Answer THREE questions
All questions carry equal marks.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions
Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.
If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.
USE ONE ANSWER BOOK FOR EACH QUESTION.
Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.
Hand in THREE answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.
1. (i) Explain what is meant by the terms surjective map, injective map and bijective map for maps \( f : D \rightarrow T \) defined for any pair of sets \( D, T \). In each case, draw an illustrative diagram and state the relationship that must exist between \( D \) and \( T \) in the case of finite sets. [6 marks]

(ii) Show that the function \( f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) (where \( \mathbb{N} \) denotes the natural numbers) defined by,
\[
f(n, m) = (2n - 1) \times 2^{m-1},
\]
is both injective and surjective. [4 marks]

(iii) Explain what it means for an infinite set \( S \) to be countable. Use part (ii) to show that the positive rational numbers are countable. [2 marks]

(iv) Show that the function,
\[
f(2n) = n, \quad f(2n - 1) = -(n - 1),
\]
defines a bijection \( \mathbb{N} \rightarrow \mathbb{Z} \) (where \( \mathbb{Z} \) denotes the set of all integers, positive and negative). [2 marks]

(v) Use the fundamental theorem of arithmetic to construct a simple bijection from the \( n \)-fold Cartesian product \( \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \) into a subset of \( \mathbb{N} \). (A brief justification that it is bijective is sufficient). [2 marks]

(vi) Explain what is meant by an algebraic number. Using parts (iv) and (v), show that the algebraic numbers are countable. [4 marks]

[TOTAL 20 marks]
2. (i) Let $A_n = \{a_1, a_2, \ldots, a_n\}$ be a finite set with $|A_n| = n$. Write down the complete list of subsets of $A_n$ for the cases $n = 1, 2, 3$. [4 marks]

(ii) Use induction to show that the total number of possible subsets of $A_n$ defined in part (i) is $2^n$. [4 marks]

(iii) Use induction to show that,

$$(1 + p)^n \geq 1 + np,$$

for $p \geq 0$ and $n$ a positive integer. [3 marks]

(iv) Use induction to show that

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$  

Deduce the resulting expression in the limit $n \to \infty$. [4 marks]

(v) Show that the number,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!},$$

cannot be rational. [Hint: Write $e = p/q$ where $p, q$ are integers. You may assume $q \geq 3$. Use part (iv) where necessary.] [5 marks]

[TOTAL 20 marks]
3. (i) Explain what it means for a sequence $a_n$ to converge to the point $a$ as $n \to \infty$. Define the terms *monotonically increasing* sequence and *monotonically decreasing* sequence. 

[3 marks]

(ii) State without proof the condition or conditions ensuring that a monotonically increasing or decreasing sequence of real numbers converges. Is this also true of a sequence of rationals? Explain your answer. 

[3 marks]

(iii) Consider a set of subintervals of $\mathbb{R}$ of the form $I_n = [a_n, b_n], n = 1, 2, 3 \cdots$, which are nested, that is

$I_1 \supseteq I_2 \supseteq I_3 \cdots$

Using your answer to part (ii), show that there exists a number $\xi$ such that $\xi \in I_n$ for all $n$. 

[5 marks]

(iv) Suppose now that the intervals defined in part (iii) satisfy the condition $|a_n - b_n| \to 0$ as $n \to \infty$. Show that $\xi$ defined in part (iii) is unique. Which property of the real numbers does this theorem guarantee? 

[5 marks]

(v) Use the theorem proved in part (iii) to show that the real numbers lying in the interval $[0, 1]$ form an uncountable set. 

[4 marks]

[ TOTAL 20 marks]
4. (i) State without proof the ratio test for the convergence of the infinite series \( \sum_{n=1}^{\infty} a_n \).

[2 marks]

(ii) Use the ratio test to assess the convergence properties of the following series:

(a) \( \sum_{n=1}^{\infty} \frac{10^n}{n!} \).

(b) \( \sum_{n=1}^{\infty} \frac{n^2}{2^n} \).

[4 marks]

(iii) In Cauchy's root test, we consider the quantity

\[ \rho = \lim_{n \to \infty} |a_n|^{1/n} . \]

Prove that the series \( \sum_{n=1}^{\infty} a_n \) converges absolutely if \( \rho < 1 \). State without proof what happens in the cases \( \rho = 1 \) and \( \rho > 1 \).

[6 marks]

(iv) Use the root test to assess the convergence properties of the following series:

(a) \( \sum_{n=1}^{\infty} \frac{2^n}{n^n} \).

(b) \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

[4 marks]

(v) Consider the series,

\( \sum_{n=0}^{\infty} (2 - (-1)^n)2^n x^n \).

Show that the ratio test is indecisive for this series. Using the root test, show that the series converges absolutely for \( |x| < \frac{1}{2} \). [Hint: Consider separately the cases of \( n \) odd and \( n \) even.]

[4 marks]

[TOTAL 20 marks]
5. (i) State the definition of a continuous function \( f : \mathbb{R} \to \mathbb{R} \). [2 marks]

(ii) State the relationship between continuity of a function \( f \) at a point and the convergence of sequences \( a_n \) to that point. [2 marks]

(iii) Use the results of part (ii) to show that the function,
\[
\theta(x) = \begin{cases} 
1 & \text{for } x \geq 0, \\
0 & \text{for } x < 0,
\end{cases}
\]
is not continuous at \( x = 0 \). [4 marks]

(iv) Suppose that the function \( f : \mathbb{R} \to \mathbb{R} \) satisfies the relation,
\[
f(x + y) = f(x) + f(y).
\] (5.1)
Assuming that \( f \) is differentiable, show that its most general form is,
\[
f(x) = Kx,
\]
where \( K \) is a constant. [4 marks]

(v) Now deduce the most general form of the function \( f \) defined in Eq.(5.1), assuming only that it is continuous (but not necessarily differentiable). [Hint: Use induction to prove that \( f(nx) = nf(x) \), explore different choices of \( x \), and use the results of part (ii).] [8 marks]

[TOTAL 20 marks]
6. (i) Confirm by explicit calculation that any \(2 \times 2\) matrix,
\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]
satisfies the equation,
\[
M^2 - (\text{Tr} M)M + (\det M)I = 0,
\]
where \(\text{Tr} M\) and \(\det M\) should be defined. Write down, without proof, expressions for \(\text{Tr} M\) and \(\det M\) in terms of the eigenvalues \(\lambda_1, \lambda_2\).

[5 marks]

(ii) Suppose that the two eigenvalues of the matrix \(M\) are equal, and denoted \(\lambda\). Show that the matrix \(G = M - \lambda I\) satisfies \(G^2 = 0\).

[3 marks]

(iii) Use the result of part (ii) to derive simple expressions for \(M^n\) and \(e^{Mt}\) (again for the case of equal eigenvalues).

[5 marks]

(iv) Compute the fourth power of the matrix,
\[
M = \begin{pmatrix} -1 & 9 \\ -1 & 5 \end{pmatrix},
\]

[4 marks]

(v) Compute \(e^{Mt}\) for the matrix defined in part (iv).

[3 marks]

[ TOTAL 20 marks]