Lectures series as a part of the activity within the frame of the Marie Curie Chair “Fundamental and Conceptual Aspects of Turbulent Flows”. PDF files available at

http://www3.imperial.ac.uk/mathsinstitute/programmes/research/turbulence/marie_curieChair

or/and http://www.eng.tau.ac.il/~tsinober/

We absolutely must leave room for doubt or there is no progress and no learning. There is no learning without posing a question. And a question requires doubt... Now the freedom of doubt, which is absolutely essential for the development of science, was born from a struggle with constituted authorities... FEYNMANN, 1964
Part I. Quasi-Gaussian manifestations of turbulent flows and their non-Gaussian nature.

Part II. Intermittency

Part III. Structure(s)
The modest purpose of the following remarks is to point out the conflict between Kolmogorov's law for the double correlations of the velocities and the normal distribution of velocity fields... The conclusion seems inescapable that the hypothesis of quasi-normality has a good chance of being outside Kolmogorov's "universal equilibrium range".

HOPF 1962

I believe real progress was made by the hypothesis that a so-called "quasi-normal joint probability" (Gaussian probability) can be assumed for the velocity field. This assumption makes it possible to express the fourth-order correlations by products of the second-order correlations. It was first suggested by Millionschikoff (who got the suggestion from A.N. Kolmogorov - his Ph.D. supervisor).

T. VON KARMAN 1958

Some significant developments in aerodynamics since 1946,

Turbulence – being essentially non-Gaussian – is such a rich phenomenon that it can ‘afford’ a number of Gaussian-like manifestations, some of which are not obvious and even nontrivial.
The non-Gaussian nature of turbulence is another ‘N’ contributing to the difficulty of turbulence – in addition to the three mentioned before non-linearity, non-integrability and non-locality.

A rather common view/assumption was that the field of turbulent fluctuations is - in some sense(s) - nearly Gaussian. For example, there were many attempts to treat the problem as a perturbation of a Gaussian field. More recently, it was proposed that some parts of a turbulent field are nearly Gaussian, so that it can be decomposed into a (nearly) Gaussian part and a (strongly) non-Gaussian one. Without entering into a comprehensive review of the whole issue, we shall argue that such an assumption is inadequate. For this purpose, we give a number of (counter) examples with the emphasis on the dynamical aspects.

For a bit more, see chapters 6 (pp. 140-147) and 7 in Tsinober 2001

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33 Most frequently it is assumed that the ‘unresolved’ (i.e. mainly the small scale) part of the turbulent field is nearly Gaussian, in contradiction to observations showing that this part of flow is the most non-Gaussian.
How much turbulence statistics is close to Gaussian and in what sense?

There is much more than just statistics.

Or is (and if when/why) $\varepsilon$ in

$$\text{Turbulence statistics} = \text{Gaussian} + \varepsilon$$

small?
QUASI-GAUSSIAN MANIFESTATIONS OF TURBULENT FLOWS EXAMPLES

VELOCITY FIELD
**LEFT**

One-dimensional probability distributions for velocity fluctuations.

\[ U = 15.7 \text{ m/s}: \bullet, u; \square, v. \]
\[ U = 7.7 \text{ m/s}: \bigcirc, u; \square, v. \]
Solid line is Gaussian distribution.

**RIGHT**

Probability distribution of instantaneous values of \(uv\).

\[ \bullet, U = 15.7 \text{ m/s}; \bigcirc, U = 7.7 \text{ m/s}. \]
Solid curve: \( p(uv) = (1/\pi\sigma_u \sigma_v) K_0(|uv|/\sigma_u \sigma_v) \).
In preparation of looking at the issues of intermittency it is naturally to ask: Can one consider the PDF on the right as a manifestation of intermittency of $uv$ (as is done sometimes)?

Probability distribution of instantaneous values of $uv$. $\bullet$, $U = 15.7$ m/s; $\bigcirc$, $U = 7.7$ m/s. Solid curve: $p(uv) = (1/\pi \sigma_u \sigma_v) K_0(|uv|/\sigma_u \sigma_v)$. 
The probability density of the product $uv$ is given by

$$p(t) = \left[ \frac{(1-r^2)^{1/2}}{\pi} \right] e^{rt} K_0(t)$$

where $t = \frac{uv}{\sigma_u \sigma_v} (1-r^2)$ and $K_0$ is the modified Bessel function of the second kind.

$r$ is the correlation coefficient $\frac{uv}{\sigma_u \sigma_v}$ and $= -0.44$ in a turbulent boundary layer.

The values of $\frac{(uv)^3}{\Gamma(uv)^2}^{3/2}$ and $\frac{(uv)^4}{\Gamma(uv)^2}^{2}$ are close to $-0.50$ and $11.0$, respectively.

For more see **ANTONIA ET AL 1963** (*Physics of Fluids, 15, 956*) and references therein.
The flatness $F_{uv}$ is approximately $11$, which is much larger than $3!$.

\[
\beta_{uw} = \frac{1}{\mu} \begin{vmatrix} R_0 \end{vmatrix} \left( \frac{R_{uw}^3}{1 - R_0^2} \right)^\frac{1}{2} \exp \left( \frac{R_{uw}^3}{1 - R_0^2} \right) \left( \frac{R_{uw}}{1 - R_0} \right) K_0 \left( \frac{R_{uw}}{1 - R_0} \right),
\]

(9)
The oddly shaped probability density distribution of the $uv$ signal is not surprising if one assumes that $u$ and $v$ are two statistically dependent random variables with correlation coefficient $R = -0.44$, each obeying the Gaussian distribution law. The joint-probability-density distribution function of the product $uv$ is

$$P(u_1, u_2) = \frac{1}{2(1-R^2)^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2(1-R^2)} (u_1^2 - 2Ru_1u_2 + u_2^2) \right\}, \quad (8)$$

where $u_1 = u/u_1$, $u_2 = v/v'$ and $R = u_1u_2 = \overline{uv}/u'v' = -0.44$.

After some transformations and integrations, see Lu & Willmarth (1972) for details, the probability density distribution of the normalized $uv/\overline{uv}$ signal can be found from (8). The result is

$$\beta_{uv}(uv/\overline{uv}) = \frac{1}{\pi (1-R^2)^\frac{1}{2}} \exp \left( \frac{R^2uv/\overline{uv}}{1-R^2} \right) K_0 \left( \left| \frac{Ruv/\overline{uv}}{1-R^2} \right| \right), \quad (9)$$

where $K_0$ is the zeroth-order $K$ Bessel function. This distribution is also included in figure 8. The agreement with measurements appears to be satisfactory. Note that, as $uv \to 0$, the Bessel function approaches infinity. Thus, $\beta_{uv} \to \infty$ as $uv \to 0$. The peak at $uv = 0$ in the measured probability density distribution is thus expected. From the shape of this distribution, the intermittent character of the $uv$ signal is expected since most of the time the $uv$ signal will stay around $uv = 0$. 
Time records of the streamwise, \( u \) and normal, \( v \), components of velocity fluctuations, and the Reynolds stress, \( u'v' \) the latter exhibiting an ‘intermittent’ behaviour. Wei & Willmarth 1989
QUASI-GAUSSIAN MANIFESTATIONS OF TURBULENT FLOWS EXAMPLES

VELOCITY INCREMENTS
PDFs of signed longitudinal velocity increments of air jet [21] left and Modane [22] right.

Represented scales (from top to bottom): $\ln(\ell/L) = -6.0069, -5.3137, -4.6206, -3.9274, -3.2343, -2.5411, -1.8480, 0.9246$ for the air jet data and $\ln(\ell/L) = -6.4137, -5.6028, -4.6645, -3.6411, -2.7501, -1.8598, -0.8685, 0.1226$ for the Modane data. All curves are arbitrarily vertically shifted for the sake of clarity.

Modified Fig. 3 from Chevillard et al. / Physica D 218 (2006) 77–82
FIG. 15. Variation of the $\delta u_r$ PDF with $r$ for $R_\lambda = 381$. From the outermost curve, $r_n / \eta = 2^{n-1} dx / \eta = 2.38 \times 2^{n-1}$, $n = 1, \ldots, 10$, where $dx = 2 \pi / 1024$. The inertial range corresponds to $n = 6, 7, 8$. Dotted line: Gaussian.
FIG. 16. Variation of PDF for $\delta v_r$ with $r$ at $R_\lambda = 381$. For various separations $r_n / \eta = 2.38 \times 2^{n-1}$, $n = 1, \ldots, 10$. Curves are for $n=1, \ldots, 10$ from the uppermost, and the inertial range corresponds to $n = 6, 7, 8$. 
QUASI-GAUSSIAN MANIFESTATIONS OF TURBULENT FLOWS EXAMPLES

LESS TRIVIAL EXAMPLES
$\omega \times u = \nabla \alpha + \nabla \times \beta$

PDFs of the angle between the Lamb vector $\omega \times u$ and its potential part for a numerically simulated turbulence and a random velocity field with the same energy spectrum.

Alignments of the vortex stretching vector $W_i = \omega_k S_{ik}$ and the eigenframe of the rate of strain tensor $\lambda_k$. Note the similarity of behaviour between the real and Gaussian fields. Try to explain the tendency $W \perp \lambda_2$ in view of two other ("contradicting") tendencies $\omega \parallel \lambda_2$ and $\omega \parallel W$. 

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**Grid turbulence**

**DNS, Periodic box**

**NSE**

**Gaussian**
Quasi-gaussian manifestations of turbulent flows. Examples. Accelerations
Figure 6.38. The correlation function of the total acceleration $\langle a(x+r)a(x) \rangle$: ◊ – grid turbulence (Hill and Thoroddsen, 1997); □ – DNS (Vedula and Yeung, 1999); —— under zero forth-cumulant assumption – Millionschikov hypothesis (Pinsky et al., 2000); △ – joint Gaussian velocities (Hill and Thoroddsen, 1997). The figure is from Pinsky et al.
FIG. 11. PDFs of $\theta(a_L, a_C)$ at $Re_\lambda$ 38 and 140 (lines A and B). Part (a) of the figure is for DNS, part (b) for GRFs.
JOINT PDFS OF $a_l = \frac{\partial u}{\partial t}$ AND $a_c = (u \cdot \nabla)u$

**NSE, $Re_\lambda = 243$**

**Gaussian**
PDFS of the cosine of the angle between

$\mathbf{a}_l = \frac{\partial \mathbf{u}}{\partial t}$ \hspace{1cm} \text{AND} \hspace{1cm} \mathbf{a}_c = (\mathbf{u} \cdot \nabla) \mathbf{u}$

Field experiment $Re_\lambda = 6800$

PTV $Re_\lambda = 80$

Fig. 11. Theoretical conditional Lagrangian acceleration PDF \( P(a|u) \) (line) and the experimental conditional Lagrangian acceleration PDF (dashed line) at velocities \( u = 0, 0.45, 0.89, 1.3, 1.8, 2.2, 2.7, 3.1 \) (from top to bottom, shifted by repeated factor 0.1 for clarity) by Mordant, et al. Lagrangian acceleration and velocity components \( a \) and \( u \) are normalized to unit variances. Aringazin, A.K 2004, Conditional Lagrangian acceleration statistics in turbulent flows with Gaussian distributed velocities, *Phys. Rev. E*70, 036301/1-8
Fig. 14. Theoretical marginal Lagrangian acceleration probability density function (84) with Gaussian distributed Lagrangian velocity (line), experimental data for the \( R = 690 \) flow (dots) by Crawford et al and the stretched exponential fit (4) (dashed line). Inset: plot of the central part of the curves of Fig. 14.
Obukhov and Yaglom (1951) employed the so-called quasi normal or Millionschikov hypothesis to treat the 2nd order correlations in pressure and its gradient, i.e. 4th order in velocity.
Pressure field exhibit quasi-Gaussian behaviour as many other even order statistics (see Holzer and Eric Siggia 1993 (Phys. Fluids, A 5, 2525 and references therein), such as negatively skewed PDFs for pressure and its gradient. A simple analytical expression can be obtained for the Laplacian of pressure in an isotropic Gaussian velocity field which is pretty close to experimental observations (Spector 1996):

\[ P(x) = \{3^{1/2}5^{5/2}\}/(4\pi) \, x^2 \, e^x \, [K_2(4x) - K_1(4x)], \quad x < 0, \]

which for large \(|x|\) has the asymptotics \(\sim |x|^{1/2} \, e^{-3|x|}\)

\[ P(x) = \{3^{1/2}5^{5/2}\}/(4\pi) \, x^2 \, e^{-|x|} [K_2(4x) + K_1(4x)], \quad x > 0, \]

which for large \(x\) has the asymptotics \(\sim x^{3/2} \, e^{-5x}\).

\[ x = \frac{\nabla^2 p}{\rho \langle \omega^2 \rangle} = \frac{\omega^2 - 2s_{ij} s_{ij}}{2\langle \omega^2 \rangle} \]
According to this hypothesis, the fourth-order velocity correlations are approximately related to the second-order correlations by expressions which are valid for normal (Gaussian) probability distributions. In other words, the hypothesis postulates that fourth-order cumulants can be neglected in comparison with the fourth-order correlation functions. For correlation functions of the form $u_i u_j u'_m u'_n = B_{ij, mn} (r, t)$, this means that we can use the equation

$$
\overline{u_i u_j u'_m u'_n} = \overline{u_i u_j} \cdot \overline{u'_m u'_n} + \overline{u_i u'_m} \cdot \overline{u'_n} + \overline{u_i u'_n} \cdot \overline{u'_m} \quad (18.1)
$$

[see Eq. (4.29) in Sect. 4.3 of Vol. 1]

Monin & Yaglom, vol 2, section 18, p. 242
Should one conclude that turbulence statistics is very close to Gaussian or that quasi-normal approximation is valid (if so in what sense)? Let us have a look at the third order correlations.

**Grid turbulence in air and water**

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Grid turbulence in air and water

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The original Millionschikov hypothesis was formulated for the four order quantities. Here is an example for the six order time correlations.

FIG. 5. Sixth-order time correlations for turbulent velocities.
Relation of Fourth-Order to Second-Order Moments in Stationary Isotropic Turbulence*

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(Received December 14, 1956; revised manuscript received June 24, 1957)

An investigation is made of the hypothesis that the fourth-order moments of the two-time velocity amplitude distribution in stationary, isotropic, incompressible turbulence are related to second-order moments as in a normal distribution. It is concluded that this hypothesis is inconsistent with the equations of motion, and the inconsistency is exhibited as a gross violation of energy conservation in the inertial range. The origin of the inconsistency is discussed. The arguments developed are used to demonstrate inconsistencies in Chandrasekhar's recent theory of turbulence. Qualitative considerations are presented with regard to the consistency of the hypothesis that fourth-order moments of the simultaneous amplitude distribution are related to second-order moments as in a normal distribution. The general validity of this restricted hypothesis is questioned also.
What conclusions can be drawn from such (or similar) “agreement” between real and Gaussian flow fields?

How meaningful is comparison of experiments with “theories” based on quasi-Gaussian stuff?

How meaningful is the agreement between the two. The right result should be for the right reason. Is the reason right in these cases?
...it would be a miracle if the usual procedure of imposing stationarity, truncating the resulting system of equations, and looking for a Gaussian solution, would lead to results much related to physics. RUELLE 1975
General (but very important) remarks.

First we mention that a purely Gaussian velocity field is dynamically impotent. Indeed, in such a field, all the odd moments vanish (therefore they are so convenient in assessing the ‘deviations’ from Gaussianity). This contradicts the Kolomogorov 4/5 law, turns the Karman-Howarth equation* into a linear one (see Monin and Yaglom, 1971), and prevents the production of enstrophy and strain (i.e. dissipation): in a Gaussian velocity field both \( \langle \omega_i \omega_k S_{ik} \rangle \equiv 0 \) and \( \langle S_{ij} S_{jk} S_{ki} \rangle \equiv 0 \). For more see section 6.8 in Tsinober 2001

* \[
\frac{\partial (f \bar{u}^2)}{\partial t} + 2(\bar{u}^2)^{3/2} \left( \frac{\partial \bar{h}}{\partial r} + \frac{4}{r} \bar{h} \right) = 2 \nu \bar{u}^2 \left( \frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right)
\]
Even if the flow field is initially Gaussian, the dynamics of turbulence makes it non-Gaussian with finite rate. This is seen by taking the mean from the equation (dropping the viscous term)

\[
\frac{D}{Dt} \omega_i \omega_j s_{ij} = \omega_j s_{ij} \omega_k s_{ik} - \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} + vt
\]

For a Gaussian velocity field

\[
\langle \omega_i \omega_j s_{ij} \rangle = \left( \omega_j s_{ij} \omega_k s_{ik} \right) - \langle \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} \rangle
\]

and

\[
\langle \omega_j s_{ij} \omega_k s_{ik} \rangle = \frac{1}{6} \langle \omega^2 \rangle^2 > 0.
\]

Note that this latter quantity it is positive pointwise for any vector field. Hence at \( t=0 \)

\[
\left\{ \frac{D}{Dt} \langle \omega_i \omega_j s_{ij} \rangle \right\}_{t=0} = \left\{ \langle \omega_j s_{ij} \omega_k s_{ik} \rangle \right\}_{t=0} > 0,
\]

i.e. at least for a short time interval \( \Delta t \), the mean enstrophy production will become positive. The essential point is that at \( t=0 \), the above relation is precise due to the freedom of the choice of the initial condition. On the other hand, we have seen that, if the initial conditions are Gaussian, the flow ceases to be Gaussian with finite rate. In other words, it is seen directly that turbulence cannot be Gaussian. A similar result is valid for \( \langle S_{ij} S_{jk} S_{ki} \rangle \)
Non-Gaussian manifestations of turbulent flows. Examples
More in the next lectures on intermittency and structure(s)
PDFs of signed longitudinal velocity increments of air jet [21] **LEFT** and Modane [22] **RIGHT**

Represented scales (from top to bottom): $\ln(\ell/L) = -6.0069, -5.3137, -4.6206, -3.9274, -3.2343, -2.5411, -1.8480, 0.9246$ for the air jet data and $\ln(\ell/L) = -6.4137, -5.6028, -4.6645, -3.6411, -2.7501, -1.8598, -0.8685, 0.1226$ for the Modane data. All curves are arbitrarily vertically shifted for the sake of clarity.

**Modified Fig. 3 from** Chevillard et al. / Physica D 218 (2006) 77–82
FIG. 15. Variation of the $\delta u_r$ PDF with $r$ for $R_\lambda = 381$. From the outermost curve, $r_n / \eta = 2^n - 1 dx / \eta = 2.38 \times 2^n - 1$, $n = 1, \ldots, 10$, where $dx = 2 \pi / 1024$. The inertial range corresponds to $n = 6, 7, 8$. Dotted line: Gaussian.
FIG. 16. Variation of PDF for $\delta v_r$ with $r$ at $R_\lambda = 381$ for various separations $r_n / \eta = 2.38 \times 2^{n-1}$, $n = 1, \ldots, 10$. Curves are for $n = 1, \ldots, 10$ from the uppermost, and the inertial range corresponds to $n = 6, 7, 8$. 
\[
\langle (\partial u_1 / \partial x_1)^n \rangle \langle (\partial u_1 / \partial x_1)^2 \rangle^{n/2}
\]

**Grid turbulence**

KLEBANOV AND FRENKIEL 1971

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**LEFT** The Reynolds number variation of the skewness of the streamwise (or longitudinal) velocity derivative $\partial u / \partial x$ from different sources. All the data from Van Atta & Antonia (1980) have been replotted here using open squares without identifying the sources, for which their original paper should be consulted. The new data are explained in the inset.

**RIGHT** The Reynolds number variation of the flatness factor of the streamwise (or longitudinal) velocity derivative $\partial u / \partial x$, from different sources. All the data from Van Atta & Antonia (1980) have been replotted here using open squares without identifying the sources, for which their original paper should be consulted. The new data are explained in the inset.

Figure 7. Comparison of the measured probability density distribution of the non-dimensional velocity gradient, $v_t$, with a Gaussian distribution. •, measured; ———, Gaussian.
**Figure 8.** Comparison of the measured third moment of the probability density distribution of $v_3$ with the third moment of a Gaussian distribution. ●, measured; ——, Gaussian.
FIG. 14. Comparison of fourth-order moments of probability distribution for turbulent velocity gradient.
FIG. 6. Variation of the longitudinal velocity derivative PDF with the Reynolds number.
FIG. 6. Variation of the longitudinal velocity derivative PDF with the Reynolds number.
PDFs of eigenvalues, $\Lambda_k$ of the rate of strain tensor $s_{ij}$

Note the positively skewed PDF of $\Lambda_2$ which is symmetric for a Gaussian velocity field.

Field experiment 2004, Sils-Maria, Switzerland, $Re_\lambda = 6800$
NONLOCALITY OF VORTICITY-STRAIN RELATION

SCATTER PLOT/JPDF. Color shows Lg No. of points.
NOTE THE GREEN CURVE

\[ B_{ij} s_{ij} \quad G_{ij} s_{ij} \quad \omega_{ij} s_{ij} \]
Field experiment, $Re_\lambda = 6800$

DNS of NSE, $Re_\Theta = 690$

PTV, $Re_\lambda = 80$

Chacin et al 2000

$Q = \frac{1}{4} \left( \omega^2 - 2s_{ik}s_{ik} \right)$

- second invariant of the velocity gradient tensor

$R = -\frac{1}{3} \left( s_{ik}s_{km}s_{mi} + \frac{3}{4} \omega_i\omega_k s_{ik} \right)$

- third invariant of the velocity gradient tensor

The first invariant is vanishing as a consequence of incompressibility
ALIGNMENTS BETWEEN THE EIGENFRAME $\lambda_i$ OF THE RATE OF STRAIN TENSOR $S_{ij}$ AND VORTICITY $\omega$

Note that the PDFs for a Gaussian field are ‘flat’: there is no tendency to such alignments, see Shtilman, L., Spector, M. and Tsinober, A. (1993) On some kinematic versus dynamic properties of homogeneous turbulence, *J. Fluid Mech.*, 247, 65–77.

Field experiment; $Re_\lambda = 10^4$  
3D-PTV; $Re_\lambda = 60$
ALIGNMENT OF VORTICITY $\omega$ AND ITS STRETCHING VECTOR $W$

$W_i = \omega_k s_{ik}$

$$\cos(\omega, W) = \omega \cdot W \ (\omega w)^{-1}$$

Note that $\omega \cdot W = \omega_i \omega_k s_{ik}$

$p(y) = 2\sqrt{3(4-y^2)^{-3/2}}$

Gaussian velocity field – symmetric PDF
IN LIEU OF CONCLUSION I
I. The above is closely related to broader issue on the role/relation of kinematics versus dynamics in turbulence

II. More in the next lectures on intermittency and structure(s)