2D metamaterials with hexagonal structure: spatial resonances and near field imaging

O. Zhuromsky and E. Shamonina
Department of Physics, University of Osnabrück, Osnabrück D-49076, Germany
azhuroms@uos.de, ekaterina.shamonina@uos.de

L. Solymar
Department of Electrical and Electronic Engineering,
Imperial College of Science, Technology and Medicine, Exhibition Road, London SW7 2BT, UK
lazslo.solymar@eng.ox.ac.uk

Abstract: The current and field distribution in a 2D metamaterial consisting of resonant elements in a hexagonal arrangement are found assuming magnetic interaction between the elements. The dispersion equation of magnetoinductive (MI) waves is derived with the aid of the direct and reciprocal lattice familiar from solid state theory. A continuous model for the current variation in the elements is introduced leading to the familiar wave equation in the form of a second order differential equation. The current distributions are shown to exhibit a series of spatial resonances for rectangular, circular and hexagonal boundaries. The axial and radial components of the resulting magnetic field are compared with previously obtained experimental results on a Swiss Roll metamaterial with hexagonal boundaries. Experimental and theoretical results are also compared for the near field image of an object in the shape of the letter M followed by a more general discussion of imaging. It is concluded that a theoretical formulation based on the propagation of MI waves can correctly describe the experimental results.

©2005 Optical Society of America

OCIS codes: (110.0110) Imaging Systems; (160.1190) Anisotropic Optical Materials; (350.5500) Propagation; (260.0260) Physical Optics

References and links

1. Introduction

Periodic inclusion of elements into a homogeneous material has been studied for well over a century. There was a lot of interest in the ‘50s and ‘60s in order to produce artificial dielectrics with increased dielectric constants [1]. New impetus to the field was given in the ‘90s by the emergence of photonic band gap materials [2,3] and in the late ‘90s when interest started to be focused on negative index materials [4,5]. In this latter class of materials, known as metamaterials, the inclusions are metallic, have distinct resonances, and both the size of the elements and the unit cell are small relative to the wavelength. One of the potential applications of metamaterials is in near-field imaging. It was shown by Pendry that a slab of indefinite media, Appl. Phys. Lett. 84, 2244-2246 (2004).

object of the shape of a letter M is calculated in Section 6, some general comments on imaging are made in Section 7 and conclusions are drawn in Section 8.

2. Derivation of the dispersion equation

![Image](image-url)

Fig. 1. (a) Lattice of resonant elements with hexagonal arrangement. (b) Direct vectors of the hexagonal lattice \((d_1, d_2)\) and vectors of the reciprocal lattice \((b_1, b_2)\).

We shall look here only at the simplest case of nearest neighbour interactions, i.e. when a current flowing in one element will induce a voltage in the neighbouring element. We may then write Kirchhoff’s equation for the voltage in the element \(n,m\) as follows

\[ ZI_{n,m} + j\omega M(I_{n,m-1} + I_{n,m+1} + I_{n-1,m} + I_{n+1,m} + I_{n-1,m+1} + I_{n+1,m+1}) = 0 \]  

(1)

where \(I_{n,m}\) is the current in the \(n,m\) element, \(Z = j\omega L + 1/\omega C\) is the self-impedance, \(L\) is the self-inductance, \(C\) is the capacitance and \(M\) is the mutual inductance between nearest neighbours. In practice the elements could take the form of capacitively loaded loops [13], Swiss Rolls [14], Split Ring Resonators [4] or one of its variants [15-17]. Note that we are considering the lossless case only: the resistance of the element has been ignored.

The position of an element \(n,m\) is given by the radius vector

\[ r_{n,m} = nd_1 + md_2 \]  

(2)

where \(d_1\) and \(d_2\) are the direct vectors of the lattice, shown in Fig. 1(b). In the \(x, y\) coordinate system they can be written as

\[ d_1 = i_x d, \quad d_2 = (i_x \cos 60^\circ + i_y \sin 60^\circ)d \]  

(3)

where \(d\) is the spacing between the elements and \(i_x\) and \(i_y\) are unit vectors in the \(x\) and \(y\) directions respectively. The corresponding vectors of the reciprocal lattice, \(b_1\) and \(b_2\) may be obtained by requiring the following conditions to be satisfied

\[ d_1 \cdot b_1 = 1, \quad d_1 \cdot b_2 = 0, \quad d_2 \cdot b_1 = 0, \quad d_2 \cdot b_2 = 1. \]  

(4)

We shall now look for the solution of Eq. (1) in the form

\[ I_{n,m} = I_{0,e} \exp(-jk \cdot r_{n,m}). \]  

(5)
$I_{0,0}$ is a constant and $\mathbf{k}$ is the wave vector which may be expressed in terms of the reciprocal lattice vector as

$$\mathbf{k} = 2\pi (f_1 \mathbf{b}_1 + f_2 \mathbf{b}_2)$$  \hspace{5cm} (6)

where $f_1$ and $f_2$ are normalized coordinates of the wave vector in the reciprocal space.

Substituting Eq. (5) into Eq. (1) we obtain the dispersion equation in the form

$$\frac{\omega}{\omega_0} = \left[1 + \kappa \left[\cos(2\pi f_1) + \cos(2\pi f_2) + \cos(2\pi f_1 - 2\pi f_2)\right]\right]^{\frac{1}{2}}$$  \hspace{5cm} (7)

where $\omega_0 = 1/\sqrt{LC}$ is the resonant frequency of the element and $\kappa = 2M/L$ is the coupling coefficient. The $\omega/\omega_0 = \text{constant}$ curves as functions of the wave vector are plotted in Fig. 2. The pass band is found to extend from $0.93\omega_0$ to $1.21\omega_0$. Note that the waves are backward waves with phase and group velocities in opposite directions.

![Dispersion curve](image)

**Fig. 2.** 2D dispersion equation of MI waves for a hexagonal metamaterial showing the contour lines of frequency as a function of $k_d x$ and $k_d y$. The boundaries of the first Brillouin zone are shown by bold lines.

### 3. Determination of the current distribution

Let us assume that there are altogether $N$ elements in our 2D hexagonal array and that one of the elements is excited by a voltage (in practice it could be done by a loop attached to a coaxial cable). The current distribution may then be determined by the relation

$$\mathbf{I} = \mathbf{Z}^{-1} \mathbf{V}.$$  \hspace{5cm} (8)

The current is written here as an $N$ dimensional column vector, $\mathbf{V}$ is also a column vector of the same dimension representing the excitation of each element. $\mathbf{Z}$ is an $N \times N$ matrix containing all the self and mutual impedances. In practice only one of the elements is excited so that $\mathbf{V}$ has only one non-zero component.

It is quite straightforward to determine numerically the current distribution from Eq. (8) when the array configuration and the excitation are known. We shall indeed do so in the examples worked out in the sections to follow. In the present section however we shall attempt to find an approximate solution which will lead us to mathematically familiar territory and offer a clear physical picture. We may expect that the propagation of MI waves, similarly to the propagation of most other waves, can be mathematically described by a second order partial differential equation. We shall therefore convert our difference equation (Eq. (1)) into a
differential equation. We can do that when the wave vectors are sufficiently small or in other words when the wavelength of the MI wave is much larger than the element spacing. Consequently we shall introduce the \( \nu, \mu \) coordinate system in the directions \( \mathbf{d}_1 \) and \( \mathbf{d}_2 \) with the continuous variables \( \nu = nd \) and \( \mu = md \), and replace the discrete function \( I_{n,m} \) by the continuous function \( I(\nu, \mu) \). A change in the subscript by unity would then be equivalent with a change of the continuous variable by \( d \) which is then regarded as an elementary change. To convert all the terms in Eq. (1) into continuous variables we need to expand the current into a Taylor series as follows

\[
I(\nu + \Delta \nu, \mu + \Delta \mu) = \left\{ 1 + \Delta \nu \frac{\partial}{\partial \nu} + \Delta \mu \frac{\partial}{\partial \mu} + \frac{1}{2} \left( (\Delta \nu)^2 \frac{\partial^2}{\partial \nu^2} + 2 \Delta \nu \Delta \mu \frac{\partial^2}{\partial \nu \partial \mu} + (\Delta \mu)^2 \frac{\partial^2}{\partial \mu^2} \right) \right\} I(\nu, \mu) \tag{9}
\]

where \( \Delta \nu \) and \( \Delta \mu \) are small deviations from \( \nu \) and \( \mu \). With the aid of Eq. (9) we may now convert Eq. (1) into the differential equation

\[
\frac{\partial^2 I}{\partial \nu^2} - \frac{\partial^2 I}{\partial \nu \partial \mu} + \frac{\partial^2 I}{\partial \mu^2} + \frac{1}{2d^2} \left( 6 + \frac{Z}{j \omega M} \right) I = 0 \tag{10}
\]

A further transformation to the \( x, y \) coordinate system yields the relationship

\[
\frac{\partial^2 I}{\partial \nu^2} - \frac{\partial^2 I}{\partial \nu \partial \mu} + \frac{\partial^2 I}{\partial \mu^2} = \frac{3}{4} \left( \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} \right) \tag{11}
\]

So we end up with the familiar wave equation

\[
\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} + k^2 I = 0 \tag{12}
\]

where

\[
k^2 = \frac{4}{d^2} \left[ 1 + \frac{1}{3k} \left( 1 - \frac{\omega^2}{k^2} \right) \right] \tag{13}
\]

and \( k = \left| \mathbf{k} \right| \). It may be easily shown that the dispersion Eq. (7) reduces to Eq. (13) when \( kd << 1 \).

A clear advantage of having the wave equation is that, at least for certain geometries, we know the solutions from past experience. For rectangular boundaries possible solutions are

\[
I = I_0 \sin(k_x x) \sin(k_y y) \tag{14}
\]

where \( I_0 \) is a constant and \( k_x \) and \( k_y \) are the \( x \) and \( y \) components of the wave vector. For a circular boundary there are circularly symmetrical solutions of the form

\[
I = I_0 J_0(kr) \tag{15}
\]

where \( J_0 \) is the zero order Bessel function of the first kind and \( r \) is the distance from the centre of the circular structure of the elements.
4. Current distributions for circular and rectangular boundaries: spatial resonances

If we choose a frequency and know which elements are excited (our formulation allows all of them to be separately excited) we can determine the current distribution from Eq. (8). The result might be in the form of odd looking current distributions because several wave vectors may coexist at a particular frequency. It is only at the spatial resonance that no more than one wave vector survives. Spatial resonance is characteristic to all wave phenomena. They exhibit the same behaviour whether they occur in vibrating membranes, organ pipes or Fabry-Perot resonators. The boundary condition to be satisfied in the present case is that the current must vanish at the boundary.

4.1 Array of circular shape

For a circle of radius $R$ the condition for spatial resonance is

$$kR = \rho_i$$

where $\rho_i$ is the $i$th root of $J_0(\rho)$.

We shall now look at the phenomenon of spatial resonance with the aid of a few examples. At this stage we need to commit ourselves as for the elements of the array, the distance between them and the resonant frequency. The element is chosen as a capacitively loaded loop that was studied before both theoretically and experimentally [13]. The loop radius, the wire diameter and the distance between the elements are taken as 10 mm, 2 mm and 22.5 mm, and the resonant frequency as 21.5 MHz. The inductance of the loop may then be determined from standard formulae [18] which give $L = 33 \text{ nH}$. The corresponding capacitance can be determined from the resonant frequency as 1.66 $\text{nF}$. The value of the mutual inductance can be obtained from Tables [18]. With our present parameters its value comes to -1.75 $\text{nH}$ yielding $\kappa = -0.106$. Owing to the hexagonal arrangement it is not possible for all the elements to lie exactly on a circular boundary. With our choice of 361 elements the deviation from the circular boundary is quite small as may be appreciated by looking at Figs. 3 (a)-(c).

Knowing the geometry the frequencies of the first 3 spatial resonances may be determined from Eq. (16) as $\omega/\omega_k = 1.207, 1.187$ and $1.155$. Numerical calculations based on the known values of the mutual impedance matrix yield 1.207, 1.191 and 1.165. As may be expected the agreement is somewhat worse for the higher spatial harmonics. The analytical results are of interest because they give good approximation for low values of $k$ and, of course, they give an immediate idea what the current distribution looks like. For the general case however it is more accurate to rely on the exact solution of the discrete problem which relies on the inversion of the impedance matrix. Note that for the numerical determination of the current distribution we need an excitation. We assume that the central element out of the 361 is excited and then proceed with the numerical solution of Eq. (8).
The numerically determined current distributions are shown by a colour code in Figs 3(a)-(c). Normalisation in each figure is to the maximum value within the figure. As may be expected at the first spatial resonance there is zero current only at the boundary. For the second and third spatial resonances the currents are approximately zero at one and two radii respectively. An interesting feature of Fig. 3(c) is that in spite of the close-to-circular boundary the hexagonal nature of the element geometry reestablishes itself further away from the centre.

In the attached Movie, using the same normalization procedure, we show the spatial variation of the current at 72 discrete values of the frequency. The frequency range is from the upper stop band to the lower stop band. If the frequency is in either stop band then the currents are concentrated to the central elements. In the upper part of the pass band (above the resonant frequency) the pattern changes quite fast with frequency whereas the change is more moderate in the lower part.

4.2 Array of rectangular shape

We shall assume the array to extend from 0 to \( L_x \) in the \( x \) direction and from 0 to \( L_y \) in the \( y \) direction. In view of Eq. (14) the wave vector components leading to spatial resonances are given by the relationships

\[
k_x L_x = p_x \pi \quad \text{and} \quad k_y L_y = p_y \pi
\]  

(17)

where \( p_x \) and \( p_y \) are integers.

![Fig. 4. Current distribution for rectangular boundary conditions at various frequencies (a-d).](image-url) (2.25 MB) Movie of current distribution in the frequency range from 1.5 to 0.7 \( \omega \) (77 frames).

We shall arrange now the same hexagonal arrangement of elements in a 19 x 19 square geometry. As in the previous example we shall compare the analytical and numerical values for the frequencies of 3 spatial resonances (fundamental plus two of the second order) shown in Figs. 4(b)-(d). The resonant frequencies are 1.2073, 1.2020 and 1.2006 from the numerical solutions. There is some ambiguity in the analytical expression: due to the jagged boundaries \( L_x \) cannot be exactly defined. The values we obtain for the resonant frequencies are 1.207, 1.202 and 1.202 but the last significant figure is probably inaccurate however the agreement between the numerical and analytical results is clearly very good.

#8585 - $15.00 USD  
Received 25 August 2005; revised 1 November 2005; accepted 1 November 2005  
(C) 2005 OSA  
14 November 2005 / Vol. 13, No. 23 / OPTICS EXPRESS  9305
Note that the establishment of the spatial resonances depends little on the exact position of the excitation as long as it is not at an expected current minimum. For the present rectangular case the excitation is taken one quarter of the way along one of the diagonals. We can obtain with this excitation all the spatial resonances but the current pattern will of course depend on the position of the excitation between the spatial resonances. At a frequency of \( \omega_1 / \omega_c = 1.2083 \) we do have an asymmetric pattern characteristic to the excitation as may be seen in Fig. 4(a).

The attached Movie shows the current distributions for the square boundaries for 77 discrete values of the frequency again from stop band to stop band. The conclusions are similar to those for the circular boundary but there is now a wider variety due to the asymmetry of the excitation.

5. Hexagonal boundaries: comparison of magnetic fields with experimental results at spatial resonances

![Hexagonal boundaries](image)

Fig. 5. Magnetic field distribution at four resonant frequencies: normal component (a-d) and tangential component (e-h). (2.1 MB, 2.55MB) Corresponding movies in the frequency range from 1.5 to 0.7 \( \omega_c \) (86 frames).
For hexagonal boundaries we have a chance to compare the theoretical results with a set of experimental ones measured by Wiltshire et al. [19]. The experiments were performed on a 2D array of 271 ‘Swiss Rolls’. The array was centrally excited on one side of the structure and the axial (perpendicular to the plane of the elements) and radial components of the magnetic field were measured on the other side. The measured 2D distribution of these two components of the magnetic field are given in their Figs. 4-7 for the first four spatial resonances. The corresponding theoretical results are obtained by the same technique as employed in our previous examples for circular and rectangular boundaries although in the present case we need to go a little further and determine the magnetic field vector from the current distribution. The results are plotted in Fig. 5. A comparison between theory and experiment shows practically the same spatial variation for both components of the magnetic field.

The question arises whether we can have good agreement for the actual frequencies of the spatial resonances as well. We would expect them to be different for the reason that the theoretical model is based on lossless capacitively loaded loops whereas lossy Swiss Rolls are used in the experiments. The main difference between the two types of elements is that for Swiss Rolls the mutual inductance between the elements decays much more slowly with distance than for capacitively loaded loops. Hence for a structure consisting of Swiss Rolls the nearest neighbour approximation for finding the propagation of magnetoinductive waves is insufficient. In fact the mutual inductance between Swiss Rolls was determined in Ref. [14] by comparing the experiments with a theory that took account of interactions between any two elements. Hence we need to modify our theory by replacing the mutual inductances relevant for capacitively loaded loops by a new set obtained from Ref. [14]. Then the impedance matrix will no longer be tri-diagonal. Including all interactions means that all elements of the impedance matrix will be different from zero. A further modification we need is to introduce losses. This is quite straightforward. It can be done by incorporating a series resistance $R$ in the expression for the self-impedance which then takes the form $Z = j\omega L + 1/ j\omega C + R$. In a resonant system loss is normally characterized by the quality factor $Q = \omega L / R$, the measure we are going to adopt.

In Fig. 6 we compare the experimental results (blue curve) for the frequencies of the spatial resonances with theoretical ones. The green curve is obtained by assuming lossless capacitively loaded loops and taking all interactions into account. The higher spatial resonances move in the right direction: higher resonances occur at lower frequencies. The detailed agreement is however poor. If we modify the theory by including losses (using the measured value of $Q = 65$) and, in addition, replace the coupling coefficients with those obtained from Ref. [14] for Swiss Rolls we obtain the red curve which shows the same kind of variation with resonance number as the experimental results. It obviously makes a difference that for Swiss Rolls the mutual inductances decline more slowly than for capacitively loaded loops. This is about as far as we can go because the Swiss Rolls used in the experiments of Refs. [14] and [19] differ somewhat from each other.
6. Imaging: comparison of theory with experiments

In another set of experiments Wiltshire et al. [9] used the hexagonally arranged Swiss Rolls for the purpose of imaging. The authors excited the 2D resonant structure (same as in Ref. [14]) by placing an M shaped wire antenna on one side of the structure and measuring the axial component of the magnetic field on the other side. The experimentally obtained image is shown in Fig. 4 of Ref. [9]. The image we have calculated from our simple model (hexagonally arranged capacitively loaded loops, no losses, nearest neighbour interactions) may be seen in Fig. 7(a) and 7(b) for $\omega_0/\omega_b = 0.98$ and 1.01 respectively. The agreement with the experimental results is very close at the lower frequency. As the frequency increases beyond the resonant frequency of the element the image quickly deteriorates. At $\omega_0/\omega_b = 1.01$ the object is unrecognizable.

7. Imaging: discussion

Subwavelength imaging has been a topic of great interest ever since Pendry proposed a negative refractive index material as a perfect lens [5]. There have been imaging experiments with silver slabs (see, e.g., Refs. [6,7]), with two or more layers of 2D arrays [20-22], and with a single layer of a 2D array as already discussed in the present paper. There is not enough evidence at the moment to say whether there are three different mechanisms at play or all of them show different facets of the same physical phenomenon. Pendry’s proof was based on the fact that with a negative index material it is possible (at least in principle) to obtain a flat transfer function in the spatial frequency region. The explanation of imaging with a single layer of Swiss Rolls was based on the existence of a negative permeability region. An alternative explanation is that in the absence of a wave mechanism the single layer of
elements simply serve to transmit the image from one side of the array to the other side. The presence of MI waves, on the other hand, might lead to strong distortion of the image, a conclusion that may be drawn from Fig. 7 above. When the structure consists of two parallel 2D arrays of metamaterial elements then MI waves may play a positive role. They may be responsible for obtaining the right current distribution on the outer array for imaging when the inner array is excited by the object. An alternative explanation of imaging for the two-layer case may be based on the curvature of the dispersion characteristics as discussed for example by Zengerle [23] or more recently by Smith et al. [20] and Belov et al. [24].

It is worth mentioning that, although the experiments [9,19] were carried out in the MHz region, the conclusions are not dependent on frequency since the structure can be scaled. Recently miniaturisation of split rings using microfabrication [25] and nanofabrication [26] was employed to create metamaterials with resonant frequencies approaching the optical range. We expect our results to be important for designing metamaterials which could be employed for near-field manipulation even at those high frequencies.

8. Conclusions

The propagation of magneto-inductive waves on a 2D array of metamaterial elements, arranged in a hexagonal form and constructed from capacitively loaded loops in the planar configuration, has been studied. A dispersion equation has been derived and the current distribution in the elements has been determined. The similarity between MI waves and other types of waves has been emphasised by deriving the generic wave equation in the form of the well known second order partial differential equation valid in the limit when the wavelength of the MI wave is large relative to the element spacing. Spatial resonances have been illustrated for rectangular and circular structures. For hexagonal boundaries the values of the axial and radial components of the magnetic field have been calculated, and have been shown to agree well with experimental results both for the spatial resonances and for imaging. The experiments were carried out in the MHz region but the conclusions are not dependent on frequency since the structure can be scaled.

Acknowledgments

The authors thank Professors Richard Syms, Ian Young, David Edwards and Mikhail Shamonin and Dr. Chris Stevens for fruitful discussions. O. Z. and E. S. acknowledge financial support by the Emmy Noether-Program of the German Research Council.