Answer THREE questions.

All questions carry equal marks.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.
1. (i) Define a group.
   Show that the group identity is unique.
   Show that there is a unique inverse for every element.
   Show that for any group \( G \), \((ab)^{-1} = b^{-1}a^{-1} \) \( \forall \ a, b \in G \).
   Let \( g_1, g_2, \ldots, g_n \in G \). Determine the inverse of the \( n \)-fold product \( g_1g_2\cdots g_n \). [9 marks]

(ii) What is an abelian group?
   Show that a group is Abelian if and only if \((ab)^{-1} = a^{-1}b^{-1} \). You need to show
   that this condition is both necessary and sufficient for the group to be Abelian. [4 marks]

(iii) Use matrix multiplication to generate an order 8 group starting from the two-
   dimensional matrices \( X_1 = -i\tau_1 \) and \( X_2 = -i\tau_2 \) (where \( i^2 = -1 \) as normal).
   The three Pauli matrices \( \tau_a \) (\( a = 1, 2, 3 \)) satisfy
   \[
   (\tau_a)^2 = I, \quad \tau_a\tau_b = -\tau_b\tau_a \ (a \neq b), \quad (1.1)
   \tau_1\tau_2 = \tau_3, \quad \tau_2\tau_3 = \tau_1, \quad \tau_3\tau_1 = \tau_2. \quad (1.2)
   \]
   You must show that your elements form a group. [7 marks]

[TOTAL 20 marks]
2. (i) Define what is meant when elements \(a\) and \(b\) of a group \(G\) are said to be conjugate. Define a conjugacy class \([a]\) where \(a \in G\).
Define the order of a group element.
Prove that the order of the identity is one but all other elements have an order of two or more.
Show that the order of all elements in a class is the same. [5 marks]

(ii) The dihedral group of eight elements \(D_4\) may be defined to be

\[
D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}\quad \text{where } a^4 = e, \quad b^2 = e, \quad ab = ba^{-1}
\]

(2.1)

where \(e\) is the identity. The group multiplication law may assumed to be associative but otherwise is defined by the relations given above.
If we denote \(a^0 = e\) and \(a^{-j} = (a^{-1})^j\) then show that for any \(i, j \in \{0, 1, 2, 3\}\) the following are true

\[
a^{-j} = (a^j)^{-1} = a^{4-j}, \tag{2.2}
\]
\[
a^i b = ba^{4-j}, \tag{2.3}
\]
\[
(a^i b)(a^j b) = a^{i-j}. \tag{2.4}
\]

Hence show that the group axioms are satisfied. [10 marks]

(iii) Find the order of each element of \(D_4\).
Find the classes of \(D_4\). [5 marks]

[TOTAL 20 marks]
3. (i) Define a product group \( G = A \times B \) in terms of ordered pairs of elements \((a, b)\) taken from the two groups \(A\) and \(B\), where \(a \in A, b \in B\).

Give a second definition of a product group \( G \) in terms of two proper subgroups \( H, K \subset G \) where \( H \cup K \neq G, H \cap K = \{e\} \) where \(e\) is the identity of \(G\).

Define a **homomorphism** and an **isomorphism**.

Show that a product group defined using the first definition always satisfies the criteria of the second definition. Start from the first definition of a product group \( G = A \times B \) and give two appropriate subsets \( H \in G \) and \( K \in G \). Then show that these subsets are in fact subgroups under the multiplication rule. Then show the remaining parts of the second definition are true. [6 marks]

(ii) In terms of the classes of \(A\) and \(B\), what are the classes of \(A \times B\)? Prove your classes of \(A \times B\) are indeed classes and state how many are there in terms of the numbers of classes in the groups \(A\) and \(B\). [5 marks]

(iii) What is the symmetry group of a rectangle in a plane?

Write down the character table of the symmetry group for the rectangle’s symmetry group.

The energy eigenfunctions for a free particle in a rectangular box of size \(L_x \times L_y\) \((L_x \neq L_y)\) are given by

\[
\varphi_{p,q}(x, y, t) = \exp\{iE_{p,q}t/\hbar\}u_{p,q}(x, y),
\]

\[
u_{p,q}(x, y) \propto \begin{cases} 
\cos(k_px) & \text{odd} \\
\sin(k_px) & \text{even}
\end{cases}
\begin{cases} 
\cos(k_qy) & \text{odd} \\
\sin(k_qy) & \text{even}
\end{cases}
\]

where \(|x| \leq L_x/2, |y| \leq L_y/2, k_p = p\pi/L_x, k_q = q\pi/L_y\), and \(p\) and \(q\) are positive integers. The notation above means that \(\cos(k_px)\) is taken if \(p\) is odd, \(\sin(k_px)\) is taken if \(p\) is even. The \((k_qy)\) factor is chosen in a similar way depending on the odd/even nature of \(q\). \(Z\) is a constant normalisation factor. The corresponding eigenvalues are

\[
E_{p,q} \propto \left( \frac{p^2}{L_x^2} + \frac{q^2}{L_y^2} \right), \quad p, q \in \mathbb{Z}^+.
\]

What are the symmetry transformations on these eigenstates. Hence relate the different irreps of the symmetry group of a rectangle to the odd/even nature of the integers \(p\) and \(q\) which label the states. [9 marks]

[TOTAL 20 marks]
4. In this question, you may use any theorems or lemmas provided you state them clearly.

There are 12 symmetry operations of a regular hexagon which form the group $D_6$. The operations fall into 6 classes which are denoted $E$, $2C_6$, $2C_3$, $C_2$, $3\sigma_v$ and $3\sigma_d$.

(i) Show that $D_4$ has four one-dimensional irreps and two two-dimensional irreps.

Explain why the character of the class $E = \{e\}$ is always the dimension of the representation.

What is the trivial representation and why are its characters always 1 for any class?

What is the parity irrep? Why is the character given in the table below for irrep is $\Gamma_1$ that of the parity irrep of $D_6$?

Do all groups have a parity irrep?  

(ii) The two-dimensional irrep $\Gamma_2$ is equivalent to the representation made from coordinate transformations in two dimensions. In an orthogonal set of orthogonal axes, a proper rotation by $\phi$ in two dimensions is given by a matrix

$$ R(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}. $$  

Show that this irrep has the character given in the table below.

(iii) Without detailed calculation explain why the following group elements in the irrep $\Gamma_2$ satisfy the following relationships


What property of the $\Gamma_2$ irrep means that the group elements in any representation must obey the same identities?

What is the special property of one-dimensional irreps that means their characters satisfy the same relationships, e.g. $\chi(C_3)\chi(C_6) = \chi(C_2)$? The character associated with group element $g$ in some representation is denoted by $\chi(g)$.

Use these relationships (4.2) to show that for the one-dimensional irreps $\chi(C_3) = 1$ while $\chi(C_3) = \chi(C_2) = \pm 1$.  

[Question 4 continued overleaf]
(iv) Write down an expression for class orthogonality and for class normalisation.

Use class normalisation and the characters already found to show that $\chi(C_6) = \pm 1$ for the second two-dimensional irrep $\Gamma''_2$.

Hence use class orthogonality on classes $E$ and $C_6$ to show that $\chi(C_6) = -1$ for all the irreps $\Gamma''_1$, $\Gamma'''_1$ and $\Gamma''_2$.

[4 marks]

(v) The character table for $D_6$ as derived so far, along with the characters for a class $3\sigma_d$ is given below. Complete this table.

<table>
<thead>
<tr>
<th>$C_{6v}$</th>
<th>$\Gamma''_1$</th>
<th>$\Gamma''_2$</th>
<th>$\Gamma'''_1$</th>
<th>$\Gamma'''_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$2C_6$</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$2C_3$</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>$C_2$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$3\sigma_v$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$3\sigma_d$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

[4 marks]

[TOTAL 20 marks]
5. Consider a linear operator, $\hat{M}$, which acts on certain functions. We will consider the vector space $\mathcal{U}$ of eigenfunctions of the linear operator $\hat{M}$, all with the same eigenvalue $\lambda$. Thus $\mathcal{U} = \{u|\hat{M}u = \lambda u\}$. Let the set of symmetry operators be $\{\hat{S}\}$ and you may assume these form a group.

(i) Let $d$ eigenfunctions $\{u_n\} (n = 1, 2, \ldots, d)$ of the same eigenvalue $\lambda$ be a basis for the vector space $\mathcal{U}$. Show that each operator $\hat{S}$ maps one basis eigenvector as follows:

$$\hat{S}(g)u_m = \sum_{n=1}^{d} R_{nm}(g)u_n, \quad \forall \hat{S}(g) \in S, m, n = 1, 2, \ldots, d \quad (5.1)$$

where distinct labels $g$ are used to label the distinct operators $\hat{S} \in S$ and their associated matrices $R(g)$.

State the relationship between the matrices $R$ and the symmetry group of the operators $\hat{S}$. [5 marks]

(ii) The equation $\nabla^2 u_n(x, y) = \lambda_n u_n(x, y)$ is solved inside a region which has the shape of an equilateral triangle with $u(x, y) = 0$ on the boundaries. Without detailed proof explain why the symmetry group of this problem, the differential equation and its boundary conditions, is $S_3$.

Suppose the coordinates are changed using a symmetry transformation, from $(x, y)$ to $(x', y')$. How are the eigenfunctions in the transformed coordinates $u_n(x', y')$ related to the eigenfunctions in the original coordinates $u_n(x, y)$?

A numerical programme solves this problem and plots of its solutions for some of the orthogonal eigenfunctions are shown below. Answer the following questions about these results making reasonable assumptions about the nature of numerical solutions. Before attempting to answer this, read the whole question as additional information is given below about these eigenfunctions and about $S_3$.

Which of the eigenfunctions $u_7, u_8, u_9, u_{10}, u_{11}$ shown below must be degenerate by symmetry?

Assign the eigenfunctions $u_7, u_8, u_9, u_{10}, u_{11}$, or combinations of them, to irreducible representations. Hence, identify any accidental degeneracy. Hint: work backwards starting from $u_{11}$.

[Question 5(ii) continued overleaf]
The programme has also been used to plot various linear combinations of the $u_7$ and $u_8$ eigenfunctions shown below. Here the

$$u'_7(x, y) = -\frac{1}{2}u_7(x, y) + \frac{\sqrt{3}}{2}u_8(x, y), \quad u'_7(x, y) = -\frac{\sqrt{3}}{2}u_7(x, y) - \frac{1}{2}u_8(x, y).$$

(5.2)

A rotation matrix in two-dimensions takes the form given in question 4 equation (4.1).

The character table of $S_3$ is

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>$E$</th>
<th>$3C_3$</th>
<th>$2\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma_2$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma_3$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

[u_7+u_8 = ]

[u_7–u_8 = ]

[u_7' = ]

[u_8' = ]

[15 marks]

[TOTAL 20 marks]
6. (i) Show that special unitary matrices form a group.

A special unitary two-by-two matrix can be written as

\[
S = \begin{pmatrix}
a_0 + ia_3 & a_2 + ia_1 \\
-a_2 + ia_3 & a_0 - ia_3
\end{pmatrix}, \quad \sum_{i=1}^{4} (a_i)^2 = 1, \quad a_i \in \mathbb{Z}, \quad (6.1)
\]

\[
= a_0 \mathbb{I} + i \sum_{i=1,2,3} a_i \tau_i. \quad (6.2)
\]

where \(\tau_i\) are the three Hermitian Pauli matrices

\[
\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.3)
\]

How does this form tell us the group is compact?

Explain how this form tells us the dimension of SU(2). How does the dimension of SU(2) compare with the dimension of this representation?

Use this form to show that the Pauli Matrices are generators of a two-dimensional SU(2) representation. \[10 \text{ marks}\]

(ii) The Pauli matrices satisfy

\[
\tau_i^2 = \mathbb{I}, \quad \tau_i \tau_j = -\tau_j \tau_i = i \varepsilon_{ijk} \tau_k \quad i \neq j \neq k \in \{1, 2, 3\}. \quad (6.4)
\]

A sum of Pauli matrices may be written as \(a \cdot \sigma = \sum_i a_i \tau_i\) where \(a\) is some three-dimensional real vector, and the Pauli matrices are components of \(\sigma = (\tau_1, \tau_2, \tau_3)\).

Show that

\[
(a \cdot \sigma)(b \cdot \sigma) = (a \cdot b) \mathbb{I} + i (a \times b) \cdot \sigma. \quad (6.5)
\]

Hence show that

\[
\exp\{\sum_a \frac{i}{2} \varepsilon^a \tau^a\}, \quad (6.6)
\]

is a special unitary matrix. What does this tell us about the relationship between the generators of the Lie algebra and the Lie group elements? \[10 \text{ marks}\]

[TOTAL 20 marks]