MULTIPLE DEFAULTS AND MERTON’S MODEL

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ABSTRACT

The issues of multiple defaults and default correlation are very relevant for risk management, credit derivatives, and credit analysis. In this paper, we extend Merton (1974) framework to accommodate multiple defaults. The aim is to present a simple, unified framework for calculating single and joint default probabilities for more than two firms in closed form. The results are relevant for various financial applications.
I. Introduction

The seminal papers of Black and Scholes (1973) and Merton (1974) pioneered the so-called structural approach for pricing defaultable bonds and assessing the credit worthiness of firms. Amongst others extending Merton (1974) model we cite, Black and Cox (1976), Geske (1977), Longstaff and Schwartz (1995), Leland and Toft (1996) and Collin-Dufresne and Goldstein (2001). In this line of research the company equity is viewed as a call option on the asset of the firm and the total liability as the strike price. The structural approach has been very popular in the industry for example, according to Ferry (2003), "Merton-type models for estimating credit default probabilities are now so common they are driving pricing in the credit market.” The considerable commercial attention given to this type of approach is due to its analytical tractability and the economic interpretation of its inputs namely, the current value and the volatility of the company’s assets, the outstanding debt and the debt maturity.

Structural models have been criticized for failing to reproduce the level of spreads that are observed in practice. The empirical literature thus far, indicates that these models under-predict spreads on corporate bonds see for example Huang and Huang (2000) and Eom, Helwege, and Huang (2003). However, recent work by Hull, Nelken, and White (2003), has highlighted the fact that the performance of the Merton’s model could be improved by calculating spreads using the implied volatility of two equity options in contrast to a more traditional approach that involves the estimation of the instantaneous equity volatility and the debt outstanding. Also Gemmill (2002) has shown that Merton’s model works well in the particular case when zero-coupon bonds are used for funding. In addition, Campbell and Taskler (2003), in their recent empirical work found that the level of volatility explains well the cross-sectional variation in corporate bond yield. Huang and Kong (2003) on the other hand, found that equity market variables have an important role in explaining credit spread changes. These findings confirm in a sense that information from equity markets is a key input in default models, which strengthen the appeal of the structural framework.

The structural approach has mainly been used to estimate relative probabilities of default and credit default swap spreads. Very few attempts have been made to extend the approach to calculate joint probabilities of default for many firms. However, the issue of default probabilities that pertain to a basket of firms is important for credit analysis, the valuation of various credit derivatives and risk
management. Credit derivatives are now among the fastest growing financial contracts in the derivatives market. In particular, default-triggered credit derivatives such as default swaps and default baskets are the most popular. These products usually involve more than one firm, hence the need for a model that can accommodate multiple defaults. Although single default probabilities are easily derived within the structural framework, derivation of joint default probability curves is more complex as correlations between the various firms need to be taken into consideration.

Amongst the recent research, that have extended the structural framework to many firms, we cite Zhou (2001). In his paper, he derives in closed form the default correlation of two firms, in a set up that allows default to occur prior to maturity. Cathcart and El-Jahel (2003) also calculate default correlation in a stochastic interest rate, hybrid structural framework where default depends on the value of the assets of the firm and on a hazard rate process. However, in these generalization attempts, the tradeoff between realistic assumptions and tractability should be noted. Although, these models overcome some limitations of the Merton’s model, they require for more than two firms, computationally intense numerical solutions which increase in difficulty with the dimensionality of the problem. In a multi-firm set up it is important to have a tractable benchmark that is easy and fast to calculate and against which, one could compare results from more realistic generalizations. In fact Merton’s model is still popular because of its tractability and simplicity. Practitioners can overcome some of its limitations by assuming for example, that a bond can only default at the time horizon they consider but not at any other time and not necessarily at the maturity of the bond.

In this paper we extend Merton (1974) model to many firms and demonstrate how various default and survival probabilities can be calculated in a tractable form. In section one, of the paper we present the Merton’s framework and its extension to $N$ firms. In section two, we implement the model, present some results and discuss its implications. In section three, we conclude.

II. Merton’s Framework

In Merton (1974) model, the value of the firm’s asset is assumed to obey a lognormal diffusion process with a constant volatility. Default occurs at maturity if the value of the firm’s asset is less than a
promised payment. We extend Merton (1974) framework to $N$ firms. In the following for every firm $i$, $i = 1, \ldots, N$, the asset value dynamics are described by the following stochastic process.

$$dV_i = \mu_i V_i dt + \sigma_i V_i dB_{V_i}$$ (1)

For each firm, $\mu_i$ and $\sigma_i$ denote respectively, the asset value drift rate and volatility. For any two firms $i, j$, $i = 1, \ldots, N$ and $j = 1, \ldots, N$, $dB_{V_i}$ and $dB_{V_j}$ are Wiener processes such that $dB_{V_i}dB_{V_j} = \rho_{ij} dt$. $\rho_{ij}$ denotes the correlation of the asset value of the two firms $i$ and $j$. Let $X_i$ denote the promised payment of firm $i$ at maturity $T$. Firm $i$ defaults at time $T$ if $V_i \leq X_i$ or if $y_i \leq 1$, where $y_i$ is defined such that $y_i = \frac{V_i}{X_i}$. $y_i$ is considered an indicator of the credit quality of firm $i$.

We also consider $D_i$ and $S_i$, $i = 1, \ldots, N$, random variables that describe the default and survival status respectively of the $N$ firms at time $T$. $D_i = 1$ if firm $i$ defaults at time $T$ and $D_i = 0$ otherwise. The probability of default of firm $i$ is $P(D_i = 1) = P(V_i \leq X_i) = P(y_i \leq 1)$. On the other hand, $S_i = 1$ if firm $i$ survives at time $T$ and $S_i = 0$ otherwise. The probability of survival of firm $i$ is $P(S_i = 1) = P(V_i > X_i) = P(y_i > 1)$. For ease of notation, we denote $P(D_i = 1) = P(D_i)$ and $P(S_i = 1) = P(S_i)$.

For a single firm the probability of default in Merton (1974), $P(D) = P(V_i \leq X_i) = P(y_i \leq 1)$ is given by,

$$P(y_i \leq 1) = \phi(d_2(y_i, \sigma_i^2)) \quad d_2(y_i, \sigma_i^2) = -\frac{\ln(y_i) + (\mu_i - \sigma_i^2/2)T}{\sigma_i \sqrt{T}}$$ (2)

$\phi$ is the cumulative density function of the standard normal distribution. In the following, we use results from probability theory to calculate default and survival probabilities for the case of two and many firms.

A. The Case of Two Issuers $N = 2$

The case of $N = 2$ illustrates the flexibility of the Merton (1974) approach and the various probabilities that can be derived in closed form within this framework.

Let $P(D_1 \cap D_2)$ represents the probability of joint default at time $T$, this probability is simply,

$$P(D_1 \cap D_2) = P(y_1 \leq 1, y_2 \leq 1) = \phi_2(d_2(y_1, \sigma_1^2), d_2(y_2, \sigma_2^2), \rho_{12})$$ (3)
$\phi_2$ is the bivariate standard cumulative normal, $d_2(y_i, \sigma_i^2)$ for $i = 1, 2$ is given by equation (2) and $\rho_{12}$ is the pairwise correlation between the assets of firms 1 and 2.

In addition to the joint probability of default a host of other useful probabilities can also be derived for example, for $i = 1, 2, j = 1, 2$ and $i \neq j$, $P(D_i | D_j)$ the probability of default of firm $i$ conditional on the default of firm $j$ at time $T$ is,

$$P(D_i | D_j) = \frac{P(D_i \cap D_j)}{P(D_j)} = \frac{\phi_2(d_2(y_i, \sigma_i^2), d_2(y_j, \sigma_j^2), \rho_{ij})}{\phi(d_2(y_j, \sigma_j^2))} \quad (4)$$

The probability of at least one default $P(D_1 \cup D_2)$ at time $T$,

$$P(D_1 \cup D_2) = P(D_1) + P(D_2) - P(D_1 \cap D_2) \quad (5)$$

$$= \phi(d_2(y_1, \sigma_1^2)) + \phi(d_2(y_2, \sigma_2^2)) - \phi_2(d_2(y_1, \sigma_1^2), d_2(y_2, \sigma_2^2), \rho_{12}) \quad (6)$$

Also one can easily deduce probabilities of survival from the probabilities of default. For $i = 1, 2$, the probability of survival of firm $i$ at time $T$,

$$P(S_i) = 1 - P(D_i) \quad (7)$$

The probability of at least one survival at time $T$,

$$P(S_1 \cup S_2) = 1 - P(D_1 \cap D_2) \quad (8)$$

The probability of joint survival at time $T$,

$$P(S_1 \cap S_2) = P(S_1) + P(S_2) - P(S_1 \cup S_2) \quad (9)$$

For $i = 1, 2, j = 1, 2$ and $i \neq j$, the probability of survival of firm $i$ conditional on the survival of firm $j$ at time $T$,

$$P(S_i | S_j) = \frac{P(S_i \cap S_j)}{P(S_j)} \quad (10)$$
The probability of default of firm $i$ and survival of firm $j$ at time $T$,

$$P(D_i \cap S_j) = P(D_i) - P(D_i \cap D_j)$$  \hspace{1cm} (11)$$

The probability of survival of firm $i$ and default of firm $j$ at time $T$,

$$P(S_i \cap D_j) = P(S_i) - P(S_i \cap S_j)$$  \hspace{1cm} (12)$$

Finally, one can also compute the default or survival correlation denoted by $\rho_D$ or $\rho_S$ of firm 1 and 2,

$$\rho_D = \rho_S = Corr(D_1,D_2) = \frac{P(D_1 \cap D_2) - P(D_1)P(D_2)}{\sqrt{P(D_1)(1-P(D_1))P(D_2)(1-P(D_2))}}$$  \hspace{1cm} (13)$$

Substituting for the relevant probabilities one gets the solution required.

**B. The Case of Many Issuers $N > 2$**

The case of $N > 2$ is more challenging as one has to take into consideration the correlation between all the firms. Nevertheless, one can still derive a host of different probabilities, the calculations are slightly more complex but involve no more than the use of the multivariate normal distribution.

**B.1. The probability of joint default $N > 2$**

Let the probability of joint $N$ defaults in a basket of $N$ firms at time $T$, be denoted by,

$$P(D_1 \cap D_2 \cap D_3 \cap \ldots \cap D_N) = P(y_1 \leq 1, y_2 \leq 1, y_3 \leq 1, \ldots, y_N \leq 1)$$  \hspace{1cm} (14)$$
This probability expression is simply given by

\[ P(y_1 \leq 1, y_2 \leq 1, y_3 \leq 1, ..., y_N \leq 1) = \phi_N(d_2(y_1, \sigma_1^2), d_2(y_2, \sigma_2^2), ..., d_2(y_N, \sigma_N^2, \rho_{12}, \rho_{13}, ...) \] (15)

\( \phi_N \) is the Nth variate standard cumulative normal, \( d_2(y_i, \sigma_i^2) \) and \( \rho_{ij} \) are specified as before. Similar expressions to equation (15) have been used in the option pricing literature see for example Johnson (1987) and Geske and Johnson (1984).

One can also calculate the probability of at least one firm surviving amongst a basket of \( N \) firms.

\[ P(S_1 \cup S_2 \cup S_3 \cup ... \cup S_N) = 1 - P(D_1 \cap D_2 \cap D_3 \cap ... \cap D_N) \] (16)

**B.2. The probability of at least one default in a basket of \( N > 2 \)**

If one wants to calculate the probability of at least one default in a basket of \( N \) firms, \( P(D_1 \cup D_2 \cup D_3 \cup ... \cup D_N) \), in this context it is not sufficient to know the probability of an individual default we also need to give complete information concerning all possible overlaps. i.e we have to derive the probability of joint default of firm \( i \) and firm \( j \), and the probability of joint default of firm \( i \), firm \( j \) and firm \( k \), etc... where \( i = 1, ..., N, j = 1, ..., N, k = 1, ..., N, \) etc... Let

\[ F_1 = \sum P(D_i), \quad F_2 = \sum P(D_i \cap D_j), \quad F_3 = \sum P(D_i \cap D_j \cap D_k), ... \] (17)

Where \( P(D_i) \) is given by equation (2), \( P(D_i \cap D_j) \) by equation (3) and \( P(D_i \cap D_j \cap D_k) \) up to \( P(D_1 \cap D_2 \cap D_3 \cap ... \cap D_N) \) by equation (15). In expression (17), two subscripts are never equal and \( i < j < k < \cdots < N \), so that in the sums each combination appears once and only once. Each \( F_m \), has \( \binom{N}{m} \) terms where

\[ \binom{N}{m} = \frac{N!}{m!(N-m)!} \] represents the number of different subpopulation of size \( m \leq N \). The last sum would be \( F_N \) and it reduces to the single term \( P(D_1 \cap D_2 \cap D_3 \cap ... \cap D_N) \) which is the probability of \( N \) joint default.
The probability of at least one default denoted by $P_1$ is then given by,

$$P_1 = F_1 - F_2 + F_3 - F_4 + -... \pm F_N$$  \hspace{1cm} (18)

For $N = 2$ we have only the first two terms $F_1$ and $F_2$ which is the two firms result mentioned previously.

From this result we can also automatically derive the probability of joint survival of $N$ firms, it is given by $P(S_1 \cap S_2 \cap S_3 \cap ... \cap S_N)$ where

$$P(S_1 \cap S_2 \cap S_3 \cap ... \cap S_N) = 1 - P_1 = 1 - F_1 + F_2 - F_3 + F_4 - +... \pm F_N$$  \hspace{1cm} (19)

**B.3. The probability of at least or exact $m$ defaults in a basket $N > 2$**

The probability of $m$ exact defaults in a basket of $N$ firms, with $1 \leq m \leq N$, is given by,

$$P_{(m)} = F_m - \binom{m+1}{m} F_{m+1} + \binom{m+2}{m} F_{m+2} - +... \pm \binom{N}{m} F_N$$  \hspace{1cm} (20)

and the probability of at least $m$ defaults in a basket of $N$ firms is given by,

$$P_m = F_m - \binom{m}{m-1} F_{m+1} + \binom{m+1}{m-1} F_{m+2} - +... \pm \binom{N-1}{m-1} F_N$$  \hspace{1cm} (21)

Where $F_m$ is defined as above.

Substituting for the relevant probabilities one get the solutions required. For proof of the above probability results see Feller (1950).
III. Implementation, Results and Implications

A. Implementation

The challenge in using the above formulas for default or survival probabilities is to evaluate the cumulative multivariate normal probability \( \Phi \). There are many routines available for this purpose. In this setup, we use the function "MULTI-NORMAL-PROB" from Numerical Algorithm Group’s (NAG) Excel add-in statistics package. This function can evaluate the multivariate normal distribution up to ten dimensions in less than a fraction of a second. For other numerical methods designed to evaluating the multivariate normal, see for example, Schervish (1985), Geske (1977) and Geske and Johnson (1984).

To illustrate our results, we construct a basket of \( N = 5 \) firms, and compare the probabilities of:

1- Joint default
2- At least one default in the basket
3- At least two defaults in the basket
4- Exactly one default in the basket
5- Exactly two defaults in the basket

We consider two main base case scenarios.

Scenario one: \( \mu_i = 0.05 \) and \( \sigma_i^2 = 0.16 \), for all \( i = 1, \ldots, 5 \) in the basket, see figures 1 – 5.

For scenario one (figures 1-5), we report the results for two values of pairwise asset correlations \( \rho_{ij} = 0.4 \) and \( \rho_{ij} = 0.1 \). For each pairwise asset correlation value, we consider three different credit qualities \( y_i = 3 \) high quality, \( y_i = 2 \) medium quality and \( y_i = 1.2 \) low quality. Once the levels of pairwise asset correlation and credit quality have been selected, these values will apply to all firms in the basket, i.e we assume that all firms within a basket have the same credit quality and the same pairwise asset correlation\(^5\).

Scenario two: \( \mu_i = 0.05, y_i = 2 \), for all \( i = 1, \ldots, 5 \) in the basket, see figures 6 – 10.
For scenario two (figures 6-10), we report the results for two values of pairwise asset correlations \( \rho_{ij} = 0.4 \) and \( \rho_{ij} = 0.1 \). For each pairwise asset correlation value, we consider three different volatility levels, \( \sigma_i^2 = 0.25 \), \( \sigma_i^2 = 0.16 \) and \( \sigma_i^2 = 0.09 \). Once again, when selected, these parameters values will apply to all the firms in the basket.

**B. Results**

For scenario one, we note two main observations.

First, credit quality is an important factor:

1- In figures 1, 2, 3, the higher the credit quality the lower the joint and the at least probabilities of default.

2- In figures 4, 5, the relationship between exact default probabilities and credit quality varies with time to maturity. In the short term, the higher the quality of the basket, the lower the probability of an exact number of defaults. In the long term, this relation is inverted. The crossing points across different quality baskets depend on the exact number of defaults we are considering. The lower the number of exact defaults, the sooner this relation will be inverted. The reason for this is that in the short term the probability of a certain number of default is high for low quality firms but over time that exact number could exceeded. Whereas in the short term, for high or medium quality baskets, the probability of a certain number of default is low but over time this exact number of default could be reached.

Second, asset correlation also plays a significant role.

1- In figure 1, we note that higher asset correlation implies higher joint default probabilities across all credit qualities.

2- In figure 2, higher asset correlation implies lower probabilities of at least one default and this result is consistent across maturities and credit qualities.
3- In figure 3, the probability of at least two defaults varies with time to maturity and the basket credit quality. For short maturities, the higher the asset correlation, the higher the default probability. For longer maturities this relationship is inverted. The timing of the crossing point across different asset correlation depends on the basket credit quality. The lower the credit quality the sooner this crossing occurs. For example, for medium and high credit quality baskets the probability of at least two defaults derived using a level of asset correlation of 0.1 becomes higher than the probability of at least two defaults derived using a level of asset correlation of 0.4, for maturities greater than 9.5 and 18 years, respectively.

4- In figure 4, the shape of the probability of exactly one default, varies with the basket credit quality and time to maturity. High credit quality baskets have typically an increasing probability of one default. Medium credit quality baskets present a humped shape. Whereas, for low credit quality baskets this probability increases for short maturities but decreases thereafter. We also observe, for all credit quality baskets, that the probability of exactly one default derived using higher asset correlation will cross the probability of exactly one default using lower asset correlation. Typically high asset correlation probabilities will be lower before the crossing point. The crossing point depends on the credit quality. The lower the quality the sooner the crossing occurs.

5- The probability of exactly two defaults in figure 5, exhibits the same characteristics as the probability of exactly one default. However, two main observations should be noted. First, the probabilities of exactly two defaults are lower. Second, the humped shape is more pronounced at longer maturities.

For scenario two, figures 6-10, the impact of asset volatility parameters is comparable to that of credit quality. High volatility generates default probability patterns comparable to those generated by low quality firms and vice versa. For example, in figure 6, the higher the asset volatility parameter, the higher the joint probability of default. This observation is consistent for all the other probabilities we consider and also across different correlation levels. For example in figure 7, higher asset correlation implies lower probabilities of at least one default and this result holds for all maturities and asset volatilities.
C. Implications

The Merton (1974) framework allows the calculation of a host of default and survival probabilities for \( N \) firms in closed form. This facilitates the study of the impact of various key parameters such as asset correlation, credit quality, time to maturity and asset volatility on these probabilities.

1- Asset correlation plays a principal role in determining probabilities of defaults within a basket. A lower asset correlation implies a lower joint probabilities of default. Therefore, diversification across regions and industries could lower significantly the probabilities of joint defaults within a basket. However, if one is considering the probabilities of an exact or at least number of defaults this relationship is not always maintained as lower asset correlation in certain cases would imply higher probabilities.

2- Credit quality of firms is also an important factor. Joint and at least probabilities of default are small for high quality firms and large for low quality firms. For the probability of an exact number of defaults this pattern is not always observed, however. The relationship between the exact probabilities and credit quality changes with the parameters of the model.

3- Default probabilities vary with maturity. Joint and at least probabilities usually start small over short horizons and then increase over time. However, the exact probability of default displays a humped shape. The time of peak hump depends mainly on the number of defaults considered, the pairwise asset correlation, the credit quality and asset volatility.

4- Asset volatilities have a great impact on the various probabilities of defaults. Joint and at least probabilities of default are small for low asset volatilities and large for high asset volatilities. This pattern, for the probability of an exact number of defaults could however, be inverted for particular maturity, credit quality and pairwise asset correlation parameter values.

IV. Conclusion

Recent empirical investigation by Campbell and Taskler (2003), and Huang and Kong (2003) highlights the importance of equity market variables in explaining credit spread movements. This makes
the argument for the structural approach even more compelling.

In this paper, we have presented a straightforward extension of Merton (1974) to accommodate multiple defaults. This extension has several advantages. First, it presents a unified framework for the calculation of single and joint default probabilities. Second, all results are in closed form and can serve as a comparison benchmark with more complex generalizations. Third, joint default probability curves can be easily build alongside various others, such as the at least and exact default probabilities curves. Fourth, as credit quality changes with time, the derived default probabilities will be dynamic and this gives a better understanding of the risk of a basket of firms. Finally, default probabilities have direct implications for risk management, credit analysis and the pricing of various credit derivatives such as nth to default swaps.
References


Notes

1 These products are considered as cash products that can be priced off default probability curves. Default probability curves describe default probabilities for various future points in time.

2 For details refer to Merton (1974).

3 $P(D_1 \cap D_2) = P(D_1 = 1 and D_2 = 1)$ where the symbols “$\cap$” stands for “and”.

4 $P(D_1 \cup D_2) = P(D_1 = 1 or D_2 = 1)$ where the symbol “$\cup$” stands for “or”.

5 This assumption could be easily relaxed.

6 We could easily allow the asset volatility to vary amongst firms within a basket.
Figure 1

\[ \mu_i = 0.05, \quad \sigma_i^2 = 0.16 \]
Figure 2

At least one default $N=5$

$m_i=0.05$, $\sigma_i^2=0.16$

Maturity

$y_i=3, \rho_{ij}=0.4$
$y_i=3, \rho_{ij}=0.1$
$y_i=2, \rho_{ij}=0.4$
$y_i=2, \rho_{ij}=0.1$
$y_i=1.2, \rho_{ij}=0.4$
$y_i=1.2, \rho_{ij}=0.1$
At least two defaults $N=5$

Figure 3

$\mu_i=0.05, \sigma_i^2=0.16$

Maturity
Figure 4

\[ \mu_t = 0.05, \sigma_t^2 = 0.16 \]

Maturity
Figure 5

Two defaults $N=5$ $y_i$=3, $\rho_{ij}=0.4$
$y_i$=3, $\rho_{ij}=0.1$
$y_i$=2, $\rho_{ij}=0.4$
$y_i$=2, $\rho_{ij}=0.1$
$y_i$=1.2, $\rho_{ij}=0.4$
$y_i$=1.2, $\rho_{ij}=0.1$

$\mu_i=0.05$, $\sigma_i^2=0.16$
Figure 6

Joint default N=5

$\mu_i=0.05, y_i=2$

Maturity
Figure 7

At least one default N=5

\( \mu_i=0.05, \gamma_i=2 \)

Maturity

Legend:
- \( \sigma_t^2=0.25, \rho_{ij}=0.1 \)
- \( \sigma_t^2=0.25, \rho_{ij}=0.4 \)
- \( \sigma_t^2=0.16, \rho_{ij}=0.1 \)
- \( \sigma_t^2=0.16, \rho_{ij}=0.4 \)
- \( \sigma_t^2=0.09, \rho_{ij}=0.1 \)
- \( \sigma_t^2=0.09, \rho_{ij}=0.4 \)
Figure 8

At least two defaults $N=5$

$\mu_i=0.05, \gamma_i=2$

Maturity

$\sigma_i^2=0.25, \rho_{ij}=0.1$
$\sigma_i^2=0.25, \rho_{ij}=0.4$
$\sigma_i^2=0.16, \rho_{ij}=0.1$
$\sigma_i^2=0.16, \rho_{ij}=0.4$
$\sigma_i^2=0.09, \rho_{ij}=0.1$
$\sigma_i^2=0.09, \rho_{ij}=0.4$
Figure 10

Two defaults $N=5$

$m_i=0.05, y_i=2$

Maturity

Legend:
- Green line: $\sigma_i^2=0.25, \rho_{ij}=0.1$
- Light blue line: $\sigma_i^2=0.25, \rho_{ij}=0.4$
- Red dotted line: $\sigma_i^2=0.16, \rho_{ij}=0.1$
- Pink dashed line: $\sigma_i^2=0.16, \rho_{ij}=0.4$
- Yellow dotted line: $\sigma_i^2=0.09, \rho_{ij}=0.1$
- Blue dotted line: $\sigma_i^2=0.09, \rho_{ij}=0.4$