Black Holes & Ricci Flow in Higher Dimensions

Student Seminar 2008

Sam Kitchen
Imperial College, London

28th Feb. 2008
Motivation.

Ricci Flow - Introduction and Explanation.


Current Work - 5D Black String & Localised Black String.

In this talk, classical GR is assumed; there is no String Theory, or even Quantum Mechanics.
A Black Hole provides the ideal testing ground for Quantum Gravity - Very high curvature.

Many theories of Quantum Gravity predict the existence of extra dimensions, some even predict non-compact ones.

Interesting results can be gleaned about important aspects of theories involving some kind of Kaluza-Klein compactification (e.g. most flavours of String Theory).

Large compact extra dimensions may still be relevant, even though they haven’t been observed (arXiv:hep-th/9803315), or even non-compact ones (e.g. Randall-Sundrum Braneworld arXiv:hep-th/9906064).

In a number of these it is still not known what form the Black Hole solutions take.

Black Holes, through AdS-CFT, give us some properties of high energy conformal field theories, see e.g. arXiv:hep-th/0507219.
The Diffusion Equation, models temperature distribution in a body with fixed boundary temperature.

$$\frac{\partial u(x, t)}{\partial t} = D \nabla^2 u(x, t)$$

Strongly parabolic PDE; means that given fixed $u$ along boundaries and $u(x, 0)$ the problem is well-posed - it can (in principle) be solved.

This equation tends to smooth out gradients in the temperature distribution.

Clearly a solution which satisfies $\nabla^2 u(x, t) = 0$ is a fixed point of the evolution.

Heat Equation can be regarded as a flow, which evolves some initial data $u_0(x) = u(x, 0)$ to a solution of $\nabla^2 u(x) = 0$. 
- Require a static (stationary) solution to Vacuum Einstein Equations, \( R_{\mu\nu} (g) = 0 \).

- Could perform a full numerical solution to Einstein’s equations, but this is generally a hard problem to solve.

- Consider a family of metrics \( g(\lambda) \) with a flow defined on it,

\[
\frac{\partial g_{\mu\nu}}{\partial \lambda} = 2R_{\mu\nu} (g)
\]

- Given some initial metric (possibly not Ricci-flat, but nearly), this flow will tend to smooth out the metric to one which is Ricci-flat.

- This only works with Riemannian metrics, although recent work has suggested it could be modified to work with Lorentzian ones.
Because GR is a diffeomorphism invariant theory, it turns out that we can add an extra term to the Ricci Flow,

\[ \frac{\partial g_{\mu\nu}}{\partial \lambda} = 2R_{\mu\nu} - \nabla_{(\mu} \xi_{\nu)} \]

where \( \xi \) is a vector field generating the diffeomorphism.

It can be shown that apart from some special cases (Ricci solitons), solutions to this will also solve \( \frac{\partial g_{\mu\nu}}{\partial \lambda} = 2R_{\mu\nu} \).

Can use this freedom to ensure the resulting system of equations is strongly parabolic by setting \( \xi_{\mu} = \Gamma^{\alpha}_{\mu\alpha} - \tilde{\xi}_{\mu} \). This gives,

\[ \frac{\partial g_{\mu\nu}}{\partial \lambda} = \nabla^2 g_{\mu\nu} + F(g, \partial g) \]
Examine the Schwarzschild solution in Euclidean signature, on a compactified time circle - Witten’s Cigar solution.

The solution is static, this severely restricts the form of the metric. We also build in the behavior of $g_{tt}$ close to the Horizon to keep $T(r)$ finite, so the metric must be,

$$ds^2 = r^2 T(r) \, dt^2 + R(r) \, dr^2 + S(r) \, d\Omega_2^2$$

One further simplification, set $\tilde{\xi}_\mu$ to be $\Gamma^\alpha_{\mu \alpha}$ evaluated for the Schwarzschild metric.

Essentially fixes the co-ordinates in which the Schwarzschild solution will be a fixed point of the flow, obviously only possible when we know the solution!
Scale the $r$ co-ordinate so that $r = 0$ on the Horizon, and place the spacetime in a box. This fixes the induced metric on the outer boundary ($r = 1$) and just leaves the transverse component free ($R(r)$).

Since $\xi$ generates diffeomorphisms, we require that $\xi|_{r=1} = 0$, otherwise the diffeomorphism would move the boundary to some other value of $r$. This gives a boundary condition for $R(1)$.

Boundary conditions $r = 0$ are given by expanding the metric functions about $r = 0$, and eliminating any singular ($\sim r^{-1}$) terms.

Finally, an initial guess is required, which need not be Ricci-flat, but should be smooth. In cases where a solution is known, an ideal candidate is present, otherwise one way to generate initial data is to solve the Laplacian, $\nabla^2 \phi = 0$, subject to the required boundary conditions.

- For a given temperature there are two solutions, one with a “small” horizon, and one with “large” horizon. The critical point separating these two types is a horizon radius of $\frac{2}{3}$.

- The smaller set of black holes are unstable under Ricci Flow, this type of instability is a linear instability due to a Gross-Perry-Yaffe mode (related to the thermal properties of Black Holes). Depending on the horizon chosen and the perturbation, the small black hole will either flow to a large black hole (with a certain horizon radius) or towards flat space.

- Can visualise the instability as a co-dimension 1 wall in the space of initial metrics. Suggests a method where we choose a surface piercing this wall, and try different initial data along this surface.

- For a given temperature there are two solutions, one with a “small” horizon, and one with “large” horizon. The critical point separating these two types is a horizon radius of $\frac{2}{3}$.

- The smaller set of black holes are unstable under Ricci Flow, this type of instability is a linear instability due to a Gross-Perry-Yaffe mode (related to the thermal properties of Black Holes). Depending on the horizon chosen and the perturbation, the small black hole will either flow to a large black hole (with a certain horizon radius) or towards flat space.

- Can visualise the instability as a co-dimension 1 wall in the space of initial metrics. Suggests a method where we choose a surface piercing this wall, and try different initial data along this surface.

- For a given temperature there are two solutions, one with a “small” horizon, and one with “large” horizon. The critical point separating these two types is a horizon radius of $\frac{2}{3}$.

- The smaller set of black holes are unstable under Ricci Flow, this type of instability is a linear instability due to a Gross-Perry-Yaffe mode (related to the thermal properties of Black Holes). Depending on the horizon chosen and the perturbation, the small black hole will either flow to a large black hole (with a certain horizon radius) or towards flat space.

- Can visualise the instability as a co-dimension 1 wall in the space of initial metrics. Suggests a method where we choose a surface piercing this wall, and try different initial data along this surface.
Extend to 5D by introducing a compact spatial direction, \( z \), into the system, for convenience we take \( z \) to range from 0 to 1. Initially concentrate on solutions constant in \( z \), such as a Black String.

Metric now has the form,

\[
 r^2 T (r, z) \, dt^2 + R (r, z) \, dr^2 + Z (r, z) \, dz^2 + rW (r, z) \, drdz + S (r, z) \, d\Omega^2_2
\]

This is significantly more complicated, not least because all the metric functions are now functions of two variables.

Employ the same methods as before, using the diffeomorphism term to get a parabolic set of equations.
Boundary conditions on $r = 0$ and $r = 1$ are the same as before, but extra boundary conditions on $z = 0, 1$ boundaries. Now the asymptotics are not flat space, but $\mathbb{R}^4 \times S^1$.

As this direction is compact, we require periodic boundary conditions on these edges. This is equivalent numerically to the vanishing of the $z$-derivative on these boundaries.

We also end up with an additional condition, $\frac{R}{T}\bigg|_{r=0} = 1$, which ensures that the time circle compactification can take place in a smooth manner all along the compact direction.
These results are mostly similar to the 4D case, once again there are two sets of solution classed by horizon radius.

- The Black String exhibits the same GPY unstable mode as the 4D Schwarzchild case, small Black Strings flow to the larger set or towards flat space. Although there is no perturbation here, the numerics excites the unstable mode.

- Explicit perturbations (here a wave in the z-direction and Gaussian in \( r \)) of the Large Black String are smoothed out by the flow.

- Flat space (\( \mathbb{R}^4 \times S^1 \)) is also stable under the Ricci Flow, and similar perturbations are smoothed out.
Results

These results are mostly similar to the 4D case, once again there are two sets of solution classed by horizon radius.

- The Black String exhibits the same GPY unstable mode as the 4D Schwarzchild case, small Black Strings flow to the larger set or towards flat space. Although there is no perturbation here, the numerics excites the unstable mode.

- Explicit perturbations (here a wave in the $z$-direction and Gaussian in $r$) of the Large Black String are smoothed out by the flow.

- Flat space ($\mathbb{R}^4 \times S^1$) is also stable under the Ricci Flow, and similar perturbations are smoothed out.
These results are mostly similar to the 4D case, once again there are two sets of solution classed by horizon radius.

- The Black String exhibits the same GPY unstable mode as the 4D Schwarzchild case, small Black Strings flow to the larger set or towards flat space. Although there is no perturbation here, the numerics excites the unstable mode.

- Explicit perturbations (here a wave in the z-direction and Gaussian in r) of the Large Black String are smoothed out by the flow.

- Flat space ($\mathbb{R}^4 \times S^1$) is also stable under the Ricci Flow, and similar perturbations are smoothed out.
These results are mostly similar to the 4D case, once again there are two sets of solution classed by horizon radius.

- The Black String exhibits the same GPY unstable mode as the 4D Schwarzchild case, small Black Strings flow to the larger set or towards flat space. Although there is no perturbation here, the numerics excites the unstable mode.

- Explicit perturbations (here a wave in the $z$-direction and Gaussian in $r$) of the Large Black String are smoothed out by the flow.

- Flat space ($\mathbb{R}^4 \times S^1$) is also stable under the Ricci Flow, and similar perturbations are smoothed out.
Future Work

Motivation  Ricci Flow  4D Schwarzschild Black Holes  Black Holes in 5D

- Look for localised (in the $z$-direction) solutions which have yet to be found. Requires a novel approach to the numerical simulation because of the need to specify two “shapes” of boundary condition because of the spherical shape of the horizon.

- Examine the behavior of Non Uniform Black Strings (NUBS) - Black Strings where the horizon radius varies in the $z$-direction. Certain classes have an unstable mode and decay into two localised black holes, how does this happen? There is some kind of topology-change during this decay.