UNIVERSITY OF LONDON
BSc/MSci EXAMINATION June 2007
for Internal Students of Imperial College of Science, Technology and Medicine
This paper is also taken for the relevant Examination for the Associateship

ELECTROMAGNETISM & OPTICS

For Second-Year Physics Students
Tuesday 5th June 2007: 10.00 to 12.00

Answer ALL parts of Section A, TWO questions from Section B and
ONE question from Section C.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FIVE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FIVE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.
You may find the following useful

\[ \epsilon_0 = 8.85 \times 10^{-12} \, \text{F} \, \text{m}^{-1} \]
\[ \mu_0 = 4\pi \times 10^{-7} \, \text{Wb} \, \text{A}^{-1} \, \text{m}^{-1} \]
\[ c = 3 \times 10^8 \, \text{m} \, \text{s}^{-1} \]
\[ e = 1.6 \times 10^{-19} \, \text{C} \]
\[ m_e = 9.1 \times 10^{-31} \, \text{kg} \]

\[ \nabla \times \nabla f = 0 \]
\[ \nabla \cdot \nabla \times \mathbf{A} = 0 \]
\[ \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \]
\[ \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]

In cylindrical coordinates \((r, \theta, z)\)

\[ \nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \]
\[ (\nabla f)_r = \frac{\partial f}{\partial r} \]
\[ (\nabla f)_\theta = \frac{1}{r} \frac{\partial f}{\partial \theta} \]
\[ (\nabla f)_z = \frac{\partial f}{\partial z} \]
\[ (\nabla \times \mathbf{A})_r = \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \]
\[ (\nabla \times \mathbf{A})_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \]
\[ (\nabla \times \mathbf{A})_z = \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \]
SECTION A

1. The equation
\[ E = E_0 \cos(kz - \omega t)\hat{x} \]
describes the electric field component of a plane electromagnetic wave.

(i) For this wave write down expressions for the
(a) wavelength
(b) frequency
(c) phase velocity
(d) direction of travel
(e) direction of polarisation. [2 marks]

(ii) Write down an expression for the magnetic field component, \( B \), of the wave taking care to show the relative strengths and directions of the 2 fields. [2 marks]

(iii) The wave strikes an interface at \( z = 0 \). Write down the boundary conditions obeyed by both the electric and magnetic components of the wave when the interface is between vacuum and
(a) a dielectric of relative permeability, \( \epsilon_r \),
(b) an ideal metal of resistivity, \( \rho = 0 \). [2 marks]

(iv) Describe qualitatively the behaviour of microwaves incident on a real metal, such as Copper or Aluminium, and state the order of magnitude of any length scales introduced. [2 marks]

(v) Write down an expression for the energy flux carried by the wave. [2 marks]

[ TOTAL 10 marks]
2.  

(i) Explain what is meant by *circularly polarised light* and write down an expression for the electric field of the wave as a function of time.  

(ii) What is meant by *critical angle* in the context of refraction of light? Evaluate the critical angle for quartz, which has a refractive index of $n = 1.46$.  

[2 marks]  

[3 marks]  

[TOTAL 5 marks]
3. Consider an infinitesimally thin sheet at \( z = 0 \) in free space carrying a spatially uniform current (in Am\(^{-1}\))

\[
I = I_0 \cos(\omega t) \hat{x}.
\]

(i) Write down general expressions for the electric and magnetic components of the generated waves for both \( z > 0 \) and \( z < 0 \). [2 marks]

(ii) Show that:

(a) The electric fields immediately to the left (\( z < 0 \)) and right (\( z > 0 \)) of the current sheet are equal. [1 mark]

(b) The magnetic field immediately to the right of the current sheet is given by

\[
B = -\frac{1}{2}\mu_0 I_0 \cos(\omega t) \hat{y}.
\]

[2 marks]

(c) The electric field close to the current sheet is given by

\[
E = -\frac{1}{2}\mu_0 c I_0 \cos(\omega t) \hat{x} = -\frac{1}{2} \left( \frac{\mu_0}{\varepsilon_0} \right)^{1/2} I_0 \cos(\omega t) \hat{x}.
\]

[2 marks]

(d) The fields for general \( z > 0 \) are given by

\[
E = -\frac{1}{2}\mu_0 c I_0 \cos[\omega(t - z/c)] \hat{x}
\]

\[
B = -\frac{1}{2}\mu_0 I_0 \cos[\omega(t - z/c)] \hat{y}
\]

and for \( z < 0 \)

\[
E = -\frac{1}{2}\mu_0 c I_0 \cos[\omega(t + z/c)] \hat{x}
\]

\[
B = +\frac{1}{2}\mu_0 I_0 \cos[\omega(t + z/c)] \hat{y}
\]

and explain the concept of ‘retarded time’. [2 marks]

(iii) Now consider a second such current sheet at \( z = -d \) with current lagging in phase by \( \phi \).

(a) Show that the electric field for \( z > 0 \) may be written

\[
E = -\frac{1}{2}\mu_0 c I_0 \cos \left[ \omega \left( t - \frac{z}{c} \right) \right] \hat{x} - \frac{1}{2}\mu_0 c I_0 \cos \left[ \omega \left( t - \frac{z}{c} \right) - \left( \omega \frac{d}{c} + \phi \right) \right] \hat{x}
\]

and for \( z < -d \) by

\[
E = -\frac{1}{2}\mu_0 c I_0 \cos \left[ \omega \left( t + \frac{z}{c} \right) \right] \hat{x} - \frac{1}{2}\mu_0 c I_0 \cos \left[ \omega \left( t + \frac{z}{c} \right) + \left( \omega \frac{d}{c} - \phi \right) \right] \hat{x}.
\]

[1 mark]

(b) Hence show how to choose the values of \( d \) and \( \phi \) such that all the radiation is emitted to the right (\( z > 0 \)). [3 marks]

(iv) Discuss the significance of the above result for improved telecommunications. [2 marks]

[TOTAL 15 marks]
4. (i) Write down Maxwell’s 4 equations as applicable to a dielectric medium without free charges or free currents, carefully defining each of the quantities involved. [4 marks]

(ii) A wave is propagating in the $z$–direction through a cylindrical coaxial cable consisting of an inner metal wire of radius, $a$, an outer cylindrical sheath of radius, $b$, and a dielectric material of relative permittivity, $\epsilon_r$, for $a < r < b$. The magnetic field, $\mathbf{B}$, inside the cable is given by

$$\mathbf{B} = B_0 \frac{1}{r} \exp [i(kz - \omega t)] \hat{\theta}.$$ 

(a) Using one of Maxwell’s equations show that the corresponding electric field $\mathbf{E}$ may be written,

$$\mathbf{E} = \frac{k}{\omega} c^2 B_0 \frac{1}{r} \exp [i(kz - \omega t)] \hat{r},$$

where $c^2 = 1/\epsilon_r \epsilon_0 \mu_0$. [2 marks]

(b) Show that the values of $\mathbf{E}$ and $\mathbf{B}$ at both $a$ and $b$ are consistent with the conditions expected at an ideal metal. [2 marks]

(c) Show that these expressions for $\mathbf{E}$ and $\mathbf{B}$ are consistent with Maxwell’s other 3 equations as long as

$$\omega^2 = c^2 k^2.$$ [4 marks]

(d) Assuming $\mathbf{B} = 0$ for $r > b$ derive expressions for the current flowing in both the inner wire and the outer sheath. [3 marks]

[TOTAL 15 marks]
5. (i) State Poynting’s theorem, and explain the physical meaning of each of the terms.

(ii) A resonator consisting of a cuboidal box with metal boundaries at $x = 0$ and $x = a$, $y = 0$ and $y = b$, and $z = 0$ and $z = d$. It has a set of modes with electric fields given by

$$E = E_0 \cos \left( \frac{\pi \ell x}{a} \right) \sin \left( \frac{\pi m y}{b} \right) \sin \left( \frac{\pi n z}{d} \right) \cos (\omega t) \hat{x},$$

where $\ell, m$ and $n$ are integers.

(a) Show that these modes obey the boundary conditions appropriate for a metal boundary. [1 mark]

(b) Show that these modes obey the wave equation in a vacuum and give an expression for the angular frequency, $\omega$. [3 marks]

(c) Derive an expression for the magnetic field $B$ corresponding to the electric field in equation (5.1). [2 marks]

(d) Equation (5.1) describes a standing wave; show that it may be rewritten as a sum of 2 modes travelling backwards and forwards respectively in the $z$ direction. [2 marks]

(e) Write down an expression for the $z$ component of the Poynting vector of one of the travelling waves in part (d). [2 marks]

(f) Hence, or otherwise, show that the force on each of the walls of the metal box at $z = 0$ and $z = d$ may be written

$$F = \frac{\pi ab}{4d} \frac{n}{\mu_0 \omega c} E_0^2.$$

[2 marks]

[TOTAL 15 marks]
6. (i) Starting from Newton’s equations with a damping term, show that the a.c. conductivity of a metal may be written as

\[ \sigma(\omega) = \frac{ne^2\tau}{m_e} \frac{1}{1 - i\omega\tau} \]

where \( \tau \) represents the time between collisions and \( n \) the density of electrons. [3 marks]

(ii) Starting from Maxwell’s equations show that in a metal the electric field \( E \) at angular frequency \( \omega \) obeys the equation

\[ \nabla^2 E + i\omega\mu_0\sigma(\omega)E + \omega^2\mu_0\varepsilon_0E = 0. \]

[4 marks]

(iii) By comparing the wave equation (6b) with that for an electromagnetic wave of the same frequency in a dielectric medium of relative permittivity \( \varepsilon_r \) show that \( \varepsilon_r \) in a metal, as a function of \( \omega \), takes the form

\[ \varepsilon_r(\omega) = 1 + \frac{i\sigma(\omega)}{\varepsilon_0\omega}. \]

[2 marks]

(iv) The conductivity of Aluminium at 1800 MHz is \( 3.5 \times 10^7 \, \Omega^{-1} \, \text{m}^{-1} \). Show that \( \varepsilon_r \) in this case may be assumed to be purely imaginary. [1 mark]

(v) Hence derive an expression for the skin depth in a good metal and state its value in Aluminium at 1800 MHz. [3 marks]

(vi) What thickness of Aluminium foil would be required to reduce the energy flux of the radiation from a mobile phone at 1800 MHz to 1%? [Hint: It is not necessary to calculate the energy flux.] [2 marks]

[TOTAL 15 marks]
7. (i) Draw a labelled diagram of an optical arrangement for observation of the interference pattern from “Young’s slits”, including all essential elements. [4 marks]

(ii) Derive an approximate expression for the spacing of the fringes formed by light of wavelength $\lambda$. Hence find the spacing of fringes on a screen placed 0.5 m away from a pair of slits whose centres are separated by 0.25 mm, when illuminated by light at 589 nm. [5 marks]

(iii) By considering diffraction at the slits, or otherwise, show that the total number of fringes observed is roughly

$$N = \frac{2d}{a},$$

where $d$ is the slit spacing and $a$ is the width of each slit. [3 marks]

(iv) What other effect limits the visibility of the fringes? [3 marks]

[ TOTAL 15 marks]
8. (i) Silicon has a refractive index of $n = 3.48$ in the infra-red. Find the reflectance of infra-red radiation incident from air onto a silicon surface at normal incidence. [3 marks]

(ii) Explain briefly, with the aid of a diagram, the principle by which simple anti-reflection coatings work. Hence derive expressions for the refractive index and the thickness of such a coating. [6 marks]

(iii) What is the ideal value of refractive index and the ideal thickness for a single layer antireflection coating on silicon designed for radiation at 1.5 $\mu$m? [3 marks]

(iv) Discuss one other common application of thin film optical coatings, taking care to explain the advantage of using this technique. [3 marks]

[TOTAL 15 marks]