(v)$. 

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\[ \kappa^Q(v) \equiv \mu_{1Q} v + \frac{\sigma_{1Q}^2}{2} + \int_{\mathbb{R}} (e^{vx} - 1 - vh(x)) M^Q(dx), \]
\[ \sigma_{1Q}^2 = \sigma_{1P}^2, \]  
\[ M^Q(dx) = (1 + \tilde{a} (1 - e^x)) M^P(dx), \]  
\[ \mu_{1Q} = \mu_{1P} - \tilde{a} \sigma_{1P}^2 + \tilde{a} \int h(x) (1 - e^x) M^P(dx). \]  

The results (26)-(28) make sense intuitively: (26) means that the volatility of the Brownian motion remains unaffected by the change of measure; all that changes are the arrival intensities of individual jumps in (27). Equation (16) gives us a clue how these intensities have to change: on a short time horizon a jump \( x \) in log return gives rise to excess return of size \( X_t + \Delta t \approx e^x - 1 \); in addition \( b_{\Delta t} \approx 1 \) and \( a_{\Delta t} \approx \tilde{a} \), with \( \tilde{a} \) given by equation (25). Therefore (16) becomes
\[ m_{t+\Delta t|t} = \frac{1 - a_{\Delta t} X_{t+\Delta t}}{b_{\Delta t}} \approx 1 - \tilde{a} (e^x - 1) = \frac{M^Q(dx)}{M^P(dx)}, \]
which is exactly (27). Finally, (28) makes sure that \( \kappa^Q(1) = r \), implying that the risk-neutral mean of the stock return is equal to the risk-free return, as required by the risk-neutral pricing formula.

### 3 Mean value process as a Fourier transform

The main idea of this paper is very simple: express the option pay-off \( H_T \) as a sum of terms which are exponential affine in \( \ln S_T \) and then calculate the risk-neutral expectation of \( H_T \) using the variance-optimal characteristic function of \( \ln S_T \), which is known from (23) and (24).

The decomposition of \( H_T \) into exponential affine terms in \( \ln S_T \) is accomplished by means of a Fourier transform of the option pay-off,
\[ H_T(\ln S_T) \equiv \max(0, e^{\ln S_T} - e^k), \]
where \( k \) is the log strike price. One has to overcome a technical obstacle at this stage because \( H_T(\ln S_T) \) is not integrable with respect to \( \ln S_T \). A simple remedy is suggested in Carr and Madan (1999): by assumption there is \( \alpha > 0 \) such that \( \mathbb{E}^Q [S_T^{1+\alpha}] \) is finite. We can therefore ‘borrow’ a factor \( S_T^{-\alpha-1}, \alpha > 0 \) from the distribution function and apply the Fourier transform to the modified payoff \( H_T(\ln S_T) S_T^{-\alpha-1} \) which is now integrable.\(^9\)

\(^9\) It is possible to weaken the integrability condition further by the following thought.
Following the notation in Carr and Madan (1999) we have

\[ H_T(s)e^{-(\alpha+1)s} = \int_{-\infty}^{+\infty} \psi(v)e^{ivs}dv, \]

where from the inverse Fourier transform we obtain:

\[ \psi(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H_T(s)e^{-(\alpha+1)s}e^{-ivs}ds \]
\[ = \frac{1}{2\pi} \int_{k}^{+\infty} (e^s - e^k) e^{-(\alpha+1)s}e^{-ivs}ds \]
\[ = \frac{e^{-(\alpha+iv)k}}{2\pi (\alpha + iv)(\alpha + 1 + iv)}. \]

Recalling the definition of the variance-optimal characteristic function (23) we can write

\[ e^{rT}H_0(\ln S_0) = E_0^Q [H_T (\ln S_0 + \ln (S_T/S_0))] \]
\[ = E_0^Q \left[ \int_{-\infty}^{+\infty} \psi(v)e^{(iv+\alpha+1)(\ln S_0 + \ln(S_T/S_0))}dv \right] \]
\[ = \int_{-\infty}^{+\infty} e^{(iv+\alpha+1)\ln S_0}\psi(v)e^{vQ(\alpha+1)T}dv. \quad (29) \]

In conclusion, the mean value process is effectively obtained by a generalized Fourier transform along the line \( \alpha+1+i\mathbb{R} \). To simplify notation let us introduce the following substitutions:

\[ \tilde{v} \equiv v - i(\alpha + 1), \quad (30) \]
\[ \tilde{\psi}(\tilde{v}) \equiv \frac{e^{-(i\tilde{v}-1)k}}{2\pi iv(i\tilde{v} - 1)}. \quad (31) \]
\[ \tilde{\psi}_T(\tilde{v}) \equiv e^{T(\kappa^Q i\tilde{v} - r)}\tilde{\psi}(\tilde{v}), \quad (32) \]

whereby from (29) we obtain

\[ H_t(\ln S_t) = \int_G \tilde{\psi}_{T-t}(\tilde{v})e^{i\tilde{v}\ln S_t}d\tilde{v}, \quad (33) \]
\[ G \equiv \mathbb{R} - i(\alpha + 1). \quad (34) \]

experiment. Suppose the issuer of the call option immediately goes long one share and hedges the modified pay-off \( H_T - S_T \). This operation will shift the optimal delta by \(-1\) and leave the hedging error intact. The modified pay-off, that of short put option, is bounded and hence only needs to be multiplied by the factor \( S_T^{-\alpha} \). This in turn imposes weaker condition on the existence of moments, namely one only needs \( \kappa^P(2) < \infty \) as opposed to \( \kappa^P(2 + 2\alpha) < \infty. \)
4 Optimal delta in continuous time

In continuous time the dynamically optimal delta (19) becomes

$$\theta^D_t = \theta^L_t + \tilde{a} \frac{H_t - V_t}{S_t},$$

where $V_t$ is the value of self-financing hedging portfolio, $\tilde{a}$ is given in (25), and the locally optimal delta is recovered from

$$\theta^L_t = \lim_{\Delta t \to 0} \frac{\text{Cov}_t(\Delta H_t, \Delta S_t)}{\text{Var}_t(\Delta S_t)},$$

in analogy to (12).

To evaluate the covariance in (35) it suffices to write down the jump-diffusion Itô formula for processes $H$ and $S$,

$$dH_t = (\cdot)dt + H'_t(\ln S_t) \sigma_{1P} dB^P_t + H_t(\ln S_t_+ + x) - H_t(\ln S_t_-),$$

$$dS_t = e^{\ln S_t}(\cdot)dt + e^{\ln S_t} \sigma_{1P} dB^P_t + e^{\ln S_t_+ + x} - e^{\ln S_t_-},$$

where the arrival intensity of jumps\footnote{To simplify notation $x$ in (36) and (37) represents random variable 'jump in $\ln S_t$ in period $(t, t + dt]$', whereas in (38) $x$ represents the realization of the jump.} of size $[x, x + dx]$ is described by the Lévy measure $M^P(dx)$ as explained in Section 2A. Since the Brownian increment and the Poisson jumps of different sizes are uncorrelated, standard rules for variance and covariance applied to (36) and (37) yield:

$$\lim_{\Delta t \to 0} \frac{\text{Cov}_t(\Delta H_t, \Delta S_t)}{S_t \Delta t} = H'_t(\ln S_t) \sigma^2_{1P}$$

$$+ \int_{\mathbb{R}} (e^x - 1) \left( H_t(\ln S_t + x) - H_t(\ln S_t) \right) M^P(dx),$$

$$\lim_{\Delta t \to 0} \frac{\text{Var}_t(\Delta S_t)}{\Delta t} = \sigma^2_{1P} S_t^2.$$

It is now a simple matter to substitute (33) into (38) and to perform the necessary differentiation to obtain:

$$\lim_{\Delta t \to 0} \text{Cov}_t(\Delta H_t, \Delta S_t) / \Delta t = S_t \int_G \tilde{\psi}_{T-t}(\tilde{v}) e^{i\tilde{v} \ln S_t} A(\tilde{v}) d\tilde{v},$$

$$A(\tilde{v}) \equiv \sigma^2_{1P} i\tilde{v} + \int_{\mathbb{R}} (e^{ix} - 1) (e^x - 1) M^P(dx).$$

13
On substituting (38) into (35) we obtain a closed-form expression for the locally optimal delta:

\[ S_t \theta_t^L = \int_G \tilde{\psi}_{T-t}(\tilde{v}) e^{i\tilde{v} \ln S_t} A(\tilde{v})/\sigma_P^2 d\tilde{v}. \]  

(42)

For computer implementation it is convenient to express the function \( A \) in terms of the objective cumulant function:

\[ A(\tilde{v}) = \kappa_P(i\tilde{v} + 1) - \kappa_P(i\tilde{v}) - \kappa_P(1). \]  

(43)

In the absence of jumps \( (M^P = 0) \) equations (35), (38) and (39) yield the standard Black–Scholes delta:

\[ \theta_t^L = \frac{H_t(\ln S_t)}{S_t} = \frac{dH(\ln S_t)}{dS_t}. \]

5 Optimal hedging error in continuous time

The most interesting part of the analysis deals with the quantification of the hedging error. To this end we define the expected squared hedging error of perfectly balanced locally optimal and dynamically optimal strategies:

\[ \varepsilon_{tL}^2 \equiv E_t^P \left[ (V_t^{\theta_L} - H_T)^2 \bigg| V_t^{\theta_L} = H_t \right], \]  

(44)

\[ \varepsilon_{tD}^2 \equiv E_t^P \left[ (V_t^{\theta_D} - H_T)^2 \bigg| V_t^{\theta_D} = H_t \right]. \]  

(45)

By construction \( \varepsilon_{tL}^2 \) and \( \varepsilon_{tD}^2 \) depend only on calendar time and the current stock price.

A The importance of hedging errors for asset pricing

Section 12.2.1 of Černý (2004b) shows that by selling the contingent claim \( H_T \) at the price \( C_0 > H_0 \) and then hedging it dynamically to maturity the trader enters into a risky position with the dynamic Sharpe ratio of
\[ \text{SR}_L = \frac{(C_0 - H_0) e^{rT}}{\varepsilon_{0L}}, \quad (46) \]

\[ \text{SR}_D = \left( \frac{e^{bT} - 1 + (C_0 - H_0) e^{rT}/\varepsilon_{0D}}{\text{basis SR}^2} \right)^{\text{option SR}^2}, \quad (47) \]

\[ \hat{b} = \lim_{\Delta t \to 0} \frac{b_{\Delta t} - 1}{\Delta t} = \left( \kappa^p (1 - r)^2 \right) / \sigma_p^2, \quad (48) \]

where \( \hat{b} \) can be interpreted as the square of the instantaneous Sharpe ratio of excess log return. The corresponding optimal size of the option contract as a proportion of the trader’s initial risk-free wealth is

\[ \alpha_L = \frac{1}{\tilde{\gamma} \left( \frac{\varepsilon_{0L}}{e^{rT} C_0} \right)^2 + \left( 1 - \frac{H_0}{C_0} \right)^2}, \]

\[ \alpha_D = \frac{1}{\tilde{\gamma} e^{bT} \left( \frac{\varepsilon_{0D}}{e^{rT} C_0} \right)^2 + \left( 1 - \frac{H_0}{C_0} \right)^2}, \]

where \( \tilde{\gamma} \) is the trader’s coefficient of local relative risk aversion.

It is obvious that the size of the hedging error \( \varepsilon_{0L}, \varepsilon_{0D} \) is crucial in determining the attractiveness of a given option deal. Note that in the standard Black–Scholes model \( \varepsilon_{0L} = \varepsilon_{0D} = 0 \) and the option must be traded at the mean value \( H_0 \). In a model where \( \varepsilon_{0D} > 0 \) the trader will only agree to sell the option if \( C_0 > H_0 \) and the size of the premium \( C_0 - H_0 \) will increase with the size of the option trade.

One can invert the relationship between the option price \( C_0 \) and the attainable Sharpe ratio \( (46, 47) \) to obtain so-called good-deal price bounds. Although it is theoretically possible for good-deal bounds based on Sharpe ratio to fall outside the no-arbitrage bounds (Cochrane and Saá-Requejo 2000), it turns out that for sensible values of Sharpe ratio (SR \( \leq 1 \)) and for options close to the money the Sharpe ratio price bounds are arbitrage-free; see Section 8B for numerical results. It is feasible to obtain tighter price bounds for deep OTM options using an exponential utility function and Hodges’ (1998) Generalized Sharpe ratio, but such an exercise is beyond the scope of the present paper.

B Evaluation of hedging errors

Let us reiterate at this stage that the derivations in this paper are informal and that mathematically rigorous derivations of the results reported here can
be found in Černý (2003b). The continuous-time limits of (11), (20), (44) and (45) read

\[ \varepsilon_{0L}^2 = \int_0^T e^{2r(T-t)}E_0^P [\tilde{\gamma}_t] \, dt, \]
\[ \varepsilon_{0D}^2 = \int_0^T e^{(2r-\tilde{b})(T-t)}E_0^P [\tilde{\gamma}_t] \, dt, \]
\[ \tilde{\gamma}_t \equiv \lim_{\Delta t \to 0} \gamma_t \left( H_{t+\Delta t} \right) / \Delta t. \]

The crux of the matter lies in the expression \( \gamma_t \left( H_{t+\Delta t} \right) \) which represents the squared hedging error of a perfectly aligned replicating portfolio (one where \( V_t = H_t \)). From (14) we obtain:

\[ \gamma_t \left( H_{t+\Delta t} \right) = \text{Var}_t^P \left( H_{t+\Delta t} \right) - \frac{\left( \text{Cov}_t^P \left( H_{t+\Delta t}, S_{t+\Delta t} \right) \right)^2}{\text{Var}_t^P \left( S_{t+\Delta t} \right)} = \text{Var}_t^P \left( \Delta H_t \right) - \frac{\left( \text{Cov}_t^P \left( \Delta H_t, \Delta S_t \right) \right)^2}{\text{Var}_t^P \left( \Delta S_t \right)}. \]

We have dealt with the covariance term, \( \text{Cov}_t^P \left( \Delta H_t, \Delta S_t \right) / \Delta t \), in equation (38). The computation of \( \text{Var}_t^P \left( \Delta H_t \right) / \Delta t \) proceeds similarly, making use of the Itô formula (36) and exploiting uncorrelatedness of the Brownian component and the individual jump components:

\[ \lim_{\Delta t \to 0} \text{Var}_t^P \left( \Delta H_t \right) / \Delta t = (H_t'(\ln S_t))^2 \sigma_{1P}^2 \]

\[ + \int_{\mathbb{R}} (H_t'(\ln S_t + x) - H_t'(\ln S_t))^2 M^P(dx). \]

Again we shall substitute for \( H_t \) from (33) and perform the necessary differentiation to obtain

\[ \lim_{\Delta t \to 0} \text{Var}_t^P \left( H_{t+\Delta t} \right) / \Delta t \]
\[ = \int_{G^2} \tilde{\psi}_{T-t} (\tilde{v}_1, \tilde{v}_2) \tilde{\psi}_{T-t} (\tilde{v}_1, \tilde{v}_2) e^{i(\tilde{v}_1 - \tilde{v}_2) \ln S_t} B(\tilde{v}_1, \tilde{v}_2) d\tilde{v}_1 d\tilde{v}_2, \]

\[ \tilde{B}(\tilde{v}_1, \tilde{v}_2) \equiv -\tilde{v}_1 \tilde{v}_2 \sigma_{1P}^2 + \int_{\mathbb{R}} \left( e^{i\tilde{v}_1 x} - 1 \right) \left( e^{i\tilde{v}_2 x} - 1 \right) M^P(dx) \]
\[ = \kappa^P (i\tilde{v}_1 + i\tilde{v}_2) - \kappa^P (i\tilde{v}_1) - \kappa^P (i\tilde{v}_2). \]
Combining expressions (39)-(41), and (51)-(54) we obtain the instantaneous expected squared hedging error as a two-dimensional Fourier transform:

$$
\tilde{\gamma}_t = \int_{G^2} \tilde{\psi}_T(t) \tilde{\psi}_T(t) e^{i(\tilde{v}_1 + i\tilde{v}_2) \ln S_t} \left( B(\tilde{v}_1, \tilde{v}_2) - \frac{A(\tilde{v}_1)A(\tilde{v}_2)}{\sigma_P^2} \right) d\tilde{v}_1 d\tilde{v}_2.
$$

Finally, we can evaluate the unconditional expectation in (49) and (50) using the definition of the objective characteristic function, $E_0^P \left[ e^{i(\tilde{v}_1 + i\tilde{v}_2) \ln S_t} \right] = \phi_t^P (\tilde{v}_1 + \tilde{v}_2) = e^{\kappa P (i\tilde{v}_1 + i\tilde{v}_2)t}$.

$$
E_0^P [\gamma_t] = \int_{G^2} \tilde{\psi}_T(t) \tilde{\psi}_T(t) e^{\kappa P (i\tilde{v}_1 + i\tilde{v}_2)t} \times e^{i(\tilde{v}_1 + i\tilde{v}_2) \ln S_0} \left( B(\tilde{v}_1, \tilde{v}_2) - \frac{A(\tilde{v}_1)A(\tilde{v}_2)}{\sigma_P^2} \right) d\tilde{v}_1 d\tilde{v}_2. \quad (56)
$$

Substituting for $\tilde{\psi}$ from (32) into (56) the total hedging error (50) of the locally optimal strategy equals

$$
\varepsilon_0^2 L = S_0^2 \int_0^T dt \int_{G^2} \left( \prod_{j=1,2} e^{\kappa^P (i\tilde{v}_j)T + (i\tilde{v}_j - 1) \ln S_0 \psi(\tilde{v}_j)} \right) e^{C(\tilde{v}_1, \tilde{v}_2)t} \times \left( B(\tilde{v}_1, \tilde{v}_2) - \frac{A(\tilde{v}_1)A(\tilde{v}_2)}{\sigma_P^2} \right) d\tilde{v}_1 d\tilde{v}_2,
$$

where

$$
C(\tilde{v}_1, \tilde{v}_2) \equiv \kappa^P (i\tilde{v}_1 + i\tilde{v}_2) - \kappa^Q (i\tilde{v}_1) - \kappa^Q (i\tilde{v}_2) = B(\tilde{v}_1, \tilde{v}_2) + \tilde{a} (A(\tilde{v}_1) + A(\tilde{v}_2)).
$$

On integrating over time we find:

$$
\varepsilon_0^2 L = S_0^2 \int_{G^2} \left( \prod_{j=1,2} e^{\kappa^Q (i\tilde{v}_j)T + (i\tilde{v}_j - 1) \ln S_0 \psi(\tilde{v}_j)} \right) \frac{e^{C(\tilde{v}_1, \tilde{v}_2)T} - 1}{C(\tilde{v}_1, \tilde{v}_2)} \times \left( B(\tilde{v}_1, \tilde{v}_2) - \frac{A(\tilde{v}_1)A(\tilde{v}_2)}{\sigma_P^2} \right) d\tilde{v}_1 d\tilde{v}_2. \quad (57)
$$

The calculation of dynamically optimal error proceeds similarly with an extra factor $e^{-b(T-t)}$ in the integrand:
Formulae (57) and (58) apply to an arbitrary contingent claim provided that the function \( \psi(\tilde{v}) \) is modified appropriately to represent Fourier transform coefficients of the contingent claim pay-off. One can easily verify that the expression

\[
B(\tilde{v}_1, \tilde{v}_2) - \frac{A(\tilde{v}_1)A(\tilde{v}_2)}{\sigma_P^2}
\]

vanishes either when \( M^P = 0 \) (Black-Scholes model), or in the pure jump case with one jump size. This result is to be expected since both cases are obtainable as a limit of a binomial model which has zero hedging error by construction.

6 Numerical computation of hedging error

The numerical implementation of the one-dimensional Fourier expressions for the mean value and optimal delta (33, 42) is well understood because similar formulae have appeared in pricing with exponential Lévy processes or in stochastic volatility models, see Heston (1993), Carr and Madan (1999), and Lee (2004). We will therefore focus our efforts on the evaluation of formulae (57, 58) which were previously unavailable in the literature. Without loss of generality we will concentrate on (58).

Let us denote the integrand in (58) by \( f \):

\[
f(v_1, v_2) \equiv S_0^2e^{-\tilde{b}T} \left( \prod_{j=1,2} e^{\alpha Q(i\tilde{v}_j)T + (i\tilde{v}_j - 1) \ln S_0} \psi(\tilde{v}_j) \right) e^{(C(\tilde{v}_1, \tilde{v}_2) + \tilde{b})T - 1} \frac{1}{C(\tilde{v}_1, \tilde{v}_2) + b} \times \left( B(\tilde{v}_1, \tilde{v}_2) - \frac{A(\tilde{v}_1)A(\tilde{v}_2)}{\sigma_P^2} \right) d\tilde{v}_1d\tilde{v}_2, \]

\( \tilde{v}_i = v_i - i(\alpha + 1) \).

We wish to evaluate the integral

\[
I \equiv \int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_2 f(v_1, v_2) = 4 \Re \int_{0}^{\infty} dv_1 \int_{-v_1}^{v_1} dv_2 f(v_1, v_2),
\]

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\]
where the second equality follows from integrand’s symmetry:

$$f(v_1, v_2) = f(v_2, v_1) = f(-v_1, -v_2) = f(-v_2, -v_1).$$

Truncating the outer integral at $v_{\text{max}}$ we obtain an expression that can be approximated by numerical quadrature:

$$I(v_{\text{max}}) \equiv 4 \Re \int_0^{v_{\text{max}}} dv_1 \int_{-v_1}^{v_1} dv_2 f(v_1, v_2).$$  \hspace{1cm} (59)$$

\textbf{A Algorithm for a single strike}

The integral (59) is most easily evaluated by a sequential application of suitably chosen Newton-Cotes formula, first running along $v_2$ axis and then along $v_1$ axis as indicated in Figure 1,

$$\int_0^{v_{\text{max}}} dv_1 \int_{-v_1}^{v_1} dv_2 f(v_1, v_2) \approx (\Delta v)^2 \sum_{j=0}^{n-1} \sum_{l=-j}^j w_{jl} f(j \Delta v, l \Delta v) \equiv S(n, v_{\text{max}})(60)$$

$$\Delta v \equiv \frac{v_{\text{max}}}{n - 1}.$$ 

Here $w_{jl}$ are integration weights; for example the trapezoidal rule gives

$$w_{jl} = \tilde{w}_j \times \tilde{w}_{jl}$$

$$\tilde{w}_0 = \tilde{w}_{n-1} = \frac{1}{2}, \quad \tilde{w}_j = 1, \quad 0 < j < n - 1$$

$$\tilde{w}_{-j} = \tilde{w}_{j}, \quad \tilde{w}_{jl} = 1, \quad -j < l < j,$$
whereas the Simpson’s 1/3 rule yields

\[ w_{jl} = \tilde{w}_j \times \tilde{w}_{jl} \]

\[ \tilde{w}_0 = \tilde{w}_{n-1} = \frac{1}{3}, \quad \tilde{w}_j = \frac{3 + (-1)^{j+1}}{3}, \quad 0 < j < n - 1 \]

\[ \tilde{w}_{j-l} = \tilde{w}_{j-l}, \quad -j < l < j. \]

A particularly pleasing feature of these schemes is apparent monotonicity of \( S_n(v_{\text{max}}) \) as a function of \( n \) for fixed \( v_{\text{max}} \). This permits further acceleration of the algorithm by extrapolating \( S_n(v_{\text{max}}) \) in powers of \( 1/n \).

We have experimented with other numerical schemes. One can get around the need to specify \( v_{\text{max}} \) by means of the transformation \( y(v) = \frac{v}{v+1} \) which maps \( \mathbb{R}_+ \) to \( (0,1] \) and makes the integration area finite while preserving boundedness of the integrand. However, we did not observe notable improvement in precision, and found an additional drawback in that the integral becomes an oscillatory function of the number of integration points. Thus despite its apparent simplicity, the Newton-Cotes scheme with either trapezoidal or Simpson’s 1/3 rule appears to be the best way forward, at least in the case of empirical distributions considered in this paper\(^{11}\). Numerical results are reported in Section 7.

### B FFT implementation for multiple strikes

If one wishes to evaluate the hedging error for a large number of strikes, it is more efficient to implement the summation (60) as a discrete Fourier transform in log strike levels and evaluate it by means of an FFT algorithm\(^{12}\). To achieve this one must perform the following change of variables in (59):

\[ (v_1, v_2) \rightarrow (v_1, v_{12}), \]

\[ v_{12} \equiv v_1 + v_2, \]

and then change the order of integration so that the inner integral is over \( v_1 \),

\[ I(v_{\text{max}}) = \int_{0}^{v_{\text{max}}} \int_{v_1}^{v_{\text{max}}} dv_1 \int_{-v_1}^{v_2} dv_2 f(v_1, v_2) \]

\[ = \int_{0}^{2v_{\text{max}}} \int_{v_1/2}^{v_{\text{max}}} dv_{12} \int_{v_{12}/2}^{v_{\text{max}}} dv_1 f(v_1, v_{12} - v_1). \]  

\(^{11}\) This may not be the case when dealing with parametric distributions such as variance gamma, see Lee (2004).

\(^{12}\) See Chapter 7 in Černý (2004b) for an introduction to DFT and FFT in Finance.
Fig. 2. Graphical illustration of the numerical integration scheme (61). Diagonal lines depict the terms in the inner summation.

The corresponding numerical integration scheme is depicted in Figure 2.

To capture the explicit dependence on the log strike level $k$ let us define

$$g(v_1, v_2) \equiv f(v_1, v_2)e^{-2k + \tilde{b}T - i(\tilde{c}_1 + \tilde{c}_2)(\ln S_0 - k)},$$

and write the numerical scheme for integration of (61) as follows:

$$S(v_{\text{max}}, n) = 2(\Delta v)^2 S_0^2 e^{2\alpha(\ln S_0 - k) - \tilde{b}T} \sum_{j=0}^{n-1} e^{2i j \Delta v (\ln S_0 - k)} \sum_{l=j}^{n-1} w_{lj} g(l \Delta v, (2j - l) \Delta v)$$

$$= 2(\Delta v)^2 S_0^2 e^{2\alpha(\ln S_0 - k) - \tilde{b}T} \sum_{j=0}^{n-1} e^{2i j \Delta v (\ln S_0 - k)} c_j. $$

Let the log strike price $k$ take $n$ equally spaced values,

$$k_l \equiv k_{\text{max}} - l \Delta k \quad l = 0, \ldots, n - 1,$$

and denote the corresponding value of $S(v_{\text{max}}, n)$ by $S_l$. Then we have:

$$S_l = \sum_{j=0}^{n-1} e^{2i j \Delta v (\ln S_0 - k_l)} c_j$$

$$= \sum_{j=0}^{n-1} e^{2i j \Delta v (\ln S_0 - k_{\text{max}} + l \Delta k)} \tilde{c}_j = \sum_{j=0}^{n-1} e^{2i j \Delta v \Delta k} \tilde{c}_j,$$  \hspace{1cm} (62)

The right hand side of (62) is a $z$-transform of vector $\tilde{c}$, with $z = \exp(2i \Delta v \Delta k)$. This transform can be evaluated efficiently using so called chirp-$z$ algorithm.
due to Bluestein (1968) and Rabiner et al. (1969), requiring three FFT transforms of length $2n$. Furthermore, if it is feasible to set

$$\triangle k = \frac{\pi}{n \triangle v},$$

then the $z$-transform simplifies further to an ordinary DFT,

$$\sum_{j=0}^{n-1} e^{i \frac{2\pi j}{n} \tilde{c}_j} = \mathcal{F}(\tilde{c})_t,$$

$$s_l = 2(\triangle v)^2 S_0^2 e^{2\alpha (\ln S_0 - k_l) - l T} \sqrt{n} \mathcal{F}(\tilde{c})_t.$$  

(64)

The decision whether to use the straightforward numerical integration (60) for each strike separately or whether to apply the $z$-transform formula (62), or even the FFT formula (64) depends on the number and spacing of strike values. Sometimes the values of $n$ and $\triangle v$ that guarantee sufficient numerical precision in (60) will lead to too wide a strike spacing in (63). Then, instead of artificially boosting $n$ to make $\triangle k$ in (63) smaller, it may be more efficient to compute hedging errors from (62) at the cost of requiring roughly 6 times the number of operations of the FFT formula (64) for the same value of $n$.

7 Comparison of continuous-time Fourier transform formula with centinomial lattice

Practical usefulness of the hedging error formula (64) is best appreciated on real data. Let us take historical log returns on FT 100 equity index in the period 1/1996-12/2002 sampled at 15 minute intervals. We divide their values into 100 equally sized bins and equate the resulting histogram with the objective probability measure. In this centinomial model we hedge a given option optimally to maturity $T = 30$ trading days. The hedging error obtained in the lattice model is then compared with the 2-D Fourier formula proposed in this paper.

The computation of the objective cumulant generating function required as the only input in the 2-D Fourier formula proceeds as follows: Suppose the log returns in the multinomial lattice take values $x_j$ with probability $p_j$, $j =$

---

13 For details of the numerical implementation of the lattice model refer to Černý (2004b).
14 GAUSS code available from the author on request.
1, 2, \ldots, N, with rebalancing frequency $\Delta t$. We then set
\[
\kappa^P(v) = \sum_{j=1}^{N} (e^{ivx_j} - 1) \frac{p_j}{\Delta t},
\]
in analogy to (22). The fitting of the empirical characteristic function from return data in equation (65) is consistent with the generalized method of moments estimation surveyed in Yu (2004).

Using GAUSS on Dell Pentium M 1.56GHz with 512Mb RAM the calculation of the unconditional squared hedging error for each strike in the lattice model takes about 40 seconds, whereas in the case of the Fourier formula all strikes together, evaluated using the chirp-z transform in equation (62), require 0.2 seconds ($n = 500$). This is true for a relatively short maturity of 30 days, and for longer time horizons the advantage of the Fourier formula is even more substantial. Table I demonstrates very good numerical precision of the Fourier formula for a wide range of strikes. Table I also suggests that the mean value and optimal delta are insensitive to excess kurtosis, and remain very close to the corresponding Black–Scholes values. The next section examines this striking result in more detail.

8 Optimal hedging in a calibrated model of FT 100 returns

A Impact of excess kurtosis on optimal hedging strategy and representative agent price

In this section we investigate how far the optimal hedging strategy in a leptokurtic model deviates from the standard Black–Scholes delta and likewise, what is the discrepancy between the Black-Scholes price and the mean value $H_0$ of the option. To examine the robustness of Black–Scholes values we will consider an exponential Lévy model based on empirical distribution of log return sampled at intervals $\Delta t = 1, 5, 15, 30$ min, normalized to have annual mean $\mu = -0.1$, or $0.1$ and annual standard deviation $\sigma = 0.2$, combined with risk-free rate $r = 0$, or $0.04$. It must be stressed that although the sampling is performed discretely, the resulting Lévy model and the optimal hedging strategy run in continuous time.

The construction of the empirical Lévy process proceeds as follows: we consider log returns in the period 1/1996–12/2002 sampled at frequency $\Delta t$. We take the highest and the lowest value in the sample and divide the interval between them into $N = 1000$ equally sized subintervals (bins). We then evaluate the relative frequency $p_j$ of log returns in each bin and obtain an empirical distri-
bution of log returns with equidistantly spaced values $\tilde{x}_j$. As the next step we re-center and re-scale the distribution so that it corresponds to annual mean $\mu$ and annual volatility $\sigma$

$$x_j = \mu \Delta t + \sigma \sqrt{\Delta t} \frac{\tilde{x}_j - \sum_{j=1}^{N} \tilde{x}_j p_j}{\sqrt{\sum_{j=1}^{N} p_j \left( \tilde{x}_j - \sum_{k=1}^{N} \tilde{x}_k p_k \right)^2}}$$

where $\Delta t$ is time in years. With the empirical distribution of log return $\{x_j, p_j\}_{j=1}^{N}$ in hand we construct the Lévy cumulant generating function $\kappa^P$ using equation (65).

For each of these calibrated models we consider options with maturity $T = 1$ month, 3 months, 6 months, or 1 year and 9 strike values corresponding to Black–Scholes delta in the set

$$D = \{0.01, 0.05, 0.10, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99\}.$$  \hspace{1cm} (66)

For each strike value we compute the corresponding values of $H_0$ and $\theta_0^L$, using the Newton–Cotes implementation of formulae (33) and (42) with $n = 1000$ and $v_{\text{max}} = 200$ as discussed in Section 6. We then identify the levels of volatility $\sigma_H$ and $\sigma_{\theta L}$ which would make the Black-Scholes price and Black-Scholes delta equal to $H_0$ and $\theta_0^L$, respectively. The mean and standard deviation of the difference between the actual volatility $\sigma$ and the implied volatilities $\sigma_H$ and $\sigma_{\theta L}$ are reported in Tables II-V.

We find that the representative agent price and optimal delta in the leptokurtic model are extremely close to the standard Black–Scholes values. This is true despite the fact that the kurtosis on 1 min horizon is as high as 730. The result is robust to changes in the risk-free rate, mean rate of stock return, sampling frequency and time to maturity. Clearly the remit of Black–Scholes formulae stretches far beyond the lognormal return distribution, to highly leptokurtic empirical equity index return distributions.

B Hedging errors and option price bounds

One can use the calibrated continuous-time exponential Lévy models of the previous section to examine the impact of excess kurtosis on hedging errors and on the width of resulting option price bounds. Since the price bounds based on Sharpe ratio can lie outside the no-arbitrage bounds (Cochrane and Saá-Requejo 2000) we restrict our attention to strike values equivalent to Black–Scholes delta between 0.2 and 0.8. Specifically, we use 25 equidistantly spaced
values of delta leading to strike values

\[
K_j = \frac{S_j}{S_0} = e^{(r+\sigma^2/2)T + \Phi^{-1}(0.2+(j-1)0.025)\sigma \sqrt{T}}, \quad j = 1, \ldots, 25
\]  

(67)

For a given empirical Lévy process, risk-free rate, time to maturity and strike price we evaluate the mean value \( H_0 \) and the dynamically-optimal hedging error \( \varepsilon_{0,D} \) using formulae (33) and (58), the latter with \( n = 500 \) integration points and \( v_{\text{max}} = 500 \). From these values we construct option price bounds based on the absence of attractive investment opportunities as measured by annualized Sharpe ratio (SR). Specifically, we compute price bounds \( C_{\pm \text{SR}} \) for \( \text{SR} = 0.5 \) and 1.0 from (47):

\[
\text{SR}^2 T \geq \left( (C_{\text{SR}} - H_0)e^{rT}/\varepsilon_{0D} \right)^2,
\]

\[
\Rightarrow C_{\pm \text{SR}} = H_0 \pm \varepsilon_{0D} e^{-rT}\text{SR}\sqrt{T}.
\]  

(68)

We use the results of Jakubenas (2002) to verify that the option price bounds obtained in (68) are consistent with no arbitrage. In practical terms this requires checking a condition numerically indistinguishable\(^\text{15}\) from Merton’s (1973) rational price bounds

\[
(S_0 - Ke^{-rT})^+ \leq C_{\pm \text{SR}} \leq S_0.
\]

The range of strike prices (67) is designed in such a way as to be non-trivial while making sure that the computed option price bounds are indeed arbitrage-free.

To make the option prices comparable across strikes and maturities we report the option price bounds in terms of Black–Scholes implied volatility. Since the historical volatility in our experiment is normalized to \( \sigma = 0.2 \) and since the mean value \( H_0 \) is very close to the Black–Scholes value, the implied volatility bounds will be centered around 0.2. It transpires that the implied volatility bounds are very robust across all strikes in the considered range (67), we therefore only report the average value of the bound across strikes and the sample standard deviation of the bound values across strikes. A typical pattern of implied volatility bounds is depicted in Figure 3. It transpires that the implied volatility bounds are very robust across all strikes in the considered range (67),

\(^\text{15}\) Jakubenas (2002) shows that, in our compound Poisson jump model, the upper no-arbitrage bound is trivial, whereas the lower no-arbitrage bound depends on the size of the smallest jump (positive or negative, depending on the sign of the drift of the log return process). In our model the smallest jumps are so small that the lower no-arbitrage bound is virtually trivial.
moneyness in terms of B–S delta

Fig. 3. Typical pattern of implied volatility bounds as a function of moneyness.

we therefore only report the average value of the bound across strikes and the sample standard deviation of the bound values across strikes. Detailed results are shown in Tables VI-IX. Table X records the smallest width of the implied volatility bounds across strikes, for each combination of parameters.

We find that the position of option price bounds is largely insensitive to changes in all of the underlying parameters, $\mu, r, T$ and the sampling frequency $\Delta t$. For annualized Sharpe ratio of 1 the bounds extend roughly 1 percentage point above and below the historical volatility; for the Sharpe ratio of $1/2$ the bounds are half as wide. The bounds are slightly narrower for out-of-the-money options and wider for options in the money.

In terms of prices, it is instructive to evaluate price bounds for ATM options. The center of the bound corresponds to Black–Scholes ATM price based on volatility $\sigma = 0.2$. Sharpe ratio of 1 implies price deviations of $\pm 3.8\%$ for $T = 1$ year, rising to $\pm 4.6\%$ for $T = 1$ month when interest rate equals 4% p.a. When the interest rate is low (0%) the price bounds are wider: at least $\pm 4.8\%$ for all examined maturities. It is important to stress that these bounds are based on continuous rebalancing without transaction costs. If one took transaction costs into account the price bounds would be considerably wider.

9 Conclusions

This paper suggests a numerically efficient method of evaluating the mean–variance trade-off of an optimally and continuously rebalanced self-financing
hedging portfolio in a frictionless market with leptokurtic stock returns generated by a general exponential Lévy process. This method is based on Fourier transform and it permits a fast and accurate evaluation of hedging errors even for large values of time to maturity and/or excess kurtosis and for many strikes simultaneously. Since the method is applicable to any contingent claim, it provides a solid base for real-time risk management and optimization of large option portfolios.

We have devised a non-parametric procedure for the calibration of stock return distribution to historical data. Using FT 100 equity index data we have evaluated the resulting hedging errors and shown how to compute the implied good-deal option price bounds. In sharp contrast to the conclusions of the Black–Scholes model, we have found that continuous option hedging is far from riskless even in the absence of transaction costs, due to excess kurtosis in stock returns. The resulting option price bounds (measured in terms of Black–Scholes implied volatility) are non-trivial and largely insensitive to changes in expected return, time to maturity, risk-free rate or sampling frequency of the historical data. Interestingly, the (locally) optimal hedging strategy differs very little from the standard Black–Scholes delta. In conclusion, with highly leptokurtic returns the Black–Scholes price is the right value to hedge towards but not the right value to price at.

The present paper does not address the question of hedging error in stochastic volatility models, neither do we consider risk diversification by means of constructing optimal option portfolios. This paper does, however, suggest a line of attack able to deal efficiently with both of these important issues.

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Lemma 1

\[
\phi_{\Delta t}^Q(v) \equiv E_t^Q \left[ e^{iv \ln(S_{t+\Delta t}/S_t)} \right] = \frac{1 + a_{\Delta t} e^{r_{\Delta t}}}{b_{\Delta t}} e^{\kappa^P(iv)_{\Delta t}} - \frac{a_{\Delta t}}{b_{\Delta t}} e^{\kappa^P(1+iv)_{\Delta t}} \\
= 1 + \kappa^Q(iv)_{\Delta t} + o(\Delta t),
\]

where \( \kappa^Q(u) \equiv \kappa^P(u) - \bar{a} \left( \kappa^P(u + 1) - \kappa^P(u) - \kappa^P(1) \right) \),

where \( \kappa^Q(u) \) is well defined on \([0, 1 + 2\bar{a}] \times \mathbb{R} \).

Proof. By direct calculation:

\[
\phi_{\Delta t}^Q(v) = E_t^Q \left[ e^{iv \ln(S_{t+\Delta t}/S_t)} \right] = E_t^P \left[ m_{t+\Delta t} \mid e^{iv \ln(S_{t+\Delta t}/S_t)} \right] \\
= \frac{1 + a_{\Delta t} e^{r_{\Delta t}}}{b_{\Delta t}} E_t^P \left[ e^{iv \ln(S_{t+\Delta t}/S_t)} \right] - \frac{a_{\Delta t}}{b_{\Delta t}} E_t^P \left[ e^{(1+iv) \ln(S_{t+\Delta t}/S_t)} \right] \\
= \frac{1 + a_{\Delta t} e^{r_{\Delta t}}}{b_{\Delta t}} e^{\kappa^P(iv)_{\Delta t}} - \frac{a_{\Delta t}}{b_{\Delta t}} e^{\kappa^P(1+iv)_{\Delta t}}.
\]

Now expand the above around \( \Delta t = 0 \) using (25) and (48):

\[
\kappa^Q(u) \equiv \frac{d}{d\Delta t} \left( 1 + \frac{a_{\Delta t} e^{r_{\Delta t}}}{b_{\Delta t}} e^{\kappa^P(u)_{\Delta t}} - \frac{a_{\Delta t}}{b_{\Delta t}} e^{\kappa^P(1+u)_{\Delta t}} \right) \bigg|_{\Delta t=0} \\
= \bar{a}r + \bar{a} - \bar{b} (1 + \bar{a}) + (1 + \bar{a}) \kappa^P(u) - (\bar{a} - \bar{b}\bar{a}) - \bar{a}\kappa^P(u + 1) \\
= \bar{a}r - \bar{b} + (1 + \bar{a}) \kappa^P(u) - \bar{a}\kappa^P(u + 1),
\]

where \( \bar{a} \equiv \lim_{\Delta t \to 0} \frac{a_{\Delta t} - \bar{a}}{\Delta t}. \) Recall from (27) that \( \bar{b} = -\bar{a} \left( \kappa^P(1) - r \right) \) which leads to (36). By Theorem 25.17 in Sato (1999) \( \kappa^P \) is well defined on \([0, 2 + 2\bar{a}] \times \mathbb{R} \) whereby it follows immediately that \( \kappa^Q \) is well defined on \([0, 1+2\bar{a}] \times \mathbb{R} \).

Lemma 2

\[ g^Q_T(v) \equiv \lim_{\Delta t \to 0} \left( \phi_{\Delta t}^Q(v) \right)^{T/\Delta t} = e^{\kappa^Q(iv)T}. \]

Proof. From (69) we have

\[
\phi^Q(v) = \lim_{\Delta t \to 0} \left( \phi_{\Delta t}^Q(v) \right)^{T/\Delta t} = \lim_{\Delta t \to 0} \left( 1 + \kappa^Q(iv)_{\Delta t} + o(\Delta t) \right)^{T/\Delta t} \\
= \lim_{\Delta t \to 0} e^{(1+\kappa^Q(iv)_{\Delta t} + o(\Delta t))T/\Delta t} \\
= \lim_{\Delta t \to 0} e^{(\kappa^Q(iv)_{\Delta t} + o(\Delta t))T/\Delta t} = e^{\kappa^Q(iv)T}.
\]

31
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Table I
Comparison of mean value, locally optimal delta and unconditional squared hedging errors between Black−Scholes model, 100−nominal lattice based on FT100 equity index data with sampling frequency $\Delta t = 15$ min and the closed form Fourier transform formula (64) based on an exponential Lévy model calibrated via (65), as a function of strike price; $T = 30$ trading days, $S_0 = 100$. The reported B−S squared hedging error is obtained from the discretely rebalanced B−S hedge by Toft (1996) formula and it is further adjusted by the multiplicative factor $(\text{kurt} − 1)/2$ to account for the excess kurtosis, see Section 12.1.9 in Černý (2004).
\[ \Delta t = 1 \text{min} \quad T = 1 \text{year} \quad T = 6 \text{months} \quad T = 3 \text{months} \quad T = 1 \text{month} \]

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**Table II**

Mean and standard deviation (in parentheses) of the difference between Black–Scholes implied volatility \( \sigma_H \), \( \sigma_{\theta L} \) and the model volatility \( \sigma = 0.2 \) across strike values corresponding to Black–Scholes delta in the set \{0.01, 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99\}, as a function of the expected rate of return \( \mu \), risk-free rate \( r \), and time to maturity \( T \). The empirical Lévy process is based on FT100 returns sampled at frequency \( \Delta t = 1 \text{ min} \).
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Table III
Mean and standard deviation (in parentheses) of the difference between Black–Scholes implied volatility $\sigma_H$, $\sigma_{\theta^L}$ and the model volatility $\sigma=0.2$ across strike values corresponding to Black–Scholes delta in the set \{0.01, 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99\}, as a function of the expected rate of return $\mu$, risk-free rate $r$, and time to maturity $T$. The empirical Lévy process is based on FT100 returns sampled at frequency $\Delta t = 5$ min.
$$\Delta t = 15\text{min}$$  
$$T = 1\text{year}$$  
$$T = 6\text{months}$$  
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Table IV
Mean and standard deviation (in parentheses) of the difference between Black–Scholes implied volatility $\sigma_H$, $\sigma_{\theta^L}$ and the model volatility $\sigma=0.2$ across strike values corresponding to Black–Scholes delta in the set \{0.01, 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99\}, as a function of the expected rate of return $\mu$, risk-free rate $r$, and time to maturity $T$. The empirical Lévy process is based on FT100 returns sampled at frequency $\Delta t = 15\text{min}$. 

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Table V

Mean and standard deviation (in parentheses) of the difference between Black–Scholes implied volatility $\sigma_H$, $\sigma_{\theta^L}$ and the model volatility $\sigma=0.2$ across strike values corresponding to Black–Scholes delta in the set $\{0.01, 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99\}$, as a function of the expected rate of return $\mu$, risk-free rate $r$, and time to maturity $T$. The empirical Lévy process is based on FT100 returns sampled at frequency $\Delta t = 30 \text{ min}$.

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Table VI
Mean and standard deviation (in parentheses) of implied volatility bounds across 25 strike values in the range (67) as a function of expected rate of return $\mu$, risk-free rate $r$ and time to maturity $T$. Empirical Lévy process based on FT100 equity index data sampled at frequency $\Delta t = 1 \text{ min}$. 

37
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Table VII
Mean and standard deviation (in parentheses) of implied volatility bounds across 25 strike values in the range (67) as a function of expected rate of return $\mu$, risk-free rate $r$ and time to maturity $T$. Empirical Lévy process based on FT100 equity index data sampled at frequency $\Delta t = 5$ min.
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Table VIII
Mean and standard deviation (in parentheses) of implied volatility bounds across 25 strike values in the range (67) as a function of expected rate of return $\mu$, risk-free rate $r$ and time to maturity $T$. Empirical Lévy process based on FT100 equity index data sampled at frequency $\Delta t = 15$ min.
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<th>$T = 6$ months</th>
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<td>20.5</td>
<td>19.1</td>
<td>21.9</td>
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Table IX

Mean and standard deviation (in parentheses) of implied volatility bounds across 25 strike values in the range (67) as a function of expected rate of return $\mu$, risk-free rate $r$ and time to maturity $T$. Empirical Lévy process based on FT100 equity index data sampled at frequency $\triangle t = 30$ min.
Table X

Minimal width of implied volatility bounds across strikes as a function of sampling frequency, expected stock return $\mu$, risk-free rate $r$ and time to maturity $T$. Bounds correspond to annualized Sharpe ratio of 1.