UNIVERSITY OF LONDON
MSci EXAMINATION May 2007
for Internal Students of Imperial College of Science, Technology and Medicine
This paper is also taken for the relevant Examination for the Associateship

COSMOLOGY

For Third- and Fourth-Year Physics Students
Tuesday 29th May 2007: 14.00 to 16.00

Answer THREE questions.
All questions carry equal marks.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions
Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.
1. In General Relativity, the Friedmann equations are:

\[ \dot{R}^2 = \frac{8 \pi G \rho}{3} R^2 - k c^2 + \frac{\Lambda}{3} R^2. \]  
(1.1)

\[ \ddot{R} = -\frac{4 \pi G}{3} \left( \rho + \frac{3 p}{c^2} \right) R + \frac{\Lambda}{3} R. \]  
(1.2)

Where \( R \) is the scale factor of the universe, \( G \) the gravitational constant, \( \rho \) the density, \( p \) the pressure, \( c \) the speed of light and \( k \) is a constant.

(i) Use the Friedmann equations to derive the fluid equation

\[ \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) = 0. \]  
(1.3)

[3 marks]

(ii) Solve the fluid equation to determine the dependence of the density on the scale factor for a universe dominated by radiation (i.e. \( p = \rho c^2 / 3 \)) and, assuming \( k = \Lambda = 0 \), determine the dependence of the scale factor with time \( R(t) \) in a radiation dominated universe.

[5 marks]

(iii) Determine the corresponding solution for a matter dominated universe with \( k = \Lambda = 0 \).

Discuss the behaviour of \( R(t) \) when \( k = -1 \) and \( k = 1 \), and sketch all three results.

[5 marks]

(iv) Show that the temperature of the radiation in the universe varies as \( T \propto 1/R \).

Derive an expression for the redshift at matter-radiation equality as a function of the present matter density and radiation temperature. Hence determine the temperature of the universe at matter-radiation equality and at the earlier time \( t = 0.01 \) s for the following current parameters:

Temperature of the cosmic microwave background \( T_{\text{CMB}} = 2.73 \) K.

Matter density parameter \( \Omega_M = 1.0 \).

Age of the universe \( t_0 = 9.1 \) Gyr.

Hubble constant \( H_0 = 72 \) km s\(^{-1}\) Mpc\(^{-1}\).

Assume throughout that \( \Lambda = 0 \) and neglect the effect of neutrinos.

[7 marks]

[TOTAL 20 marks]

[The radiation constant \( a = 7.565 \times 10^{-16} \) J m\(^{-3}\) K\(^{-4}\), \( 1 \) pc = \( 3.09 \times 10^{16} \) m.]
2. (i) What is the meaning of the term $\Lambda$ in Equations 1.1 and 1.2 in Question 1? Give examples of the interpretation of this term and discuss briefly the evidence that, at the current epoch, $\Lambda > 0$. [5 marks]

(ii) For a pressureless universe with $\Lambda > 0$ and $k \leq 0$, discuss and sketch the behaviour of the scale factor $R(t)$ and derive the late-time behaviour of $R(t)$. [5 marks]

(iii) Show that, in a universe with $\Lambda > 0$ and $k = 1$ there exists a critical $\Lambda$ for which $\ddot{R} = \dot{R} = 0$ at some value $R_{\text{crit}}$. Derive expressions for $\Lambda_{\text{crit}}$ and $R_{\text{crit}}$ in terms of the scale factor, $R_0$, and density, $\rho_0$, at the present time. Describe and sketch the possible time evolution of universes with $k = +1$ and

(a) $\Lambda = \Lambda_{\text{crit}}$,

(b) $0 < \Lambda < \Lambda_{\text{crit}}$, and

(c) $\Lambda > \Lambda_{\text{crit}}$. [7 marks]

(iv) Assuming our universe is currently completely dominated by the cosmological constant, how long will it take to expand by a factor $10^6$? Express your answer both in terms of the Hubble Time $\tau_0$ and in Gyr. Assume $H_0 = 72$ km s$^{-1}$ Mpc$^{-1}$. [3 marks]

[TOTAL 20 marks]

$[1 \text{ pc} = 3.09 \times 10^{16} \text{ m.} ]$
3. (i) Describe the meaning of the terms in the Robertson-Walker Metric:

\[ ds^2 = dt^2 - \frac{R^2(t)}{c^2} \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) . \]

State why \( k \) can take values of 0, −1 and +1 without loss of generality and, re-expressing the metric as necessary, describe the geometry of universes defined by these three cases.

[5 marks]

(ii) Define the terms proper distance, luminosity distance and diameter distance. By considering a spherical surface in an expanding universe centered on a source of photons, show that the luminosity distance is related to the redshift \( z \) of the source by:

\[ d_{lum} = R_0 r_0 (1 + z) , \]

where \( r_0 \) is the radial comoving coordinate and \( R_0 \) is the scale factor when the photons are received.

[5 marks]

(iii) Show that in an Einstein-de-Sitter model (\( k = 0, \Lambda = 0 \)) the luminosity distance is given by:

\[ d_{lum} = 2 c \tau_0 \left( 1 + z - \sqrt{1 + z} \right) , \]

where \( \tau_0 = 1/H_0 \) is the Hubble time.

[4 marks]

(iv) Derive or deduce the corresponding diameter distance:

\[ d_{diam} = \frac{2c}{H_0} \frac{(1 + z)^{1/2} - 1}{(1 + z)^{3/2}} . \]

[3 marks]

(v) For an Einstein-de-Sitter universe, determine the redshift at which the minimum angular size occurs.

[3 marks]

[TOTAL 20 marks]
4. Write an essay describing how fluctuations in the cosmic microwave background (CMB) can be used to constrain the cosmological parameters. Include in your answer:

(i) A discussion of the physical processes in the early universe relevant to CMB fluctuations;

(ii) A description of how to calculate the CMB power spectrum and sketches which illustrate the differences expected for different parameters;

(iii) Details of the observations which have been used to constrain the parameters;

(iv) A description of the most important results which have been obtained by comparing these observations to models.

[TOTAL 20 marks]
5. (i) State the *cosmological principle*, defining any terms as necessary, and discuss briefly the empirical evidence on which this principle is based. [5 marks]

(ii) The age of the universe can be expressed as:

\[ t_0 = \int_0^{t_0} dt = \int_0^{R_0} \frac{dR}{R} . \]

In a flat \((k = 0)\) cosmology where the density parameter associated with the cosmological constant, \(\Omega_\Lambda\), and the matter density \(\Omega_M\), are both non-zero, show from this and equation 1.1 [on page 2] that the age of the universe is given by:

\[ t_0 = \frac{2\tau_0}{3} \sqrt{\frac{1}{1 - \Omega_M}} \sinh^{-1} \sqrt{\frac{1 - \Omega_M}{\Omega_M}} . \]

You may wish to use the following substitutions

\[ x = \frac{R}{R_0}, \quad y = x^{3/2} \quad \text{and} \quad \sqrt{\frac{1 - \Omega_M}{\Omega_M}} y = \sinh \theta . \]

[9 marks]

(iii) Evaluate the age when \(k = 0\) for the cases \(\Omega_M = 0.01\), \(\Omega_M = 0.5\) and \(\Omega_M = 0.99\) for a Hubble constant \(H_0 = 72 \text{ km s}^{-1} \text{ Mpc}^{-1}\). Compare them to any observational estimates of the age of the universe of which you are aware, and draw appropriate conclusions. [6 marks]

[1 pc = 3.09 \times 10^{16} \text{ m.}]
6. (i) Describe the flatness, horizon and monopole problems associated with the hot big bang theory. Discuss what is meant by the concept of cosmological inflation and how it might solve these problems. [5 marks]

(ii) CMB measurements from WMAP suggest that the total density parameter at the present time, $\Omega_0 = 1.0 \pm 0.02$. Use the Friedmann equation to estimate the limits on the density parameter at the time of matter radiation equality ($t_{eq} = 10^{11}$ s).
Assume the present age of the Universe is 13.9 Gyr and ignore the effects of $\Lambda$ on the expansion of the Universe. What are the limits at the Planck time (assumed to be $t = 10^{-43}$ s)? [5 marks]

(iii) Show that structures in the early universe will form only if they exceed the Jeans length:

$$L_J \approx v_s \left( \frac{G\rho}{c^2} \right)^{1/2},$$

where $v_s$ is the sound speed and $\rho$ the density. [3 marks]

(iv) In a matter-dominated universe (with $\Lambda = p = 0$) the density contrast $\delta$ can be shown to obey the equation:

$$\ddot{\delta} + 2\frac{R}{R} \dot{\delta} - 4\pi G \rho \delta = 0.$$

By trying a power-law solution, solve for the growing and decaying modes in an Einstein-de-Sitter universe ($\Omega_0 = 1$). Derive the solution for $\Omega_0 = 0$, and describe the late-time behaviour for both cases. Draw appropriate conclusions about how galaxy redshift surveys can be used to constrain the cosmological parameters. [7 marks]

[TOTAL 20 marks]

End