UNIVERSITY OF LONDON
MSci EXAMINATION May 2007
for Internal Students of Imperial College of Science, Technology and Medicine
This paper is also taken for the relevant Examination for the Associateship

UNIFICATION

For Fourth-Year Physics Students
Tuesday 15th May 2007: 14.00 to 16.00

Answer TWO questions.
All questions carry equal marks.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions
Write your CANDIDATE NUMBER clearly on each of the TWO answer books provided.
If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.
USE ONE ANSWER BOOK FOR EACH QUESTION.
Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.
Hand in TWO answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.
1. Consider the Lagrangian density for a single real scalar field \( \phi(x) \in \mathbb{R} \) and a single complex scalar field, \( \Phi(x) \in \mathbb{C} \)

\[
\mathcal{L}(\phi, \Phi) = (\partial_\mu \Phi^\dagger)(\partial^\mu \Phi) - m^2|\Phi|^2 - \frac{1}{2}\mu^2\phi^2 - g\phi|\Phi|^2
\]  

where \( m^2, \mu^2, g \in \mathbb{R} \).

(i) Assuming we are working in four space-time dimensions, show that the dimension of the field \( \Phi \) must be 1 in natural units.
Assume that \( \phi \) has usual dimensions for a scalar field in four space-time dimensions.
Give the dimension for each of the parameters \( m, \mu \) and \( g \).
Does power counting of the coupling constants suggest this theory is renormalisable or not? [5 marks]

(ii) Write down the symmetry transformations for the fields \( \Phi \) and \( \phi \). Demonstrate that they are symmetries.
Hence what is the continuous internal symmetry group (so not the space-time symmetry) of this theory? Is it a local or global symmetry? [5 marks]

(iii) The equations of motion for a field \( f_i(x) \) are given by

\[
\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu f_i)} - \frac{\partial \mathcal{L}}{\partial f_i} = 0.
\]  

What are the equations of motion for the \( \Phi(x) \) and the \( \phi(x) \) fields?
Starting from the Lagrangian (1.1) and the equations of motion, find the conserved current. [6 marks]

(iv) By inspection of 1.1 state why the \( \phi \) field cannot be a physical degree of freedom.
Use the equation of motion for \( \phi \) to eliminate \( \phi \) from \( \mathcal{L}(\phi, \Phi) \) of (1.1) to produce a Lagrangian \( \mathcal{L}(\Phi) \) that contains only \( \Phi(x) \). Why is this a legitimate operation in terms of physical solution?
Hence identify the true propagating degrees of freedom for this theory. Indicate the kinetic, mass and interaction terms for the physical degrees of freedom in \( \mathcal{L}(\Phi) \).
Why must \( \mu^2 < 0 \) for a physical theory? [4 marks]

[TOTAL 20 marks]
2. (i) The classical behaviour of \( N \) Dirac fermion fields \( \Psi_i(x) \) \((i = 1, 2, \ldots, N)\) of mass \( M \) is described by the Lagrangian

\[
\mathcal{L}_\Psi = \bar{\psi} i \gamma^\mu \partial_\mu \psi - M \bar{\psi} \psi \quad (2.1)
\]

Show that this is invariant under a global \( U(N) \) transformation

\[
\psi(x) \rightarrow \psi'(x) = U \psi(x) \quad (2.2)
\]

where \( U \) is an \( N \)-dimensional unitary matrix. \([2 \text{ marks}]\)

(ii) The adjoint representation of \( SU(N) \) can be written in terms of the similarity transformation of traceless Hermitian matrices, \( \Phi \). Thus if \( S \) is an \( N \)-dimensional special unitary matrix then the transformation

\[
\Phi' = S \Phi S^\dagger \quad (2.3)
\]

respects the group multiplication properties of \( S \) and so \( SU(N) \), and both \( \Phi \) and \( \Phi' \) are traceless Hermitian matrices.

Consider an \( N \)-dimensional complex traceless Hermitian matrix of scalar fields \( \Phi(x)_{ij} \) \((i, j = 1, 2, \ldots, N)\), so \( \Phi^\dagger = \Phi \) and \( \Phi(x)_{ii} = 0 \). Show that

\[
\mathcal{L}_\Phi = \text{Tr}\{(\partial_\mu \Phi)(\partial^{\mu} \Phi)\} - V(\text{Tr}\{\Phi \Phi\}) \quad (2.4)
\]

has an \( SU(N) \) global symmetry under \( \Phi \rightarrow \Phi' = S \Phi S^\dagger \).

Show that \( \Phi \rightarrow \Phi' = e^{i\theta} \Phi \) is not a symmetry of this Lagrangian. \([3 \text{ marks}]\)

(iii) Consider the following Lagrangian of a doublet of Dirac fermions, \( \psi_i \) \((i = 1, 2)\) and one two-dimensional traceless Hermitian matrix of complex scalar fields \( \Phi_{ij} \) \((i, j = 1, 2)\)

\[
\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi - M \bar{\psi} \psi + \text{Tr}\{(\partial_\mu \Phi)(\partial^{\mu} \Phi)\} - m^2 \text{Tr}\{\Phi \Phi\} - \lambda \text{Tr}\{\Phi \Phi\}
\]

\[
+ g \bar{\psi} \cdot \Phi \cdot \psi
\]

where in the last term \( \bar{\psi} \cdot \Phi \cdot \psi \equiv \bar{\psi}_i \gamma^0 \Phi_{ij} \psi_j \) with spin indices suppressed, and similar in other terms.

Given that a unitary matrix can be written as a phase multiplied by a special unitary matrix, show that for \( g \neq 0 \) this Lagrangian has continuous symmetry \( U(1) \times SU(2) \).

Give the symmetry transformations for the fields. \([4 \text{ marks}]\)

[Question 2 continued overleaf]
(iv) Any two-dimensional Hermitian traceless matrix field such as \( \Phi \) can be written as

\[
\Phi = \frac{1}{2} \begin{pmatrix}
\phi_3 & \phi_1 - i\phi_2 \\
\phi_1 + i\phi_2 & \phi_3
\end{pmatrix} = \phi_a T^a, \quad \phi_a(x) \in \mathbb{R}, \quad a \in \{1, 2, 3\}, \quad \text{(2.6)}
\]

where the \( T^a \) are the generators of the two-dimensional representation of SU(2)(A3). Show that \( \text{Tr}\{\Phi \Phi\} = \frac{1}{2} \phi_a \phi_a \).

Assuming there is no symmetry breaking, identify the physical degrees of freedom and their masses. A detailed derivation is not needed here but you should indicate how you arrive at your answer. The appendix contains information on the representations of SU(2).

[4 marks]

(v) For \( m^2 < 0 \) the vacuum expectation value of the scalar fields, \( \Phi_0 \), satisfies \( \text{Tr}\{\Phi_0 \Phi_0\} = v^2/2 = -m^2/(2\lambda) \).

By using the form (2.6) and by shifting the \( a = 3 \) scalar component by an appropriate amount \( v \), so \( \phi_3(x) = v + \sigma(x) \), show that the physical degrees of freedom in the theory (2.5) in the presence of symmetry breaking are one massive scalar, two massless scalars and two Dirac fermions of different masses. Find expressions for the masses.

What is the stability or little group (no detailed proof required for this)?

[7 marks]

[TOTAL 20 marks]
3. In this question we look at the pattern of masses generated by the symmetry breaking in the electro weak model.

The contribution to the electroweak Lagrangian coming from the gauge boson, lepton (first generation only) and scalar sectors can be written as

$$\mathcal{L} = \mathcal{L}_{W,B} + \mathcal{L}_l + \mathcal{L}_\Phi$$  \hspace{1cm} (3.1)

where

$$\mathcal{L}_l = \bar{l} \mathcal{D}_l \gamma_\mu l + \bar{e} \mathcal{D}_e \gamma_\mu e + g_\ell \Phi^\dagger \bar{e} R l L + \text{(h.c.)},$$  \hspace{1cm} (3.2)

$$\mathcal{L}_{W,B} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G^{a}_{\mu\nu} G^{a,\mu\nu},$$  \hspace{1cm} (3.3)

$$\mathcal{L}_\Phi = (\mathcal{D}_\mu \Phi)^\dagger (\mathcal{D}_\mu \Phi) - m^2 (\Phi^\dagger \Phi) - \lambda (\Phi^\dagger \Phi)^2, \quad m^2 < 0, \lambda > 0.$$  \hspace{1cm} (3.4)

The field strengths are

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,$$  \hspace{1cm} (3.5)

$$G^{a}_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu,$$  \hspace{1cm} (3.6)

and the fields are

$$l_L = \begin{pmatrix} \nu_L(x) \\ e_L(x) \end{pmatrix}, \quad e_R = e_R(x), \quad \Phi = \begin{pmatrix} \Phi^{(+)}(x) \\ \Phi^{(0)}(x) \end{pmatrix}.$$  \hspace{1cm} (3.7)

The field $e_R$ and the components of $l_L$ are the spinors representing the appropriate chiral eigenstates of the relevant particles ($e(x) = \text{electron}$, $\nu(x) = \text{neutrino}$). Spinor indices are suppressed. $\Phi^{(+)}(x)$ and $\Phi^{(0)}(x)$ are complex fields. The fields $l_L$ and $\Phi$ are vectors in the two-dimensional SU(2) representation.

The covariant derivatives are

$$\mathcal{D}_L^\mu = \partial^\mu - i g \frac{1}{2} \tau^a W^{a,\mu}(x) + i g' \frac{1}{2} B^\mu(x),$$  \hspace{1cm} (3.8)

$$\mathcal{D}_R^\mu = \partial^\mu + ig' B^\mu(x),$$  \hspace{1cm} (3.9)

$$\mathcal{D}^\mu = \partial^\mu - i g \frac{1}{2} \tau^a W^{a,\mu}(x) - i g' \frac{1}{2} B^\mu(x).$$  \hspace{1cm} (3.10)

The SU(2) weak isospin generators for the Higgs and left handed lepton fields are $T^a = \frac{1}{2} \tau^a$ ($a \in \{1, 2, 3\}$), as given in the appendix in (A.3). Note that $e_R$ is an SU(2) singlet so its SU(2) generators are equal to zero.

[Question 3 continued overleaf]
The fourth generator is the abelian $U_Y(1)$ weak hypercharge generator, $Y$. In these conventions $Y$ is equal to the unit matrix multiplied by the weak hypercharge $q_Y$,

$$Y = q_Y I$$

(3.11)

leaving a factor of a half with the $g'$ factor in the covariant derivatives. As always with generators of abelian groups, $Y$ does not have to satisfy the usual non-abelian generator normalisation conventions.

There is one unbroken generator

$$Q = T^3 + \frac{1}{2} Y.$$  

(3.12)

We will work with a Higgs field, which can be defined as

$$\Phi = U(x) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} (v + h(x)) \end{pmatrix}, \quad v, h(x) \in \mathbb{R},$$

(3.13)

$$U(x) = \exp\{i\theta^a(x)T'^a\}, \quad \theta^a(x) \in \mathbb{R}, \quad (a \in \{1, 2, 3\})$$

(3.14)

where the $T'^a$ are the broken generators.

(i) State, without proof, Goldstone’s theorem regarding the number of massless scalar modes and the number of broken generators when a global symmetry is broken. State, in one sentence, what happens to the massless particles and the masses of gauge particles when the symmetry is local. Hence state a relationship between broken generators and massive gauge bosons.

List the number of degrees of freedom in each distinct bosonic mode in the electroweak model (i) before symmetry breaking ($v = 0$) and (ii) after symmetry breaking. You should note the nature and number of each distinct type of bosonic particle involved (spin 0 or 1, massive or massless). Hence show that the total number of bosonic degrees of freedom is the same in the cases of both (i) unbroken and (ii) broken symmetry.

[5 marks]

(ii) Explain qualitatively how the factor of $U(x)$ in (3.14) can be removed and hence why it is unphysical.

[5 marks]

(iii) Find the masses of the leptons in terms of $g_e$ and $v$. You may use without proof the facts that a Dirac fermion can be split into its left and right handed parts as $\psi = \psi_R + \psi_L$ and also that

$$\bar{\psi} \psi = \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R.$$  

(3.15)

[5 marks]

(iv) By calculating the relevant part of $D^\mu \Phi$ or otherwise, find the combinations of the $W^1, W^2, W^3$ and $B$ gauge bosons which correspond to the gauge boson mass eigenstates and calculate their masses in terms of $v$, and some combination of $g, g'$ and/or the Weinberg angle $\theta_w$ where $\tan(\theta_w) = g'/g$.

[5 marks]

[TOTAL 20 marks]
4. In this question, you may quote any relevant theorems or standard results without proof provided you state such principles clearly.

Consider a theory of one complex scalar field \( \Phi(x) \in \mathbb{C} \), two Dirac fermions \( \psi(x), \eta(x) \) (spinor indices are suppressed in this question), and two gauge Boson fields \( A^\mu(x), B^\mu(x) \in \mathbb{R} \). Their dynamics is described by the Lagrangian

\[
\mathcal{L} = (D(\Phi) \mu \Phi)^{\dagger} (D(\Phi) \mu \Phi) + i \bar{\psi} \gamma^\mu D(\psi) \mu \psi + i \bar{\eta} \gamma^\mu D(\eta) \mu \eta
\]

\[
- \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} G^{\mu\nu} G_{\mu\nu}
\]

\[
- M^2 |\Phi|^2 - m_2 \bar{\psi} \psi - m_3 \bar{\eta} \eta^2 - \lambda |\Phi|^4 - g (\Phi \bar{\psi} \eta + \Phi^\dagger \bar{\eta} \psi)
\]

(4.1)

where

\[
F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu, \quad G^{\mu\nu} := \partial^\mu B^\nu - \partial^\nu B^\mu
\]

(4.2)

\[
D(\phi) \mu := \partial^\mu - ie_a A^\mu(x) - eb B^\mu(x)
\]

(4.3)

\[
D(\psi) \mu := \partial^\mu - 2ie_b B^\mu(x)
\]

(4.4)

\[
D(\eta) \mu := \partial^\mu + ie_a A^\mu(x) - eb B^\mu(x)
\]

(4.5)

and \( M^2, m_2^2, m_3^2, e_a, e_b, \lambda, g \) are all real constants.

Suppose \( M^2 < 0 \) but the remaining constants \( m_2^2, m_3^2, e_a, e_b, \lambda, g \) are all positive. Find the masses and charges of the physical particles.

You may find the following useful:

\[
\begin{pmatrix} a & b \\ b & d \end{pmatrix} = B^{-1} \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} B
\]

(4.6)

\[
B = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad \tan(2\alpha) = \frac{2b}{a-d}
\]

(4.7)

\[
\lambda_{\pm} = \left( \frac{a+d}{2} \right) \pm \left[ \left( \frac{a-d}{2} \right)^2 + b^2 \right]^{1/2}
\]

(4.8)

[TOTAL 20 marks]

End of questions.

Please turn over for Appendix.
Appendix: Representations of SU(2) and SO(3)

SU(2) is three dimensional with generators $T^a_{ij}$ where $a, b, c = 1, 2, 3$ while $i, j$ range over the dimension of the representation. Here we work with representations which satisfy

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab} \quad \text{(A.1)}$$

The rank of SU(2) is 1 so the Cartan sub-algebra is dimension one generator and only one SU(2) generator can be diagonal.

Two-dimensional representations of SU(2) only

The generators are half the Pauli matrices $T^a = \frac{1}{2} \sigma^a$, and as well as (A.1) satisfy

$$(T^a)^\dagger = T^a, \quad \text{Tr}(T^a) = 0, \quad [T^a, T^b] = i \epsilon^{abc} T^c, \quad \{T^a, T^b\} = \delta^{ab} \mathbb{I}, \quad \text{(A.2)}$$

where $\epsilon^{abc}$ is the totally anti-symmetric tensor with $\epsilon^{123} = +1$. Thus

$$T^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \text{(A.3)}$$

Three-dimensional representations of SU(2) and SO(3)

The adjoint representation is present in all Lie Algebras, where one can write $T^a_{ij} = -if^{aij}$, $a, i, j = 1, \ldots, \dim(G)$ and the $f^{abc}$ are the structure constants for the Lie algebra. The adjoint representation is the three-dimensional representation of $SO(3)$ and $SU(2)$. The two common forms given here, related to each other by a unitary transformation, both satisfy

$$[T^a, T^b] = i \epsilon^{abc} T^c. \quad \text{(A.4)}$$

where $\epsilon^{abc}$ is the totally anti-symmetric tensor with $\epsilon^{123} = +1$. Note that the structure constants differ from those used in the two-dimensional case, (A.2), by a factor of 2.

The first three dimensional representation has $T^3$ in diagonal form.

$$T^1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad \text{(A.5)}$$

A second way of writing the three dimensional representation has no generator in diagonal form. If we think in terms of rotations of real three-dimensional vectors, the definition of SO(3), this second representation is quickly found. It is therefore natural to use this when wanting to find the representation of SO(3) in terms of real fields. It is also the natural way to write the three dimensional representation in the way which is common for the adjoint representation. Here this gives

$$T^b_{bc} = -\frac{i}{2} \epsilon^{abc}. \quad \text{(A.6)}$$

which can be written out as

$$T^1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{(A.7)}$$