Phonon-like dispersion curves of magnetoinductive waves

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The dispersion characteristics of magnetoinductive (MI) waves for a one-dimensional array of metamaterial elements are investigated for the case where the element properties vary in a bi-periodic manner. It is shown that, by this means (analogously to acoustic waves in a solid), a variety of dispersion curves can be obtained including those displaying an “optical” branch. The flexibility of the metamaterial design makes it possible to apply this approach for obtaining specified dispersion properties. A design permitting parametric amplification is proposed. © 2005 American Institute of Physics. [DOI: 10.1063/1.2011789]

It was proposed by Pendry et al.\(^1\) that Split Ring Resonators as metamaterial elements can produce negative permeability, confirmed experimentally shortly afterwards by Smith et al.\(^2\) Magnetoinductive (MI) waves came about as a by-product of this research. They owe their existence to magnetic (inductive) coupling between resonant elements. Their properties have been investigated both theoretically\(^3,4\) and experimentally.\(^5,6\) Applications for delay lines,\(^7\) phase shifters,\(^8\) and producing images\(^9\) have been recently reported. Their main advantages are ease of production and great flexibility meaning that all the design parameters of each one of the elements can be independently controlled.

The simplest metamaterial realization of MI waves is in the form of a one-dimensional array of identical resonant elements in which each element is coupled magnetically to all the other elements. The propagation problem was indeed formulated in these very general terms in Ref. 4. There are by now a large variety of such elements available: The Swiss Roll,\(^1\) the Capacitively Loaded Loop,\(^3,5\) the Spiral Resonator,\(^10\) the Broadside-Coupled SRR,\(^11\) the Singly Split Double Ring,\(^12\) and the Twin Split Ring\(^13\) to give a few examples. The common feature between them is that with good approximation they can be treated as coupled resonant circuits characterized by an inductance \(L\), capacitance \(C\), resonant frequency \(\omega_0\) and mutual inductance, \(M\). If the further approximation is made that there is interaction between nearest neighbours only, then all these structures can be regarded as realizations of one of the periodic filters of Atabekov\(^14\) or one of the slow wave structures of Silin and Sazonov.\(^15\) In the lossless case the dispersion equation may then be written in the form\(^1,4,14,15\)

\[
1 - \omega_0^2/\omega^2 + \kappa \cos kd = 0
\]

where \(\omega_0 = 1/\sqrt{LC}\) is the resonant frequency, \(d\) is the distance between the elements, \(\kappa = 2M/L\) is the coupling coefficient and \(kd\) is the phase change per element.

Depending on the sign of \(M\), which is determined by the orientation of the elements, MI waves are either forward (with co-directional group and phase velocities) or backward (group and phase velocities in opposite directions). They are backward for the planar configuration (elements in the same plane) and forward for the axial configuration (plane of the elements perpendicular to the axis).

So far we have discussed just two types of dispersion characteristics both restricted to a narrow band. However, in practice, when a particular device is to be designed it is desirable to have considerable freedom in choosing the array’s dispersion properties. Ideally we wish to tailor the dispersion to whatever the specifications demand. Luckily the great flexibility of metamaterial engineering provides such an opportunity. The size of each element and the distance between them can be easily controlled enabling us to design arrays with quite different intrinsic properties.

The way forward is suggested by the close analogy between MI waves and acoustic waves in solids. It is well-known that the dispersion characteristics of diatomic solids differ greatly from those having identical elements (see, e.g., Ref. 16). It is not only that the width of the pass band and the initial slope of the curve are different but also a new band appearing well known as the “optical” branch. The new band is a direct consequence of bi-periodicity well known in Solid State Physics (see, e.g., Ref. 17). It may, therefore, be expected that arrays of metamaterials made up by elements in a bi-periodic arrangement will follow similar trends and will give us some additional flexibility in design.

There are two obvious ways for achieving double periodicity: (i) To change some parameter of the element (\(L\) or \(C\) resulting in a change of the resonant frequency) and (ii) to vary the distance between the elements. These possibilities are shown schematically in Fig. 1 for arrays of capacitively loaded loops, the structures we are going to investigate in more detail. Planar configurations are shown in Figs. 1(a) and 1(b) and axial ones in Figs. 1(c) and 1(d).

Denoting the impedances of the loops by \(Z_{01}\) and \(Z_{02}\), the distances between neighboring elements by \(d_1\) and \(d_2\) and the mutual inductances corresponding to those distances by \(M_1\) and \(M_2\) the dispersion equation can be easily obtained in the form...
The axial configuration we choose is the periodic one only the capacitances vary from element to element; (b) and (d) distance varies between neighboring elements.

$$\frac{\cos d_1 + d_2}{2} = \pm \frac{1}{2 \sqrt{M_1 M_2}} \sqrt{-\frac{Z_{01} Z_{02}}{\omega^2} - (M_1 - M_2)^2}.$$

(2)

Let us now look at two practical examples, one for the axial and one for the planar configuration. In both cases we shall assume that the distances and the inductances are the same and only the capacitances vary from element to element. The parameters chosen are as follows: Loop radius, \( r_0 = 10 \text{ mm} \), wire thickness, \( r_w = 2 \text{ mm} \). The circuit parameters are \( L = 33 \text{ nH} \), \( C_1 = 208 \text{ pF} \), \( C_2 = 177 \text{ pF} \). Then \( \phi_{01} = 0.95 \phi_{02} \) and \( \phi_{02} = 1.05 \phi_{01} \) with \( \phi_{01}/(2\pi) = 63.87 \text{ MHz} \). For the axial configuration we choose \( d = 10 \text{ mm} \) resulting in \( M/L = 0.149 \) and for the planar case \( d = 20.5 \text{ mm} \) corresponding to \( M/L = -0.104 \).

In a practical case there are of course losses making \( Z_{01}, Z_{02} \) complex. Consequently \( k \) must also be complex and must be written in the form \( k = \beta - j\alpha \) where \( \beta \) and \( \alpha \) are the propagation and attenuation constants. The losses will be characterized by choosing a quality factor of 150. The dispersion curves with both propagation and attenuation constants are shown in Figs. 2 and 3. Note that there is a stop band in the middle for both cases for the range \( \omega_{01} < \omega < \omega_{02} \).

The physical reason is that within this band the loops’ impedances have different characters, one is inductive and the other one is capacitive.

We may immediately see from Figs. 2 and 3 that the major distinction between the axial and planar configurations is no longer there. We have forward propagating waves in the lower and backward propagating waves in the upper branch independently of the configuration. There is however a difference if we consider the phases of the currents within a pair. For an axial line the currents of the neighboring elements in the upper branch are in anti-phase and the currents in the lower branch are in phase. Using the analogy with the diatomic model we can refer to the upper branch of the dispersion characteristics as “optical” and to the lower branch as “acoustic” (Fig. 2). For a planar line the situation is reversed: The currents of the lower branch are now in anti-phase thus this is the one that we should call “optical” (Fig. 3). It is worth noting that both branches of the MI wave dispersion curve may be responsible for interaction with electromagnetic waves.\(^{18}\)

We are not aware of any device which makes use of both the acoustic and of the optical branches but it may very well be the case for MI waves. They may be employed as waveguides providing two distinct pass bands. A particular application may be in Magnetic Resonance Imaging (MRI) when the image is required at two different frequencies far from each other.

Another set of potential applications is for nonlinear wave interaction exemplified by parametric amplification. Such amplifiers can be realized (see, e.g., Ref. 19) by replacing the capacitance by a varactor diode. The parametric amplification mechanism involves transfer of power from the pump wave at \( 2\omega \) to the signal wave at \( \omega \), its main merit being low noise amplification. The requirement is then to ensure pass bands for both waves and to satisfy the condition of synchronism so that the two waves have identical phase velocities. The main application we have in mind is again in MRI. In addition to the guiding of the signal, that has already been demonstrated,\(^{20}\) we are thinking about the parametric amplification of the signal close to the source improving thereby the signal-to-noise ratio.

For a concrete design we choose the signal frequency to be amplified as \( \omega_{01}/(2\pi) = 63.87 \text{ MHz} \) corresponding to the proton’s magnetic resonance frequency at 1.5 T. The propagation constant may then be chosen within the pass-band where the attenuation is comparatively low, say \( 2\beta d = \pi/3 \). The pump wave should then have the frequency \( 2\omega_{01}/(2\pi) = 127.74 \text{ MHz} \) and the propagation constant \( 2\beta d = 2\pi/3 \). The above requirements can be satisfied with an axial structure where the distance between the elements is \( d = 0.5 r_0 (M = 0.336L) \), \( C_1 = 164 \text{ pF} \), \( C_2 = 56 \text{ pF} \), \( L = 33 \text{ nH} \) yielding \( \omega_{01} \)
amplification.  

The corresponding dispersion curve is shown in Fig. 4. It can be seen that both the signal (k, ω) and pump (2k, 2ω) waves required for parametric amplification can be met by tailoring the dispersion characteristics.

All the calculations so far have been for an infinite line. In order to have the same conditions in a finite line it must be terminated by a matched load. For a single periodic line its value was determined in Ref. 4. Using a similar technique the matching impedances (there are two now depending on the element terminating the line) may be obtained as

\[ Z_{1,2}^{(1,2)} = -\frac{M_{1,2}Z_{01,02}}{M_{1,2} + M_{2,1} \exp[i k(d_1 + d_2)]}. \]  

We have demonstrated, using the analogy with acoustic waves, that bi-periodic structures may lead to a variety of MI wave characteristics and illustrative examples have been given for both axial and planar configurations. It has been shown in a further example that the requirements of parametric amplification can be met by tailoring the dispersion characteristics.

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