Multirate-Transfer Dual-Porosity Modeling of Gravity Drainage and Imbibition

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Summary

We develop a physically motivated approach to modeling displacement processes in fractured reservoirs. To find matrix/fracture transfer functions in a dual-porosity model, we use analytical expressions for the average recovery as a function of time for gas gravity drainage and countercurrent imbibition. For capillary-controlled displacement, the recovery tends to its ultimate value with an approximately exponential decay (Barenblatt et al. 1990). When gravity dominates, the approach to ultimate recovery is slower and varies as a power law with time (Hagoort 1980). We apply transfer functions based on these expressions for core-scale recovery in field-scale simulation.

To account for heterogeneity in wettability, matrix permeability, and fracture geometry within a single gridblock, we propose a multirate model (Ponting 2004). We allow the matrix to be composed of a series of separate domains in communication with different fracture sets with different rate constants in the transfer function.

We use this methodology to simulate recovery in a Chinese oil field to assess the efficiency of different injection processes. We use a streamline-based formulation that elegantly allows the transfer between fracture and matrix to be accommodated as source terms in the 1D transport equations along streamlines that capture the flow in the fractures (Di Donato et al. 2003; Di Donato and Blunt 2004; Huang et al. 2004).

This approach contrasts with the current Darcy-like formulation for fracture/matrix transfer based on a shape factor (Gilman and Kazemi 1983) that may not give the correct average behavior (Di Donato et al. 2003; Di Donato and Blunt 2004; Huang et al. 2004). Furthermore, we show that recovery is exceptionally sensitive to parameters that describe the physics of the displacement process, highlighting the need to make careful core-scale measurements of recovery.

Introduction

Di Donato et al. (2003) and Di Donato and Blunt (2004) proposed a dual-porosity streamline-based model for simulating flow in fractured reservoirs. Conceptually, the reservoir is composed of two domains: a flowing region with high permeability that represents the fracture network and a stagnant region with low permeability that represents the matrix (Barenblatt et al. 1960; Warren and Root 1963). The streamlines capture flow in the flowing regions, while transfer from fracture to matrix is accommodated as source/sink terms in the transport equations along streamlines. Di Donato et al. (2003) applied this methodology to study capillary-controlled transfer between fracture and matrix and demonstrated that using streamlines allowed multimillion-cell models to be run using standard computing resources. They showed that the run time could be orders of magnitude smaller than equivalent conventional grid-based simulation (Huang et al. 2004). This streamline approach has been applied by other authors (Al-Huthali and Datta-Gupta 2004) who have extended the method to include gravitational effects, gas displacement, and dual-permeability simulation, where there is also flow in the matrix. Thiele et al. (2004) have described a commercial implementation of a streamline dual-porosity model based on the work of Di Donato et al. that efficiently solves the 1D transport equations along streamlines.

In this paper, the streamline-based formulation will be extended to incorporate both capillary- and gravity-driven displacement using transfer functions based on semianalytic solutions to the flow equations. To account for subgridblock variability, we introduce a multirate model that accounts for heterogeneity in the fracture network in each gridblock, following the approach of Ponting (2004). The multirate model allows for transfer between fracture and matrix to occur over different time scales and accommodates situations where there is a variation of fracture spacing or matrix permeabilities that cannot be explicitly captured on the scale of a gridblock (Daly and Mueller 2004). We also describe a more sophisticated model, in which the transfer rates are linked to the parts of the fracture network that are water-saturated. Then, the approach is applied to a fractured Chinese field study to determine the optimal development strategy. We show how fracture geometry, reservoir wettability, and heterogeneity control recovery.

Our approach is to study each displacement process separately and to derive, rigorously, the correct averaged transfer rate from fracture to matrix. This contrasts with the current Darcy-like shape-factor approach (Gilman and Kazemi 1983) that may not capture the correct upscaled properties (Di Donato et al. 2003; Di Donato and Blunt 2004; Huang et al. 2004). Our approach is similar to that pioneered by Terez and Firoozabadi (1999), Civan and Rasmussen (2001), and Civan et al. (1999). They also used a semianalytical approach to derive transfer functions for different recovery processes and proposed a multirate model to capture the subtleties of matrix to fracture flow. Our conceptual model is different: we use multiple rates to capture the influence of distinct fracture sets. The details of our implementation are also different, and we use streamline-based simulation for the field-scale studies.

We will show that the physics of the displacement, rather than simply the fracture geometry (represented by the shape factor), has a significant impact on recovery, which indicates the need to make careful measurements of displacement efficiency at the small scale.

Streamline Formulation

In a dual-porosity simulator, the flowing (fracture) and stagnant (matrix) domains are separately defined. We assume that there is no significant fluid flow between matrix gridblocks. Transport occurs through exchange of fluid from the matrix to the fractures and in the fracture network itself. In the streamline-based dual-porosity model, high-permeability matrix is combined into the flowing fraction conceptually by considering its contribution to the effective permeability of the flowing fraction.

The streamline code assumes incompressible two-phase flow. The volume conservation equations for oil/water flow are:

\[
\frac{\partial S_{wf}}{\partial t} + \mathbf{v}_w \cdot \nabla f_{wf} + \nabla \cdot \mathbf{g} G = -T, \quad \text{(1)}
\]

\[
\frac{\partial S_{wm}}{\partial t} = T, \quad \text{(2)}
\]

where \( T \) is the transfer function. Define a time-of-flight \( \tau(s) \) as the time taken for neutral tracer to move a distance \( s \) along a streamline (Gelhar and Collins 1971):

\[
\tau(s) = \int_0^s \frac{\phi_l}{|W_i|} ds. \quad \text{(3)}
\]

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Then we can transform Eq. 1 into an equation along a streamline as follows:

\[
\frac{\partial S_{sw}}{\partial t} + \frac{\partial f_{sw}}{\partial t} + \frac{1}{\phi_i} \nabla \cdot (gG) = \frac{T}{\phi_i},
\]

(4)

Dual-porosity streamline simulation is similar to standard single-porosity simulation (Batycky et al. 1997) and only differs in its treatment of the 1D conservation equations (Di Donato et al. 2003; Di Donato and Blunt 2004). Both the flowing and stagnant fracture saturations are mapped onto streamlines, and we solve the following two transport equations along each streamline:

\[
\frac{\partial S_{sw}}{\partial t} + \frac{\partial f_{sw}}{\partial t} = \frac{T}{\phi_i},
\]

(5)

\[
\frac{\partial S_{swm}}{\partial t} = \frac{T}{\phi_m}.
\]

(6)

\( S_{swf} \) and \( S_{swm} \) are then mapped back to the grid, and the simulation follows the single-porosity methodology.

**Single-Rate Transfer Functions**

**Capillary-Controlled Imbibition.** A number of authors have performed countercurrent imbibition experiments in which water-wet cores have been surrounded by water (Zhang et al. 1996; Morrow and Mason 2001). The recovery of oil can be matched by a simple exponential function of time (Aronofsky et al. 1958).

We will use a transfer function based on a semianalytical solution for countercurrent imbibition proposed by Barenblatt et al. (1990) reviewed in Appendix A. The recovery, \( R \), is:

\[
R = R_s \left(1 - e^{-\beta t}\right),
\]

(7)

where \( R_s \) is the ultimate recovery. The rate constant \( \beta \) is defined by:

\[
\beta = 3 \sqrt{\frac{K_m \sigma_m \mu \mu_m}{\phi_m L_c^2} \lambda},
\]

(8)

where \( \sigma_m \) is the oil/water interfacial tension. The mobilities \( \lambda = k_i / \mu \) are defined at \( S_{swm} \), the maximum saturation reached in the matrix during imbibition; we assume that the system is not strongly water-wet and so this saturation is lower than 1-\( S_{swm} \) where \( S_{swm} \) is the residual oil saturation in the matrix after waterflooding, including forced displacement. The case where \( S_{swm} = 1 \)—\( S_{swm} \) giving \( \beta = 0 \), has been considered by Tavassoli et al. (2005a). \( J \) is the dimensionless gradient of the capillary pressure at \( S_{swm} \), defined by

\[
\frac{\partial P}{\partial S_{swm}} = \frac{\phi_m}{\sqrt{K_m \sigma_m}} \left[ \frac{\partial f(S_{swm})}{\partial S_{swm}} \right]_{S_{swm} = \beta} = -\frac{\phi_m}{\sqrt{K_m \sigma_m}} J. \]

(9)

The definition of the rate constant is similar to that derived by Zhou et al. (2002), although here we evaluate the mobilities at the end of imbibition.

The problem with the use of the rate constant (Eq. 8) is that it is assumed that the mobilities at the end of imbibition are known. In many cases, these are not measured. Later, we will present an empirical expression for the transfer rate that does not require an estimate of mobility; however, it does involve an empirical parameter to match experiment.

In the analytical analysis of Barenblatt et al. (1990), only a linear system of length \( L_c \) was considered. To account for more complex fracture geometries, we use an effective length \( L_e \) given by (Zhang and Morrow 1996):

\[
L_e^2 = \frac{V}{\sum \frac{A_i}{l_i}}.
\]

(10)

where \( V \) is the matrix block volume, \( A_i \) is the area open to flow in the \( i \)th direction, and \( l_i \) is the distance from the open surface to a no-flow boundary.

If we view \( S_{swm} \) as the average saturation in the matrix, then we can write:

\[
\frac{R}{R_s} = \frac{S_{swm} - S_{swi}}{S_{swm} - S_{swi}},
\]

(11)

where \( S_{swi} \) is the initial water saturation in the matrix. Thus, from Eq. 7:

\[
S_{swm} = S_{swi} + (S_{swm} - S_{swi}) (1 - e^{-\beta t}).
\]

(12)

Then from Eq. 12:

\[
T = \beta \phi_m (S_{swm} - S_{swi}) \text{ for } S_{swi} > 0
\]

\[
= 0 \text{ for } S_{swi} = 0.
\]

(13)

We assume that the transfer function is independent of the flowing saturation, as long as \( S_{swi} > 0 \). This is consistent with assuming that the capillary pressure in the low-permeability matrix is much higher than in the fractures—the driving force for imbibition is the matrix capillary pressure, and imbibition continues until the matrix and fracture capillary pressures are equal—when \( S_{swi} = 0 \) and \( S_{swm} = S_{swi} \). Thus, from Eq. 13, we take:

\[
T = \beta \phi_m (S_{swm} - S_{swi}) \text{ for } S_{swi} > 0
\]

\[
= 0 \text{ for } S_{swi} = 0.
\]

(14)

The transfer function varies linearly with matrix saturation. We will call this the linear transfer function.

Note that the transfer rate constant \( \beta \) is a function of the fracture spacing, interfacial tension, and matrix permeability. It is also a strong function of wettability and viscosity contrast through the introduction of mobilities in Eq. 8. In particular, for mixed-wet systems, the transfer rate can be 100 to 10,000 times lower than for similar strongly water-wet media (Morrow and Mason 2001) because of very low water mobility (Behbahani and Blunt 2005). Note that contact angle does not appear explicitly in the expression for the rate constant—its effect on the rate through changing mobility is complex and not readily captured in any simple dependence. Furthermore, in the pore space there is normally a range of contact angles, so there is no single representative value (Behbahani and Blunt 2005).

Our suggestion in fracture modeling is to input an appropriate rate constant \( \beta \) dependent on the matrix, fracture, and fluid properties. The traditional approach of using a shape factor to represent fracture geometry is physically opaque because it does not relate easily to a transfer rate (Gilman and Kazemi 1983). To account for wettability and mobility in a conventional model, the matrix relative permeabilities and capillary pressure can be adjusted. However, the impact of such changes on recovery is obscured by the unnecessarily complex nonlinearity of the resultant transfer function that simply adds computational difficulty while failing to represent the correct physics.

**Gravity Drainage.** We also consider gas injection, where gas overcomes a capillary barrier to enter a matrix block and then oil drains until the gas and oil are in capillary/gravity equilibrium. We assume that the gas is incompressible and that water is everywhere at its irreducible saturation. The assumption of gas incompressibility is only reasonable if \( P \Delta P \approx 1 \), where \( P \) is the average pressure and \( \Delta P \) is the pressure drop across the reservoir, or the pressure drop during the field life. Therefore, this may be applicable to gas injection processes, but is not suitable to model gas cap expansion during primary production. However, the transfer function we derive could still be used in a compressible code—it is not limited to the incompressible streamline model used in this paper. We treat each matrix block separately and do not allow for capillary continuity between blocks (Horie et al. 1990; Por et al. 1989). In the governing transport Eqs. 5 and 6, we simply substitute gas for the water phase.

As before, we base our transfer function on analytical solutions to the flow equations in an idealized geometry—in this case, ver-
tical gravity drainage of oil in the presence of gas. The ratio of gravitational to capillary forces in each matrix block, \( r \), is defined as:

\[
\frac{r}{H} = \frac{\Delta \rho g L}{\sigma_{ng} v^*} \frac{K_m}{\phi_m},
\]

where \( L \) is the block height, \( H \) is the amount of capillary rise of oil in the presence of gas, and \( v^* \) is the dimensionless entry pressure for gas invasion into oil. The density difference \( \Delta \rho = \rho_o - \rho_g \).

For \( r \ll 1 \), gas cannot enter the block and there is no gravity drainage. For \( r \gg 1 \), gravitational forces dominate and we can write a solution for the average recovery at late times due to Hagoort (1980):

\[
R = 1 - \frac{1}{S_g^{*}} \left( \frac{\alpha}{(1 - a^*)} \right) \frac{S_g}{S_g^{*}},
\]

where the rate constant \( \alpha \) is

\[
\alpha = \frac{a K_m}{(1 - a^*)} \frac{\sigma_{ng}}{\phi_m L},
\]

and the exponent \( a \) (it is assumed that \( a \gg 1 \)) is defined such that at low oil saturation the oil relative permeability is

\[
k_{ro} = k_{ro}^{max} (S_{em} - S_{orgm})^a,
\]

where \( S_{orgm} \) is the residual oil saturation in the presence of gas. Note that we define the ultimate recovery by finding the maximum gravitational to capillary forces in each matrix block, \( L \).

Coupling With the Fracture Fractional Flow. The multirate model leads to the regions of the matrix in contact with fractures having the largest surface area and consequently the highest transfer rates being drained first, while the regions with the lowest transfer rates are drained last. The lowest transfer rates may come from the largest aperture fractures. Therefore, during waterflooding, the largest fractures become saturated by water first and the smaller aperture fractures, with the largest surface area, are only

Extensions to the Approach. It is possible to extend this approach to study other physical processes, such as transfer caused by fluid expansion [implicitly included in the traditional formulation (Gilman and Kazemi 1983; Barenblatt et al. 1960; Warren and Root 1963)], displacement caused by a combination of capillary and gravitational forces, and interaction between blocks due to capillary continuity (Horie et al. 1990; Por et al. 1989). We treat each matrix block as independent.

**Multirate Transfer**

**Simple Model.** The single-rate model assumes that within each gridblock there is a single set of fracture and matrix properties. In many cases there may be several different fracture sets with different spacing within a single gridblock, and the matrix permeability may have small-scale variations (Daly and Mueller 2004). Lumping all the transfer between fracture and matrix in a single rate may therefore be inaccurate and, as we show later, will tend to overpredict oil recovery at late time because a single-rate model will not adequately account for the regions of the matrix that are in poor communication with the fractures.

**Fig. 1** illustrates intersecting fracture sets in a limestone outcrop near the Bristol Channel in England (Belayneh and Cosgrove 2004). In dual-porosity simulation, a single gridblock tens of hundreds of meters in extent would include many fractures. We would like to capture the transfer from these different fracture sets.

To account for subgridblock heterogeneity, we follow the approach of Ponting (2004) and propose a multirate model wherein the matrix is assumed to be composed of \( N \) domains, each with different transfer rates. The transfer into each of the domains is handled separately:

\[
T = \sum_{k=1}^{N} T_k; \quad \phi_m = \sum_{k=1}^{N} \phi_m; \quad \phi_m S_{em} = \sum_{k=1}^{N} \phi_m S_{emk};
\]

\[
T_k = \beta_k \phi_m (S_{emk} - S_{em}), \quad S_{em} > 0
\]

\[
= 0, \quad S_{em} = 0
\]

Mathematically, Eqs. 24 through 26 with two or three domains are similar to models proposed by other authors to describe multiple transfer rates in a single block, although the physical interpretation is very different (Daly and Mueller 2004; Terez and Firouzabadi 1999; Civan and Rasmussen 2001; Civan et al. 1999).

The same equations can be used for gas injection with water substituted by gas, and the transfer function is

\[
T_k = \frac{\alpha_k \phi_m}{a_k - 1} (S_{emk} - S_{em})^{a_k} S_{emk} > 0
\]

\[
= 0, \quad S_{emk} = 0
\]

**Fig. 1**—Intersecting fracture sets in a limestone outcrop in the Bristol channel (Belayneh and Cosgrove 2004). The fracture spacing is between 0.3 and 1 m. We use a multirate model to accommodate transfer from different fracture sets.
We propose another model where in the upscaled fracture fractional flow curve different average saturation regions represent the flooding of different fracture sets. Corresponding to each fracture set would be transfer into the matrix—in this scenario, the smaller matrix blocks, with the highest transfer rates, may be recovered last because imbibition will not start until their surrounding fractures are waterflooded. This would occur only when the average fracture saturation was sufficiently high. The multirate model, where the portions of the matrix that are recovered are controlled by the average fracture saturation that in turn represents which fracture sets are flooded.

Mathematically, the fractional flow curve is divided into $N$ sections $S_{wfk}(k = 1, \ldots, N)$ by definition $S_{wfN} = 1$ and $S_{wf0} = 0$, assuming no residual saturation in the fractures with corresponding matrix porosities and transfer rates. The expressions are only a simple extension of Eqs. 24 through 26 as described previously, but now have the structure to represent very complex interactions between the single-rate version of this model and matrix. We also allow for reduced transfer if a fracture set is not completely saturated, assuming now that locally fractures are either fully saturated or dry and that fractures at the small scale are rarely partially saturated. For gravity drainage, Eq. (29) becomes:

$$T_k = \frac{\alpha k}{a_k - 1} (S^g_k - S_{wmk})^{\alpha_k} \quad S^g_k \geq S_{wf}$$

$$= \frac{\alpha k}{a_k - 1} S^g_k - S_{wfk} \left( S^g_k - \frac{S_{wf} - S_{wfk-1}}{S_{wf} - S_{wfk}} \right)^{\alpha_k} \quad S^g_k \geq S_{wf} > S_{wfk-1}$$

$$= 0 \quad S^g_k > S_{wf} > S_{wfk-1}$$

$$S_{wf} \leq S_{wfk-1}$$

The numerical implementation of these transfer functions in the 1D transport equations along streamlines is described in Appendix B. In streamline-based simulation, most of the run time is taken updating the pressure field; in comparison, solving the transport equations is fast. As a consequence, adding a multirate model increases the run times by less than 1%.

Reservoir Description

To illustrate our simulation approach, we will run simulations on a model of the Liu7 oil field. This small field—the estimated initial oil in place is 22 million m$^3$—is located in a remote, mountainous region of northwest China and was discovered in 1980. The hydrocarbons are located 4000 to 5000 m below the ground surface; there is no connecting aquifer or gas cap. The reservoir is composed of layers of extensively fractured dolomite with 31 major partially-sealing faults crossing the reservoir. There are three main fracture sets: vertical fractures within layers with a density of 0.2 to 45 m$^{-1}$ (average 4.1 m$^{-1}$); oblique fractures with a density of 0.2 to 5.7 m$^{-1}$ (average 1.3 m$^{-1}$); and vertical fractures between layers of density 0.1 to 3 m$^{-1}$ (average also 1.3 m$^{-1}$). The fractures have typical apertures in the range of 15 to 200 $\mu$m and provide good connectivity in all directions. The overall fracture porosity is approximately 0.4%. The matrix porosity lies in the range of 10 to 18% with permeabilities between 1 and 10 mD.

The reservoir description shown in Fig. 3 illustrates a layered system with wide variations in fracture permeability both within and between layers, with a tendency for the bottom of the field to connect by water later. Thus the multirate model does not capture the proper sequence of displacement within a gridblock.

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$$= \frac{\alpha k}{a_k - 1} S^g_k - S_{wfk} \left( S^g_k - \frac{S_{wf} - S_{wfk-1}}{S_{wf} - S_{wfk}} \right)^{\alpha_k} \quad S^g_k \geq S_{wf} > S_{wfk-1}$$

$$= 0 \quad S^g_k > S_{wf} > S_{wfk-1}$$

$$S_{wf} \leq S_{wfk-1}$$

Fig. 2—Schematic of the conceptual model used to develop our dual-porosity model. The fracture network is considered to be composed of different fracture sets with different spacing (as in Fig. 1); in this example, three fracture sets are present. The left-hand figure would represent a single gridblock in a field-scale simulation. The average fracture fractional flow (right-hand figure) has contributions from all fracture sets—the larger aperture fractures are filled first at low average saturation, with smaller aperture fractures filled at higher saturations. Associated with each fracture set is a different matrix/fracture transfer rate. We represent the complex interaction between fracture and matrix by a series of linear functions representing transfer from different fracture sets at different rates.
have higher permeability. The fracture permeability varies from over 4 D to a minimum of 2 mD (the average matrix permeability). The model uses a Cartesian grid with 116 x 74 x 126 blocks of size 40 m by 39 m with a variable height of average 4 m. The average x direction permeability (in the direction of the main fracture set) is 440 mD, and the average y direction permeability is 150 mD, while the average z direction (vertical) permeability is 27 mD —this is a value averaged over several layers; within each layer, the x and z direction permeabilities are similar because they are controlled by vertically oriented fractures. There are a total of 102,000 active gridblocks; the model has been downscaled from an original simulation model with only 13,000 active blocks. For waterflooding simulations, we have five vertical injectors with a constant rate of 2,000 m³/day each and 13 vertical producers with a constant bottomhole pressure of 17 MPa, as are currently used in the field. For gas injection, we use two vertical injectors near the top of the field with a pressure of 17 MPa. The average density, we find 1/40.4 m² and β = 8×10⁻⁸ s⁻¹ = 1/140 days⁻¹. This means that imbibition will take approximately 140 days to give significant recovery. This timescale is short relative to the typical periods for water injection and implies that imbibition will be an effective displacement process. However, the analysis relies on using a single effective rate, which is high because the average fracture density is large. A more conventional analysis using a shape factor would come to the same conclusion. In reality, though, there is a huge variation in the fracture spacing from 2.5 cm to 5 m within each simulation gridblock that should be reflected in the model.

To capture the observed range in fracture spacing, we consider two multirate models. In both cases, we constrain the parameters such that \( \phi \beta = \sum_{j=1}^{N} \phi_{av} \beta_{j} \) where \( \beta_{j} \) is the rate constant for the single-rate case, \( \beta_{j} = 8 \times 10^{-8} \) s⁻¹. Then, the initial imbibition rate

\[
T = \sum_{j=1}^{N} T_{j} = \left(1 - S_{wmi} - S_{mi}ight) \sum_{j=1}^{N} \beta_{j} \phi_{av} = \left(1 - S_{wmi} - S_{mi}\right) \phi_{av} \beta_{0}
\]

is the same as for the single-rate model. This allows us to compare the impact of the multirate model on recovery—by definition, recovery will be similar to begin with, as we demonstrate later, will show considerable variations at later times. The first, two-rate model has one third of each gridblock containing matrix with fractures whose spacing is approximately 30 times larger than average—representing a fracture spacing close to the maximum observed value of 5 m. We then have \( \phi_{av} = 0.043, \beta_{1} = 8 \times 10^{-11} \) s⁻¹ (1,000 times less than \( \beta_{0j} \)), \( \phi_{av} = 0.087, \beta_{2} = 1.2 \times 10^{-11} \) s⁻¹ (chosen to obey Eq. 33). The second, three-rate model is similar, with one third of the matrix contacted by fractures with a spacing of 5 m and another third having a fracture spacing of approximately 1 m: \( \phi_{av} = 0.043, \beta_{1} = 8 \times 10^{-11} \) s⁻¹ (30 times less than \( \beta_{0j} \)), \( \phi_{av} = 0.043, \beta_{2} = 2.7 \times 10^{-9} \) s⁻¹, and \( \beta_{3} = 2.4 \times 10^{-9} \) s⁻¹. For a model coupled to the fracture fractional flow, we use the same three-rate model with \( S_{wmi} = 0.13 \) and \( S_{wmi} = 0.23 \).

To reduce our uncertainty in the assignment of transfer rates, we need to characterize the variability in fracture spacing accurately. This could be done through a careful analysis of fracture patterns [see, for instance, Daly and Mueller (2004)]. A more rigorous approach would be to simulate displacement through a realistic fracture geometry at the gridblock scale, where all the

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**Parameters for Imbibition Modeling.** We now discuss how to assign parameters in the transfer functions. As mentioned earlier, the problem with Eq. 8 is that for this case, we do not know the mobilities at the end of imbibition because relative permeabilities were not measured. Instead, we define a transfer rate based on the correlation proposed by Zhang et al. (1996) that works well for cases where the oil and water viscosities are not very different from each other:

\[
\beta = b \sqrt{\frac{K_m}{\phi_m \mu_o} L_s^2 \sqrt{\frac{1}{\mu_w}}}, \quad \text{.......................... (32)}
\]

where the parameter \( b \) depends on the reservoir wettability and varies from 0.05 for strongly water-wet systems (Zhang et al. 1996) to 10⁻⁵ and lower for mixed-wet systems (Morrow and Mason 2001). Measurements of waterflooding capillary pressure for this field indicate an Amott water wettability index of approximately 0.5—the amount of water displaced by spontaneous imbibition is similar to that displaced by subsequent forced water injection. Experimentally (albeit in sandstones) imbibition for such systems is approximately 100 times slower than for water-wet media (Morrow and Mason 2001; Bebahani and Blunt 2005). Hence, we take \( b = 5 \times 10^{-5} \). Then in Eq. 32 and using the parameters listed in Table 1, we find that \( \beta = 2.0 \times 10^{-6} L_s^2 \) s⁻¹ with \( L_s \) measured in m. Using Eq. 10, where \( I \) is half the fracture spacing, and assuming that we have three independent fracture sets with the average density, we find \( 1/L_s^2 = 40.4 \) m² and \( \beta = 8 \times 10^{-8} \) s⁻¹ = 1/140 days⁻¹. This means that imbibition will take approximately 140 days to give significant recovery. This timescale is short relative to the typical periods for water injection and implies that imbibition will be an effective displacement process. However, the analysis relies on using a single effective rate, which is high because the average fracture density is large. A more conventional analysis using a shape factor would come to the same conclusion.

In reality, though, there is a huge variation in the fracture spacing from 2.5 cm to 5 m within each simulation gridblock that should be reflected in the model.

To capture the observed range in fracture spacing, we consider two multirate models. In both cases, we constrain the parameters such that \( \phi \beta = \sum_{j=1}^{N} \phi_{av} \beta_{j} \) where \( \beta_{j} \) is the rate constant for the single-rate case, \( \beta_{j} = 8 \times 10^{-8} \) s⁻¹. Then, the initial imbibition rate

\[
T = \sum_{j=1}^{N} T_{j} = \left(1 - S_{wmi} - S_{mi}\right) \sum_{j=1}^{N} \beta_{j} \phi_{av} = \left(1 - S_{wmi} - S_{mi}\right) \phi_{av} \beta_{0}
\]

is the same as for the single-rate model. This allows us to compare the impact of the multirate model on recovery—by definition, recovery will be similar to begin with, as we demonstrate later, will show considerable variations at later times. The first, two-rate model has one third of each gridblock containing matrix with fractures whose spacing is approximately 30 times larger than average—representing a fracture spacing close to the maximum observed value of 5 m. We then have \( \phi_{av} = 0.043, \beta_{1} = 8 \times 10^{-11} \) s⁻¹ (1,000 times less than \( \beta_{0j} \)), \( \phi_{av} = 0.087, \beta_{2} = 1.2 \times 10^{-11} \) s⁻¹ (chosen to obey Eq. 33). The second, three-rate model is similar, with one third of the matrix contacted by fractures with a spacing of 5 m and another third having a fracture spacing of approximately 1 m: \( \phi_{av} = 0.043, \beta_{1} = 8 \times 10^{-11} \) s⁻¹ (30 times less than \( \beta_{0j} \)), \( \phi_{av} = 0.043, \beta_{2} = 2.7 \times 10^{-9} \) s⁻¹, and \( \beta_{3} = 2.4 \times 10^{-9} \) s⁻¹. For a model coupled to the fracture fractional flow, we use the same three-rate model with \( S_{wmi} = 0.13 \) and \( S_{wmi} = 0.23 \).

To reduce our uncertainty in the assignment of transfer rates, we need to characterize the variability in fracture spacing accurately. This could be done through a careful analysis of fracture patterns [see, for instance, Daly and Mueller (2004)]. A more rigorous approach would be to simulate displacement through a realistic fracture geometry at the gridblock scale, where all the

---

**Table 1—Parameters Used in the Simulations**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi_{w} )</td>
<td>0.004</td>
</tr>
<tr>
<td>( K_m )</td>
<td>( 2 \times 10^{-16} ) m²</td>
</tr>
<tr>
<td>( \rho_o )</td>
<td>( 1.2 \times 10^{-3} ) Pa.s</td>
</tr>
<tr>
<td>( \rho_w )</td>
<td>850 Kg m⁻³</td>
</tr>
<tr>
<td>( \varepsilon_{w} )</td>
<td>0.03 N m⁻¹</td>
</tr>
<tr>
<td>( \epsilon_{o} )</td>
<td>0.23</td>
</tr>
<tr>
<td>( L_c )</td>
<td>2.4 m</td>
</tr>
<tr>
<td>( \phi_{m} )</td>
<td>0.13</td>
</tr>
<tr>
<td>( a )</td>
<td>2, 4; ( k_{im} ) = 1</td>
</tr>
<tr>
<td>( S_{wmi} )</td>
<td>0.544</td>
</tr>
<tr>
<td>( \mu_{w} )</td>
<td>( 3 \times 10^{10} ) Pa.s</td>
</tr>
<tr>
<td>( \rho_{w} )</td>
<td>( 1026 ) Kg m⁻³</td>
</tr>
<tr>
<td>( \rho_{o} )</td>
<td>( 250 ) Kg m⁻³</td>
</tr>
<tr>
<td>( \sigma_{w} )</td>
<td>( 0.011 ) N m⁻¹</td>
</tr>
<tr>
<td>( \sigma_{o} )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

---

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fractures are explicitly modeled. From this, the effective fracture fractional flow could be computed and rate constants for transfer determined to match the observed average recovery in the discrete fracture model. While the tools for such an upsampling approach have been developed (Matthi et al. 2005), we have not performed this analysis for this study. Mathematically, we would be reproducing average recovery by a series of linear functions, which should be a reliable and robust approach. In this work, we use plausible values of parameters based on experimental data, and the results illustrate the uncertainty in our performance predictions from using different models.

Parameters for Gravity Drainage. For gravity drainage, we only consider a single-rate model. The transfer rate is less sensitive to variations in gridblock size than in imbibition—α is independent of areal fracture spacing and only scales as 1/L as against 1/L^2 for β. Therefore, consideration of multirate models has less impact than in imbibition. The base case has the height of the matrix block, L = 4 m, corresponding to the height of each gridblock, and the oil relative permeability exponent α = 2. Using Eq. 15 and the parameters listed in Table 1, we find r = 9.73, so gravitational forces dominate over capillary effects. The rate constant α in Eq. 17 is 3×10^-8 s^-1 = 1 year^-1. We ran two other cases for comparison. Because there is layering within each gridblock, we consider a case with L = 2 m (r = 4.86 and α = 6×10^-8 s^-1). Also, we assess the impact of oil relative permeability by running a case with α = 4 and L = 4 m (r = 9.73 and α = 2.8×10^-7 s^-1).

Results

Comparison With Grid-Based Simulation. To check our streamline method, we first compared the results with a conventional, commercial, grid-based code. We used the original reservoir description with 13,000 active blocks that were the same size as our downscaled model—the reservoir was overall approximately 10 times smaller. We set an average transfer rate βav = 8×10^-7 s^-1. In a conventional model, we need to specify a shape factor. We found a shape factor in the conventional model that gave the same transfer rate for an average saturation as our linear model; a description of how this is done is given by Di Donato et al. (2003). The shape factor is 0.056 m^-2. Note that this is a very different value than would be inferred from the fracture geometry alone (approximately 40 m^-2). The value we used implicitly includes the effects of rock and fluid properties as well as fracture spacing. Fig. 4 demonstrates that when given the same transfer rates, grid-based and streamline-based models give very similar results; moreover, using either the linear or the conventional models for transfer also gives virtually identical predictions if an appropriate shape factor is used. This agrees with a comprehensive series of comparison studies performed by Huang et al. (2004). However, using a shape factor based on fracture spacing alone would not be correct.

Imbibition. Fig. 5 shows horizontal slices through the field showing the fracture and matrix saturations after 1,800 days of water-flooding for the single-rate model, the three-rate model, and for a water-wet case. The water-wet system allows more imbibition, having S_w* = 1−S_m = 0.77 and b = 0.05 with a single rate of β = 8×10^-6 s^-1. In all cases, water breaks through very rapidly as it channels through the fractures, and the water-cut is over 90% beyond 2,000 days. The volume swept by the water in the fractures is controlled by the reservoir heterogeneity. Where there is water in the fracture, imbibition can occur and oil is recovered from the matrix.

Fig. 6 shows the total oil rate from all wells for the one-, two-, and three-rate models. The single-rate model gives higher recoveries because imbibition is a rapid process—taking on the order of 140 days—and regions of the reservoir contacted by water in the fractures see recovery from the associated matrix.

For the two- and three-rate models, one-third of the matrix has an imbibition time 1,000 times larger, of approximately 140,000 days (400 years), and so effectively this portion of the matrix is never recovered. This results in lower matrix water saturations (see Fig. 5). In the three-rate model, the imbibition time for the second region is approximately 4,200 days, which is similar to the overall time for the simulation, allowing some of this section of the matrix to be recovered. This still gives a slightly lower recovery than the two-rate model, where recovery is rapid in two-thirds of the matrix.

Fig. 7 shows the oil rate for the model coupled to the fractional flow (the fff model). Also shown for comparison is the Kazemi model—this is the fractional-flow model with a single rate βav that corresponds to the model proposed by Kazemi et al. (1992). Coupling with the fractional flow leads to even lower recoveries because rapid transfer only occurs once the fracture saturation is above ½. Interestingly, though, the Kazemi model gives a higher recovery than the equivalent single-rate model (Fig. 5) even though, by definition, the transfer rate in this model is always the same or lower (see Eq. 29). This has been discussed previously (Di Donato et al. 2003; Di Donato and Blunt 2004)—the Kazemi model predicts a more smeared-out saturation profile in the fractures, which lessens the apparent mobility contrast between injected and displaced fluids, leading to a higher overall sweep efficiency. Because in both cases imbibition is rapid, the improved sweep efficiency more than compensates for the slightly slower transfer rates in the Kazemi model.

Fig. 8 illustrates the effect of wettability on the results. The water-wet case gives higher recovery because more oil can be recovered from the matrix by spontaneous imbibition (S_w* is higher, see Fig. 5). Note that in Fig. 5, fewer of the fractures are swept by water in the water-wet case. This is because the large amount of imbibition slows down the advance of water in the fractures. Water is continuing to invade more and more of the fractures after 2,000 days, resulting in the higher oil rate evident in Fig. 7. Once water occupies the fracture, imbibition is exceptionally rapid, with a time constant of only 1.4 days.

Fig. 9 shows the recovery curves for the single-rate, three-rate, fff, and water-wet cases as a function of pore volumes injected. One pore volume injected (including fracture and matrix volumes) corresponds to nearly 10,000 days. The water-wet case gives the best recovery because more water can enter the matrix—it is also the only case wherein recovery continues beyond 1 pore volume injected because more of the fractures continue to be swept. Including a plausible amount of heterogeneity in the subgrid-scale fracture spacing leads to much less favorable predictions of recovery.

Gravity Drainage. Fig. 10 shows a vertical slice through the field at two different times for simulations of gravity drainage using the base-case model. Gas is injected at the top of the field and slowly displaces oil downward. Where gas has entered the fractures, slow drainage allows oil to be recovered from the matrix.
Fig. 11 shows the recovery curves for the three cases considered. Notice that in contrast to imbibition, recovery continues at late times. This is a result of the power-law functional form for recovery (Eq. 16). The oil relative permeability has a dramatic impact—changing the exponent $a$ in Eq. 18 from 2 to 4 results in approximately half the recovery, even though the rate constant in the latter case, Eq. 17, is higher. For $a=4$, the oil relative permeability is much lower at low oil saturation and gravity drainage is

Fig. 5—Horizontal slices through the reservoir showing the fracture (left) and matrix (right) saturations after 1,800 days of waterflooding. The top figure (a) shows the single-rate model, the middle figure (b) the three-rate model, and the bottom figure (c) a water-wet case.

Fig. 6—Oil rate from all wells for simulations of imbibition for the one-, two-, and three-rate models. In all cases, water breakthrough is very rapid, but the multirate models give lower recovery because portions of the matrix are effectively inaccessible to water over the time scale of the simulation.

Fig. 7—Oil rate from all wells for simulations of imbibition for different models. The fff model refers to a case where the transfer rates are coupled to the fracture fractional flow. The Kazemi model is equivalent to the fff model with a single average transfer rate.
1. To model displacement in a fractured reservoir, use transfer as follows:
The main conclusions and recommendations of this work are
Conclusions
an effect on performance as the fracture geometry.

very slow. Decreasing the block height $L$ also leads to lower recovery by reducing the fraction of oil that can be recovered from each block. However, overall, the recoveries are more favorable than for waterflooding.

Recap of the Field Results
For this field, gravity drainage is more efficient than waterflooding. This is because we have a tilted reservoir where an artificial gas cap can be maintained. The ultimate recovery for gravity drainage is high, giving final recoveries of up to 0.4. The power-law recovery in each matrix block allows oil to continue to be recovered, albeit slowly, for long times. The dolomite rock is mixed-wet, and only half the oil initially in place can be recovered by spontaneous imbibition. This, coupled with a poor sweep efficiency in the very heterogeneous fracture network, makes water-flood recoveries rather low.

The recovery is controlled by both the distribution of fracture permeability and the matrix properties—in particular, the matrix wettability (for waterflooding) and oil relative permeability (for gravity drainage) control the recovery rate and have as significant an effect on performance as the fracture geometry.

Conclusions
The main conclusions and recommendations of this work are as follows:
1. To model displacement in a fractured reservoir, use transfer functions based on theoretical analysis backed up by experiment; simply using the traditional shape-factor approach may be inaccurate because it does not explicitly capture the correct averaged behavior.
2. View transfer in terms of a rate, and estimate its value carefully. The rate depends on both the fracture geometry (conventionally captured through a shape factor) and the physics of the process. The recovery predictions are very sensitive to the rate constant chosen, which suggests that core-scale measurements of the displacement processes of interest should be performed to reduce uncertainties in recovery estimates.
3. To account for subgridblock heterogeneity, a multirate model allows the effects of different fracture sets to be accommodated. Assuming transfer at a single averaged rate will tend to overestimate recovery.

Nomenclature

\[ a = \text{oil relative permeability exponent, dimensionless} \]
\[ A = \text{area, } L^2, m^2 \]
\[ A_i = \text{area open to flow in the } i\text{th direction, } L^2, m^2 \]
\[ b = \text{dimensionless rate constant} \]
\[ f = \text{fractional flow, dimensionless} \]
\[ f_{wf} = \text{fractional flow in fracture, dimensionless} \]
\[ F = \text{fraction, dimensionless} \]
\[ g = \text{acceleration because of gravity, } Lt^{-2}, \text{ms}^{-2} \]
\[ G = \text{gravity fractional flow, dimensionless} \]
\[ H = \text{capillary rise, } L, m \]
\[ J^* = \text{entry dimensionless capillary pressure} \]
\[ J' = \text{derivative of } J \text{ function} \]
\[ k_r = \text{relative permeability} \]
\[ K = \text{permeability, } L^2, m^2 \text{ or D} \]
\[ K_f = \text{fracture permeability, } L^2, m^2 \text{ or D} \]
\[ K_m = \text{matrix permeability, } L^2, m^2 \text{ or D} \]
\[ L, l = \text{length, } L, m \]
\[ L_e = \text{effective length, } L, m \]
\[ P_c = \text{capillary pressure, } mL^{-1}t^{-2}, \text{Pa} \]
\[ P_{cm} = \text{matrix capillary pressure, } mL^{-1}t^{-2}, \text{Pa} \]
\[ Q = \text{injection rate, } L^3t^{-1}, \text{m}^3s^{-1} \]
\[ r = \text{gravity/capillary ratio, dimensionless} \]
\[ R = \text{recovery} \]
\[ R_u = \text{ultimate recovery} \]
\[ S = \text{saturation, dimensionless} \]
\[ S^* = \text{maximum saturation of invading fluid} \]
\[ S_g = \text{gas saturation, dimensionless} \]
\[ S_{gfk} = \text{fracture gas saturation associated with the } k\text{th fracture set, dimensionless} \]
\[ S_{om} = \text{matrix gas saturation, dimensionless} \]
\[ S_{genk} = \text{matrix gas saturation associated with the } k\text{th fracture set, dimensionless} \]
\[ S_{orr} = \text{matrix residual oil saturation, dimensionless} \]
\[ S_{cor} = \text{residual oil saturation, dimensionless} \]
\[ S_{orgm} = \text{matrix residual gas saturation, dimensionless} \]
\[ S_w = \text{water saturation, dimensionless} \]
\[ S_w^* = \text{maximum saturation of invading water} \]
\[ S_{swf} = \text{fracture water saturation, dimensionless} \]
\[ S_{swf_k} = \text{fracture water saturation associated with the } k\text{th fracture set, dimensionless} \]
\[ S_{swf_k}^* = \text{maximum saturation of invading water in the } k\text{th fracture set} \]
\[ S_{swm} = \text{matrix water saturation, dimensionless} \]
\[ S_{swm} = \text{matrix irreducible water saturation, dimensionless} \]
\[ S_{swm} = \text{matrix water saturation associated with the } k\text{th fracture set, dimensionless} \]
\[ t = \text{time, } T, s \]
\[ t_D = \text{dimensionless time} \]
\[ T = \text{transfer function, } T^{-1}, s^{-1} \]
\[ T_k = \text{transfer function in domain } k, T^{-1}, s^{-1} \]
\[ v_t = \text{total velocity, } LT^{-1}, \text{ms}^{-1} \]
\[ V = \text{volume, } L^3, m^3 \]
Fig. 10—Vertical slices through the reservoir showing the fracture (left) and matrix (right) gas saturations for simulations of gravity drainage at two different times. 1 PVI corresponds to almost 10,000 days.

\[ x = \text{distance to the boundary}, \ m \]
\[ x_D = \text{dimensionless length}, \ x/L \]
\[ x_D^* = \text{distance of water moved into the block} \]
\[ \alpha = \text{rate in gravity transfer function}, \ T^{-1}, s^{-1} \]
\[ \alpha_k = \text{rate in gravity transfer function in domain } k, \ T^{-1}, s^{-1} \]
\[ \beta = \text{imbibition transfer rate}, \ T^{-1}, s^{-1} \]
\[ \beta_k = \text{imbibition transfer rate in domain } k, \ T^{-1}, s^{-1} \]
\[ \lambda = \text{mobility}, \ M^{-1}Lt, \ Pa^{-1}s^{-1} \]
\[ \lambda_o = \text{oil phase mobility}, \ M^{-1}Lt, \ Pa^{-1}s^{-1} \]
\[ \lambda_i = \text{total mobility}, \ M^{-1}Lt, \ Pa^{-1}s^{-1} \]
\[ \lambda_w = \text{water phase mobility}, \ M^{-1}Lt, \ Pa^{-1}s^{-1} \]
\[ \mu = \text{viscosity}, \ ML^{-1}t^{-1}, \ Pa.s \]
\[ \mu_g = \text{gas viscosity}, \ ML^{-1}t^{-1}, \ Pa.s \]
\[ \mu_o = \text{oil viscosity}, \ ML^{-1}t^{-1}, \ Pa.s \]
\[ \mu_w = \text{water viscosity}, \ ML^{-1}t^{-1}, \ Pa.s \]
\[ \rho = \text{density}, \ ML^{-3}, \ kg.m^{-3} \]
\[ \rho_g = \text{gas density}, \ ML^{-3}, \ kg.m^{-3} \]
\[ \rho_o = \text{oil density}, \ ML^{-3}, \ kg.m^{-3} \]
\[ \rho_w = \text{water density}, \ ML^{-3}, \ kg.m^{-3} \]
\[ \sigma = \text{interfacial tension}, \ L^{-1}t^{-2}, \ Nm^{-1} \]
\[ \sigma_{so} = \text{gas and oil interfacial tension}, \ L^{-1}t^{-2}, \ Nm^{-1} \]
\[ \sigma_{ow} = \text{oil and water interfacial tension}, \ L^{-1}t^{-2}, \ Nm^{-1} \]
\[ \tau = \text{time of flight}, \ t, \ s \]
\[ \phi = \text{porosity}, \ \text{dimensionless} \]
\[ \phi_f = \text{fracture porosity}, \ \text{dimensionless} \]
\[ \phi_m = \text{matrix porosity}, \ \text{dimensionless} \]
\[ \phi_{mk} = \text{domain } k \text{ matrix porosity}, \ \text{dimensionless} \]

Subscripts
- \( c \) = effective
- \( D \) = dimensionless
- \( f \) = flowing or fracture
- \( g \) = gas
- \( i \) = initial
- \( j \) = gridblock label
- \( k \) = fracture set label
- \( m \) = matrix or stagnant
- \( N \) = number of fracture sets
- \( o \) = oil
- \( r \) = residual
- \( t \) = total
- \( w \) = water
- \( \infty \) = ultimate (at infinite time)

Superscript
- \( n \) = time level

Acknowledgments
We would like to thank Shell, the DTI, the EPSRC, and the sponsors of the ITF project on Improved Simulation of Flow in Fractured and Faulted Reservoirs for supporting this research. We are very grateful to Petro-China for providing us with data on the Liu7 oil field.

References


### Appendix A—Analytical Solution for Countercurrent Imbibition From Barenblatt et al. (1990)

Here we review the analytic solution for countercurrent imbibition in one dimension derived by Barenblatt et al. (1990). The analysis follows that presented by Tavassoli et al. (2005b). Conservation of water volume in one dimension with no overall flow can be expressed as follows:

\[
\frac{d S_w}{d x} + \frac{\partial}{\partial x} \left( \frac{\lambda_w \lambda_t}{K} \frac{\partial P_w}{\partial S_w} \frac{\partial S_w}{\partial x} \right) = 0. \quad \text{(A-1)}
\]

Eq. A-1 is rewritten in terms of dimensionless variables: the normalized saturation, \( S = \frac{S_n - S_{ini}}{1 - S_{ini} - S_{w}} \) [Eq. A-2]

and a dimensionless length defined by \( x_0 = x/L \). The boundary conditions for flow in the region \( 0 \leq x_0 \leq 1 \) are as follows: Continuity of capillary pressure at the inlet face \( (P_0 = 0) \) gives \( S = S^* \) at \( x_0 = 0 \), where \( S^* \) is the normalized maximum water saturation after spontaneous imbibition \( (S^* > 1) \). The outlet is a no-flow boundary, which implies that \( \frac{dS}{dx_0} = 0 \) at \( x_0 = 1 \). The conservation equation, assuming uniform absolute permeability, is then:

\[
\frac{dS}{dt} + \frac{K}{\phi(1 - S_m - S_w) L^2} \frac{\partial}{\partial x_0} \left( \frac{\lambda_w \lambda_t}{\lambda_t} \frac{\partial S}{\partial x_0} \right) = 0. \quad \text{(A-3)}
\]

Rather than attempt a solution of the nonlinear Eq. A-3 directly, a solution of the weak or integral form of the equation is constructed. Integrating Eq. A-3 over the domain and noting that the flux vanishes at \( x_0 = 1 \) yields

\[
\int_{x_0}^{1} \left[ \frac{K}{\phi(1 - S_m - S_w) L^2} \frac{\lambda_w \lambda_t}{\lambda_t} \frac{\partial S}{\partial x_0} \right] dx_0 = 0, \quad \text{(A-4)}
\]

where the mean saturation \( \bar{S} \) is defined by

\[
\bar{S} = \int_{0}^{\infty} S dx_0.
\]

Following Eq. 9,

\[
\frac{dP_w}{dS} \bigg|_{x_0=0} = -\sigma_{cw} \sqrt{\frac{\phi}{K}} J_e = \frac{1}{(1 - S_m - S_w)} \frac{dP_w}{dS} \bigg|_{x_0=S^*}. \quad \text{(A-5)}
\]
Then, using Eqs. A-4 and A-5:
\[
\frac{\partial S}{\partial t} = -\sqrt{\frac{K}{\phi} \frac{\sigma_{\omega f}^J}{L^2}} \left( \frac{\lambda_\omega \lambda_f}{\lambda_f} \right) \frac{\partial S}{\partial x}\bigg|_{t=0}.
\] (A-6)

Next, a dimensionless time \( t_D \) is defined:
\[
t_D = t \sqrt{\frac{K}{\phi} \frac{\sigma_{\omega f}^J}{L^2}} \left( \frac{\lambda_\omega \lambda_f}{\lambda_f} \right), \quad s = S^o.
\] (A-7)

and Eq. A-7 takes the form
\[
\frac{\partial S}{\partial t_D} = -J^f \frac{\partial S}{\partial x}\bigg|_{t=0}.
\] (A-8)

**Early-Time Solution.** First, an early-time solution is derived that is valid until the advancing water front reaches the far boundary. A quadratic form for the saturation profile is assumed:
\[
S(x_D, t_D) = S^o - A(t_D)x_D + B(t_D)x_D^2,
\]
where \( x_D^2(t_D) \) is the distance the water has moved into the block at time \( t_D \). This form automatically obeys the boundary condition at \( x_D = 0 \). By definition, the quadratic function in Eq. A-9 must vanish at \( x_D^2(t_D) = 1 \), so that the saturation profile is continuous. The saturation gradient is also continuous at \( x_D^2(t_D) \). Two conditions allow us to find the time-dependent coefficients \( A \) and \( B \):
\[
S(x_D, t_D) = S^o - \left( \frac{1}{2} \left( \frac{x_D}{x_D^2} - 1 \right) \right), \quad S(x_D, t_D) = S^o - \left( \frac{1}{3} \frac{J^f}{t_D} \right) x_D^2.
\] (A-10)

and \( S = 0 \) for \( x_D^2 \leq x_D^2 \leq 1 \). It is then possible to derive:
\[
x_D^2(t_D) = \sqrt{2} t_D^2, \quad S(x_D, t_D) = S^o - \frac{4J^f}{3} t_D^2.
\] (A-11)

Both the penetration distance and the mean saturation grow as the square root of the dimensionless time. This solution is valid for \( t_D \leq 1 \), where \( t_D \) is the time at which the water first reaches the far (closed) boundary. This time is found by setting \( x_D^2(t_D = 1) = 1 \), which gives \( t_D = 1/\sqrt{2J^f} \).

**Late-Time Solution.** At late times, \( t_D > t_D^* \), a second-order polynomial for the saturation profile, Eq. A-9 is assumed as before. The no-flux condition at the far end implies that the saturation gradient vanishes at \( x_D = 1 \), which leads to the condition \( 2B(t_D) = A(t_D) \). The average saturation is
\[
\overline{S}(t_D) = S^o - \frac{A(t_D)}{3}.
\] (A-13)

Eq. A-8 is used to derive a differential equation for \( A(t) \) that is solved to find:
\[
A(t) = A(t_J) e^{-\eta(t-t_J)},
\] (A-14)

where \( \eta = 3J^f \) and \( A(t_J) \) is a constant whose value is determined by requiring that the early-time and late-time solutions coincide when \( t_D = t_D^* \). Equating the early-time solution and the late-time solution at \( t_D = t_D^* \) leads to \( A(t_J) = 2S^o \). Therefore:
\[
\overline{S}(t_D) = S^o - \frac{2}{3} \left( e^{-\eta(t-t_J)} \right).
\] (A-15)

For simplicity, when we derive a transfer function, we ignore the early-time behavior. Identifying the average saturation with the recovery, we find Eq. 7 where the constant \( A \) is now found to ensure that \( \overline{S}(t_D = 0) = R(0) = 0 \).

The crucial issue in this derivation is the boundary condition at the inlet. We assume that the system is not strongly water-wet and so the oil mobility is finite at the end of imbibition. If the system is strongly water-wet and \( S^o = 1 \), the functional form of the recovery is very different (Tavassoli et al. 2005a). This is a subtlety overlooked in the original analysis (Barenblatt et al. 1990).

**Appendix B—Numerical Implementation**

We first solve for an intermediate saturation in the fractures, ignoring transfer using single-point upstream weighting:
\[
S^m_{ij} = S^m_i - \frac{\Delta \tau}{\Delta t}(f_{ij}(S^a_{ij}) - f_{ij}(S^m_{ij-1})),
\] (B-1)

where \( j \) labels the gridblock and \( n \) labels the time level. \( \Delta \tau \) is the timestep size and is chosen to ensure that solutions to Eq. B-1 are stable and accurate, as in single-porosity simulation. In streamline simulation, many timesteps may be taken to transport fluid along a streamline before the pressure field is recomputed (Batycky et al. 1997).

**Single-Rate Model.** For \( T \) given by Eq. 14, if \( S^m_{ij} = 0 \) then \( T = 0 \) and \( S^m_{ij} = S^m_{ij-1} \). If \( S^m_{ij} > 0 \), we can solve for the saturation at the next time level analytically from its previous value using Eq. 12:
\[
S^m_{ij} = S^m - (S^m - S^m_{ij-1}) e^{-\beta \Delta t},
\] (B-2)

\[
S^m_{ij} = S^m - \phi_{mf} \frac{S^m_{ij-1} - S^m_{ij}}{\phi_{mf}}.
\] (B-3)

If \( S^m_{ij} < 0 \), set \( S^m_{ij} = 0 \) and find the matrix saturation that is consistent with this and mass balance:
\[
S^m_{ij} = S^m_{ij} + \frac{\phi_{mf}}{\phi_{mf}} S^m_{ij}.
\] (B-4)

**Single-Rate Model for Gravity Drainage.** For \( T \) given by Eq. 23, again, if \( S^m_{ij} = 0 \), then \( T = 0 \) and \( S^m_{ij} = S^m_{ij-1} \). If \( S^m_{ij} > 0 \), we can solve for the saturation at the next time level analytically using Eq. 21:
\[
S^m_{ij} = S^m - \left( (S^m - S^m_{ij-1}) e^{-\beta \Delta t} \right) \frac{1}{\phi_{mf}}.
\] (B-5)

\[
S^m_{ij} = S^m - \phi_{mf} \frac{S^m_{ij-1} - S^m_{ij}}{\phi_{mf}}.
\] (B-6)

If \( S^m_{ij} < 0 \), set \( S^m_{ij} = 0 \) and find the matrix saturation that is consistent with this and mass balance:
\[
S^m_{ij} = S^m_{ij} + \frac{\phi_{mf}}{\phi_{mf}} S^m_{ij}.
\] (B-7)

**Simple Multirate Model.** Again, if \( S^m_{ij} = 0 \), then \( T = 0 \) and \( S^m_{ij} = S^m_{ij-1} \). If \( S^m_{ij} > 0 \), we use expressions similar to Eq. B-2:
\[
S^m_{ij} = S^m_{ik} - (S^m_{ik} - S^m_{ij}) e^{-\gamma \Delta t}.
\] (B-8)

\[
S^m_{ij} = S^m_{ij} - \phi_{mf} \sum_{k=1}^N \frac{\phi_{mk}}{\phi_{mf}} (S^m_{ik} - S^m_{ij} - S^m_{ij} - S^m_{ik}).
\] (B-9)

If \( S^m_{ij} < 0 \), set \( S^m_{ij} = 0 \) and find the matrix saturations that are consistent with this and mass balance. There are a number of ways to do this—we choose a simple, approximate approach. Define:
\[
\Delta S_{ijk} = (S^m_{ik} - S^m_{ij})(1 - e^{-\gamma \Delta t}); \quad \phi_{mf}\Delta S_{ijk} = \sum_{k=1}^N \phi_{mk}\Delta S_{ijk}.
\] (B-10)

\[
F = \frac{\Delta S_{ijk}}{S_{ij}}; \quad F > 1.
\] (B-11)

Then:
\[
S^m_{ij} = S^m_{ij} + \Delta S_{ijk} F.
\] (B-12)

**Simple Multirate Model for Gravity Drainage.** Again, if \( S^m_{ij} = 0 \), then \( T = 0 \) and \( S^m_{ij} = S^m_{ij-1} \). If \( S^m_{ij} > 0 \), we use expressions similar to Eq. B-5:
Fracture Fractional Flow Model. The numerical implementation is more involved, because we need to check each fracture saturation \( S_{\text{fract}} \), and invoke mass balance. Again, if \( S_{\text{fract}} < 0 \), set \( S_{\text{fract}} = 0 \) and find the matrix saturations that are consistent with this and mass balance. Again:

\[
\Delta S_{\text{fract}} = S_{\text{fract}}^n - S_{\text{fract}}^{n+1} = (S_{\text{fract}}^n - S_{\text{fract}}^{n+1})^{1-n} + \alpha \Delta t \frac{1}{u_{\text{fract}}}.
\]  

\[
\phi_{\text{fract}} \Delta S_{\text{fract}} = \sum_{k=1}^{N} \phi_{nk} \Delta S_{\text{fract}}
\]

\[
F = \frac{\Delta S_{\text{fract}}}{S_{\text{fract}}} ; \quad F > 1.
\]  

Then:

\[
S_{\text{fract}}^{n+1} = S_{\text{fract}}^n + \frac{\Delta S_{\text{fract}}}{F}.
\]  

Fracture Fractional Flow Model for Gravity Drainage. As before, if \( S_{\text{fract}}^n = 0 \), then \( T = 0 \) and \( S_{\text{fract}}^{n+1} = S_{\text{fract}}^n \); \( S_{\text{fract}}^{n+1} = 0 \). We find the fracture saturation at time level \( n+1 \) that is consistent with Eqs. 30 and 31:

\[
f = \frac{S_{\text{fract}}^{n+1} - S_{\text{fract}}^{n+1-i}}{S_{\text{fract}}^n - S_{\text{fract}}^{n-i}}.
\]  

Our initial estimate for the fracture gas saturation at time level \( n+1 \) in Eq. B-21 is \( S_{\text{fract}}^{n+1} = S_{\text{fract}}^n \). We find another value of \( S_{\text{fract}}^{n+1} \) using Eqs. B-22 and B-23. If \( S_{\text{fract}}^{n+1} \geq 0 \), then set \( S_{\text{fract}}^{n+1} \) to be half its previously estimated value. The updated value of \( S_{\text{fract}}^{n+1} \) is then put into Eq. B-21, and we iterate until we reach a converged solution.