COUPLED MODE DEVICES

10.1 INTRODUCTION

Until now, we have been concerned mainly with the properties of perfect, uniform guides. We have shown them all to support a set of characteristic modes, which are (by definition) field structures that propagate down the guide with only a change in phase. We have encountered some exceptions to this rule - most notably, waveguide tapers, bends, gratings and beam couplers, where the mode amplitudes also change due to power redistribution between guided modes, or conversion from guided to radiation modes - but we have managed to avoid any quantitative calculations of the amplitude changes concerned. It is now the time to rectify this omission.

It turns out that a common physical principle underpins the behaviour of nearly all perturbed waveguide structures: namely, that the introduction of a perturbation to an otherwise perfect guide can cause an interchange of energy amongst the modes of that guide. This interchange can be highly significant, if a number of conditions are satisfied. In fact, one individual mode can be converted into another with close to 100% efficiency in many cases. This principle, known as mode coupling, is ubiquitous, and occurs in many branches of physics other than guided wave optics (most notably, in quantum mechanics). As far as we are concerned, it is important to note that besides explaining the performance of waveguides suffering from some imperfection or distortion, the principle can also be exploited for use in new guided wave devices. We shall illustrate this point by considering some of the wide range of devices that operate through coupling between co- and contra-directional modes.

10.2 THE DIRECTIONAL COUPLER - BASIC PRINCIPLES

We will start with one of the most successful and versatile devices in both integrated and fibre optics, the directional coupler, which works by coupling together two modes travelling in the same direction. In its simplest form, this acts as a beam splitter, but more complicated devices can be used as two-way switches or modulators; further variants can be used as filters or polarizers. In broad outline, the device works as follows.

From previous experience, we know that there is an evanescent field extending outside any dielectric waveguide. If two parallel guides are placed sufficiently close together, these parts of the field must overlap spatially. Usually, the interwaveguide gap required for this overlap to be significant is of the order of the guide width. For example, Figure 10.2-1 shows the situation for the lowest-order modes in two identical, adjacent symmetric slab waveguides. The refractive index distribution of the twin-guide system is also shown. The indices of the two guide cores are \( n_{g1} \) and \( n_{g2} \), respectively, and the index in the substrate material outside the guide has the uniform value \( n_s \).

Because of the intermodal overlap, it is hard to decide which guide any light in the overlap region actually belongs to. This implies that there must be a mechanism for light in one guide to be transferred to the other. It turns out (as we shall see later) that, if the two guides satisfy a particular set of conditions, and run parallel for sufficient distance, this interchange of power can be highly significant, reaching almost 100% in many cases. The light then starts coupling back, so the power transfer process must be periodic with distance. We can represent this as shown in Figure 10.2-2.
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Figure 10.2-1 Overlap of the transverse fields of two adjacent parallel symmetric slab guides.

Here the input is to Guide 1, at \( z = 0 \). We shall assume that complete coupling occurs at \( z = L \), when all the light has been transferred to Guide 2. At \( z = 2L \), therefore, the light will have been completely coupled back to Guide 1, and so on. In fact, the process is a continuous, bidirectional one, of a type known as **multiple scattering**. The condition the guides must satisfy for 100% power transfer is therefore that they are **synchronous** (i.e. have the same propagation constants) so that all the scattered contributions add up in-phase. This usually requires them to be identical.

Figure 10.2-2 Conceptual representation of the coupling process.

We can use this effect to make a device that can switch light between two waveguides, as follows. We start with two identical single-moded guides, and choose the length of the coupling region so that the device runs from \( z = 0 \) to \( z = L \). Under normal circumstances, an input to Guide 1 will then be completely coupled across, and emerge as an output from Guide 2. If we can now arrange to desynchronise the guides under electrical control, we can prevent the power transfer process occurring. In this case, the light emerges from Guide 1, without any cross coupling. Since the choice of output guide is now electrically alterable, the device functions as a two-way switch.

**INTEGRATED OPTIC DIRECTIONAL COUPLERS IN INSULATING CRYSTALS**

Figure 10.2-3 shows the cross-section of a structure suited to realisation in integrated optics. This is made from two channel waveguides buried in an insulating, electro-optic
substrate, with a pair of surface electrodes. The Ti:LiNbO$_3$ process is now well developed for fabrication of such devices. We have already discussed the effect of this type of electrode pair in Chapter 9. When a D.C. voltage is applied between the two electrodes, the result is a static electric field, represented by the field lines in Figure 10.2-3. Note that the field is directed mainly downwards through the left-hand guide, and upwards through the right-hand one. This will produce a change in the refractive index through the linear electro-optic effect. The exact result will depend on the precise field variation, but the crystal orientation will also affect matters. For simplicity, let us assume that the crystal is chosen so that the vertical field component is the important one for the particular polarization of light involved.

![Figure 10.2-3 Cross-section of an electro-optic directional coupler.](image)

Now, the linear electro-optic effect is (by definition) linear. In this case, therefore, we can deduce that the index change must be of opposite sign in the two guides. Consequently, any change in guide propagation constant that results must also be of opposite sign. The application of a D.C. voltage to the electrodes can therefore be used to desynchronise two otherwise identical guides. The difference between the propagation constants is often written as $\Delta \beta = \beta_2 - \beta_1$, and is proportional to voltage. The electrode structure shown is consequently often known as a pair of $\Delta \beta$ electrodes. An electro-optic device using such electrodes is called a **directional coupler switch**, shown in plan view in Figure 10.2-4. Here, additional bend sections have been included to separate the waveguides at the input and output, for example for coupling to pairs of fibres (which have a large core-to-core separation due to their cladding). Integrated optic directional couplers in semiconducting crystals require a rather different electrode construction, which will again be covered in Chapter 12.

![Figure 10.2-4 Plan view of a directional coupler switch.](image)
10.3 THE DIRECTIONAL COUPLER - THEORETICAL ANALYSIS

We will now form a theoretical model of the directional coupler. The derivation of the relevant equations is, unfortunately, rather lengthy, so for simplicity we base the analysis on scalar theory, and concentrate on the central region of the device where the guides run parallel. We start by assuming that the two guides are oriented in the z-direction, and are described in isolation by the refractive index distributions $n_1(x, y)$ and $n_2(x, y)$, respectively. Figure 10.3-1 shows how these distributions might look for a pair of symmetric slab waveguides. Here the refractive index outside the guides is everywhere $n_s$, while the indices in the two guiding layers are $n_{g1}$ and $n_{g2}$ respectively. Assuming that the electric field is polarized in the y-direction, the scalar wave equation governing the variation of the electric field $E_{yi}(x, y, z)$ for each guide in isolation is then given by:

$$\nabla^2 E_{yi}(x, y, z) + n_i^2(x, y)k_0^2 E_{yi}(x, y, z) = 0$$

Both guides are now assumed to be single-moded, with guided eigenmode solutions defined by:

$$E_{yi}(x, y, z) = E_i(x, y) \exp(-j\beta_i z)$$

Where $i = 1$ or $2$, and where $E_i(x, y)$ is the transverse field of the mode in the $i^{th}$ guide, and $\beta_i$ is its propagation constant. These solutions must satisfy the individual waveguide equations:

$$\nabla^2_{xy} E_i(x, y) + [n_i^2(x, y)k_0^2 - \beta_i^2] E_i(x, y) = 0$$

Here the subscripts on the Laplacian imply that the differentiation is to be performed with respect to $x$ and $y$ only. We take it that these equations have previously been solved by some means, so that we know both of the transverse field distributions $E_i(x, y)$ and both propagation constants $\beta_i$.

![Diagram of refractive index distributions](image)

Figure 10.3-1 The refractive index distributions $n_1(x)$ and $n_2(x)$ for two isolated guides.
Now, we assume that the complete coupler is described by a refractive index distribution $n_T(x, y)$, which corresponds to both guides together. We have already shown in Figure 10.2-1 how this distribution might look for the slab waveguide model used above. The scalar wave equation for the complete coupler is then:

$$\nabla^2 E_y(x, y, z) + n_T^2(x, y)k_0^2 E_y(x, y, z) = 0$$

10.3-4

Now, because we expect power to be coupled slowly from one guide to the other, we assume that we can describe the solution for $E_y(x, y, z)$ as a superposition of the modes in the two isolated guides, i.e. as a linear combination of the two modal solutions previously found. We therefore put:

$$E(x, y, z) = A_1(z) E_1(x, y) \exp(-j\beta_1 z) + A_2(z) E_2(x, y) \exp(-j\beta_2 z)$$

10.3-5

Here the functions $A_1(z)$ and $A_2(z)$ are the local amplitudes of the two modes. Clearly, these must vary with distance, to describe the slow changes in power associated with the coupling process.

Carrying out the necessary differentiation, we can show that:

$$\nabla^2 E_y(x, y, z) = [A_1 \nabla^2_{xy} E_1 + (d^2 A_1/dz^2 - 2j\beta_1 dA_1/dz - \beta_1^2 A_1) E_1] \exp(-j\beta_1 z) + [A_2 \nabla^2_{xy} E_2 + (d^2 A_2/dz^2 - 2j\beta_2 dA_2/dz - \beta_2^2 A_2) E_2] \exp(-j\beta_2 z)$$

10.3-6

We now substitute this expression, and the assumed solution of Equation 10.3-5, into the scalar wave equation (Equation 10.3-4). The resulting equation can then be drastically simplified, by eliminating the $\nabla^2_{xy}$ terms using the waveguide equations for the individual guides (10.3-3). The relevant terms are merely collected together, and set equal to zero. The result is:

$$[ d^2 A_1/dz^2 - 2j\beta_1 dA_1/dz + k_0^2 (n_T^2 - n_1^2) A_1 ] E_1 \exp(-j\beta_1 z) + [ d^2 A_2/dz^2 - 2j\beta_2 dA_2/dz + k_0^2 (n_T^2 - n_2^2) A_2 ] E_2 \exp(-j\beta_2 z) = 0$$

10.3-7

Note the appearance in Equation 10.3-7 of two new functions, $n_T^2 - n_1^2$ and $n_T^2 - n_2^2$. Bearing in mind the square-law relationship between refractive index and relative dielectric constant, $n^2 = \varepsilon$, we can interpret these as follows. The first represents a perturbation in dielectric constant $\Delta\varepsilon_1$ to Guide 1, caused by the introduction of a neighbouring waveguide, Guide 2. This will clearly be non-zero only in the neighbourhood of Guide 2. The second is a similar perturbation $\Delta\varepsilon_2$ caused to Guide 2 by Guide 1, and will in turn be significant only near Guide 1.

Of course, we can make further simplifications. For example, we can neglect second derivatives of the amplitudes (e.g. $d^2 A_i/dz^2$), on the grounds that these represent modal envelopes, which will probably vary only slowly with distance. We can also divide through by $\exp(-j\beta_1 z)$. This yields:

$$[-2j\beta_1 dA_1/dz + k_0^2 (n_T^2 - n_1^2) A_1] E_1 + [-2j\beta_2 dA_2/dz + k_0^2 (n_T^2 - n_2^2) A_2] E_2 \exp(-j\Delta\beta z) = 0$$

10.3-8

where $\Delta\beta = \beta_2 - \beta_1$, represents the mismatch in propagation constant between the guides. We are now nearly there; Equation 10.3-8 is almost (but not quite) useable. The problem is that we really wish to find an equation governing the changes in the wave amplitudes $A_i$ with distance. Since these are one-dimensional functions, it is not sensible to solve a three-dimensional equation. Somehow, we must remove the x- and y-variations contained in the
quantities $E_i$ and $n_i^2 - n_{r1}^2$. We do this in two steps. First, we multiply the whole of Equation 10.3-8 by the complex conjugate of the transverse field in Guide 1 ($E_1^*$), and integrate over the device cross-section. This yields the following new equation:

$$[-2j\beta_1 < E_1 , E_1 > dA_1 /dz + < k_0^2 (n_{r1}^2 - n_{r2}^2) E_1 , E_1 > A_1] + [-2j\beta_2 < E_2 , E_1 > dA_2 /dz + < k_0^2 (n_{r2}^2 - n_{r2}^2) E_2 , E_1 > A_1] \exp(-j\Delta\beta z) = 0$$

10.3-9

Here we have used the inner product notation introduced in Chapter 6 to represent all the integrals that arise. These are of course independent of $x$ and $y$, so the operation has had the desired effect.

Secondly, we now make a number of assumptions concerning the relative sizes of the remaining terms. (1) We assume that the overlap term $< E_2 , E_1 >$ is negligible, i.e. that the spatial overlap of the two modes is small. This is reasonable, if the guides are not too close together. (2) We also assume that the self-coupling term $< k_0^2 (n_{r2}^2 - n_{r2}^2) E_1 , E_1 >$ is negligible. This is physically reasonable, since the integral is only non-zero in the region of the perturbation $\Delta\varepsilon_r$, which lies in the evanescent region where the field function $E_1$ is itself small. (3) We assume on the other hand that the self-overlap term $< E_1 , E_1 >$ is non-negligible, since $E_1$ must of course overlap completely with itself. (4) We also assume that the coupling term $< k_0^2 (n_{r2}^2 - n_{r2}^2) E_2 , E_1 >$ is non-negligible, since $E_1$ is significant in the region where $\Delta\varepsilon_{r2}$ is non-zero, the core of Guide 1. Neglecting terms (1) and (2), and retaining terms (3) and (4), leaves Equation 10.3-9 in the form:

$$-2j\beta_1 < E_1 , E_1 > dA_1 /dz + < k_0^2 (n_{r2}^2 - n_{r2}^2) E_2 , E_1 > A_2 \exp(-j\Delta\beta z) = 0$$

10.3-10

Now, we could of course repeat the process above, this time multiplying Equation 10.3-8 by $E_2^*$. We can guess that the result will be the slightly different equation:

$$-2j\beta_2 < E_2 , E_2 > dA_2 /dz + < k_0^2 (n_{r2}^2 - n_{r2}^2) E_1 , E_2 > A_1 \exp(+j\Delta\beta z) = 0$$

10.3-11

We now make two more simplifications. Firstly, since the propagation constants will not change by very much during the switching process, we assume that $\beta_1 \approx \beta_2 \approx \beta_0$ for the purpose of division. Secondly, we assume that the symmetry of the device will ensure that:

$$< k_0^2 (n_{r2}^2 - n_{r2}^2) E_2 , E_1 > / < E_1 , E_1 > \approx < k_0^2 (n_{r2}^2 - n_{r2}^2) E_1 , E_2 > / < E_2 , E_2 >$$

10.3-12

With these assumptions, Equations 10.3-10 and 10.3-11 become:

$$dA_1 /dz + j\kappa A_2 \exp(-j\Delta\beta z) = 0$$

$$dA_2 /dz + j\kappa A_1 \exp(+j\Delta\beta z) = 0$$

10.3-13

Here we have introduced a new quantity $\kappa$, which is known as the coupling coefficient. This is arguably the most important parameter of the coupler, and is defined as:

$$\kappa = (k_0^2 /2\beta_0) < (n_{r2}^2 - n_{r2}^2) E_2 , E_1 > / < E_1 , E_1 >$$

10.3-14

Equations 10.3-13 are known as coupled mode equations, and represent the final stage in the derivation. Essentially, they imply that variations in the amplitude of the mode in Guide 1 are linked to the amplitude of the mode in Guide 2 through the coupling coefficient $\kappa$, and vice-versa. In other words, the mode amplitudes are coupled together. We shall see how to solve the equations very shortly; first, we will briefly consider factors affecting the value of the coupling coefficient.
FACTORs AFFECTING THE COUPLING COEFFICIENT

The expression for $\kappa$ in Equation 10.3-14 is relatively complicated. However, we note that the first term, $k_0^2/2\beta_0$, is really a constant, since it depends mainly on the optical wavelength. Similarly, the last term, $< E_1, E_1 >$, is a normalisation factor. The salient features are all contained in the central term, $< (n_2^2 - n_1^2) E_2, E_1 >$. This is, in effect, an overlap integral between $\Delta \varepsilon_r$ (the dielectric constant perturbation seen by Guide 2 due to Guide 1) and the two fields $E_1$ and $E_2$.

In our slab guide model, $\Delta \varepsilon_r$ is non-zero only within the core of Guide 1, so this region will give the only contribution to the integral. The main requirement for $\kappa$ to be large is therefore that the evanescent tail of the field $E_2$ penetrates Guide 1 to a significant extent. We can then say that $\kappa$ is most strongly affected by the interwaveguide gap $g$, which should be small (but not too small, or our equations will be invalid). Because evanescent fields fall off roughly exponentially, it turns out that $\kappa$ also depends exponentially on $g$, to good approximation. Typically, $g$ should be of the order of the guide width (usually, a few $\mu$m).

$\kappa$ is also affected by the confinement of the modes. For example, a poorly confined mode will have an evanescent field that extends a long distance from the guide core, which will result in a strong coupling coefficient. However, weak confinement may be undesirable because it increases bend loss, so the optimum confinement is something of a compromise. Finally, $\kappa$ is affected by polarization. Even though we have assumed a scalar model here, the use of a birefringent electro-optic substrate generally causes different coupling rates for TE and TM modes. This is clearly unfortunate, since the device will no longer be polarization-independent. Fortunately, careful choice of the fabrication parameters can equalise $\kappa_{TE}$ and $\kappa_{TM}$ in some cases (for example, in the Ti:LiNbO$_3$ process).

10.4 SOLUTION OF THE EQUATIONS AT SYNCHRONISM

We shall see first what happens when the two guides are synchronous, i.e. when there is no propagation constant mismatch, so that $\Delta \beta = 0$. The two coupled mode equations then reduce to:

$$\frac{dA_1}{dz} + j\kappa A_2 = 0 \quad ; \quad \frac{dA_2}{dz} + j\kappa A_1 = 0$$  \hspace{1cm} 10.4-1

We can solve these equations as follows. Differentiating the left-hand one, we get:

$$\frac{d^2 A_1}{dz^2} + j\kappa \frac{dA_2}{dz} = 0$$  \hspace{1cm} 10.4-2

Substituting for $dA_2/dz$ using the right-hand one then gives:

$$\frac{d^2 A_1}{dz^2} + \kappa^2 A_1 = 0$$  \hspace{1cm} 10.4-3

This is a standard second-order differential equation, with the general solution:

$$A_1 = C_1 \cos(\kappa z) + C_2 \sin(\kappa z)$$  \hspace{1cm} 10.4-4

Here $C_1$ and $C_2$ are constants that must be chosen to satisfy the boundary conditions. If we initially launch light only into Guide 1, with unit modal amplitude, these must be that:
A_1 = 1, and A_2 = 0 on z = 0  

Now, from Equation 10.4-1, we know that:

\[ A_2 = (-1/j\kappa) dA_1/dz \]  

So the boundary conditions could be written in the alternative form:

A_1 = 1, and dA_1/dz = 0 on z = 0  

It is easy to see the solution must then be:

A_1 = \cos(\kappa z)  

The solution for A_2 can be found by differentiation, using Equation 10.4-6. The result is:

A_2 = -j \sin(\kappa z)  

Note that A_2 is not real in Equation 10.4-9, showing that A_2 is in quadrature with A_1. It is probably more useful to consider the powers in the two guides, rather than mode amplitudes. These can be found in the usual way, via Poynting's Theorem. Normalising relative to the input power, we get:

\[ P_1 = A_1A_1^* = \cos^2(\kappa z) \quad ; \quad P_2 = A_2A_2^* = \sin^2(\kappa z) \]  

This demonstrates one of the nicer features of coupled mode theory. After a string of complicated manipulations, we still end up with a physically convincing result: total power, which is of course given by P_1 + P_2, is conserved by the device. This is what we would expect, since there is no loss. We can plot the variations of P_1 and P_2 with distance as shown in Figure 10.4-1.

As can be seen, the variations are in complete agreement with our earlier qualitative discussion, and the power distribution between the guides is an oscillatory function of z. All the power is transferred from Guide 1 to Guide 2 at the point z = L, when \( \kappa L = \pi/2 \). Complete transfer also occurs when \( \kappa L = 3\pi/2, 5\pi/2 \) and so on. Generally, however, the device parameters are chosen for the lowest of these values. We can arrange this either by altering the length L or the coupling coefficient \( \kappa \). For Ti:LiNbO_3 channel waveguide devices, L_{\text{eff}} is typically 5 - 10 mm.
10.5 ASYNCHRONOUS SOLUTION

What happens if there is a mismatch in propagation constant, so that $\Delta \beta$ is non-zero? Well, we can still solve the coupled mode equations. First, it is best to make the substitutions:

$$A_1 = a_1 \exp(-j\Delta \beta z/2) \quad \text{and} \quad A_2 = a_2 \exp(+j\Delta \beta z/2)$$

10.5-1

Clearly, we can obtain derivatives of these new variables, in the form:

$$dA_1/dz = [da_1/dz - j\Delta \beta /2 \ a_1] \exp(-j\Delta \beta z/2)$$

10.5-2

and so on. Substituting into Equations 10.3-13, we find that all the exponentials cancel out, leaving only constant terms, so that:

$$da_1/dz - j\Delta \beta /2 \ a_1 + j\kappa \ a_2 = 0$$

$$da_2/dz + j\Delta \beta /2 \ a_2 + j\kappa \ a_1 = 0$$

10.5-3

These equations are exactly equivalent to Equations 10.3-13, but much easier to solve. We won’t repeat the mathematics involved, but merely quote the general solution. For the same input as before, the power output from Guide 2 is given by:

$$P_2 = \sin^2[\sqrt{(\nu^2 + \xi^2)}] / (1+\xi^2/\nu^2)$$

10.5-4

where we have introduced two new parameters, a normalised coupling length $\nu$ and a normalised dephasing parameter $\xi$, given by:

$$\nu = \kappa L \ ; \ \xi = \Delta \beta L / 2$$

10.5-5

Equation 10.5-4 looks a bit like a sinc$^2$ function (remember that sinc(x) = sin(x) / x). We can plot it out as a function of $\xi$ (together with the power in Guide 1, given by $P_1 = 1 - P_2$), for our design value of $\nu = \pi / 2$. This gives the typical switching characteristic shown in Figure 10.5-1. When $\xi = 0$ (i.e., at synchronism) all the power emerges from Guide 2 as before, but as $|\xi|$ increases the power transfer is destroyed and $P_2$ falls, with a sort of 'filter' characteristic. Similarly, $P_1$ rises as $|\xi|$ increases, so that total power is once again conserved. Now, the first zero in Equation 10.5-4 occurs when:

$$\sqrt{(\nu^2 + \xi^2)} = \pi$$

10.5-6

Given that we chose a normalised coupling length of $\nu = \pi / 2$, this implies that:

$$\xi = \pi \sqrt{3} / 2, \quad \text{or} \quad \Delta \beta L = \pi \sqrt{3}$$

10.5-7

To make a switch, we therefore need only to be able to define two states:

i) state A, when no voltage is applied to the electrodes (so $\Delta \beta L = 0$). The light then emerges from Guide 2.

ii) state A’, when a voltage is applied such that $\Delta \beta L = \pi \sqrt{3}$. The light then emerges from Guide 1.
DESIGN EXAMPLE

We can compare the voltages required to switch a coupler and a Mach-Zehnder interferometer, as follows. For example, we know from Chapter 9 that the output of the interferometer will fall to zero when there is \( \pi \) radians phase difference between the arms. The equivalent value in a coupler can be found from Equation 10.5-7 as \( \pi \sqrt{3} \) rads. At first sight, therefore, the coupler seems less efficient, since it will need \( \sqrt{3} \) times the voltage for the same electrode geometry. However, coupler electrodes provide push-pull modulation from a narrow electrode gap. This would not be practical in the Mach-Zehnder modulator, since the guides are further apart (a bipolar drive is actually required in Figure 9.9-1 for push-pull operation). A more realistic figure for the voltage ratio is therefore \( \sqrt{3}/2 \).

PRACTICAL SWITCHES

The astute reader will have noticed that one switch state in Figure 10.5-1 is electrically-adjustable, but the other (at \( \xi = 0 \)) is set by the coupling length. Since \( \kappa \) is dependent on modal confinement, an error in fabrication may easily lead a value of \( \nu \) different from \( \pi/2 \).

It is natural to ask whether the two switch states may then still be obtained. The short answer is no: the cross-coupled state will in general suffer crosstalk. As a result, an alternative set of electrodes, known as stepped \( \Delta \beta \) electrodes, are often used. Each electrode is divided into two at its midpoint, giving two pairs of electrodes, both of length \( L/2 \). Using two separate control voltages (one for each pair) it is then possible to tune the device so that both states are always achievable. However, this can only be done if the value of \( \nu \) is greater than \( \pi/2 \).

Directional coupler switches are often integrated together as a switch array. If a number of couplers are suitably arranged on a substrate, they can connect \( 2N \) inputs to \( 2N \) outputs very simply. Figure 10.5-2 shows an example of a 4 x 4 switch, connecting four input fibres to four outputs. 8 x 8 Ti:LiNbO\(_3\) switch arrays are now common, but there is a limit to array size set by the dimensions of available crystal substrates.
10.6 FIBRE DIRECTIONAL COUPLERS

Directional couplers can also be made in single-mode optical fibre form. The fibre is first mounted in a silica or glass block, which has a curved groove cut in it, and the block is then polished down until the fibre core is just exposed. Two such blocks are then placed together, so that the polished areas are in contact, with a little index-matching fluid at the join. The evanescent fields in the two fibres then overlap, so power is transferred between them much as in the integrated device above. This arrangement is known as a **polished fibre coupler**, and is shown in Figure 10.6-1.

The main differences from the integrated optic device are that the coupling is non-uniform, because the spacing between the fibre cores varies with distance, and that the fibre version is not switchable. However, the coupling efficiency may easily be tuned, simply by sliding the blocks across each other as shown in Figure 10.6-2a. This alters the core-to-core separation, and hence the coupling coefficient, allowing simple variation of the splitting ratio. Often, this is adjusted for a coupling length of $\nu = \pi/4$, so the device acts as a 50 : 50 splitter. Figure 10.6-2b shows a tunable single-mode fibre coupler.

If ordinary circular fibre is used to fabricate a polished coupler, the resulting component will effectively scramble the input polarization, because there will be coupling between orthogonal polarization modes. To make a **polarization-preserving coupler**, this process must be suppressed. The easiest way is to use polarization-preserving fibre, so that orthogonal polarization modes are not synchronous. D-fibre and bow-tie fibre (described in Chapter 8) are both suitable. The only complication is that the polarization axes of the
fibres in the two blocks must be parallel; the usual approach is to arrange them to be parallel and perpendicular to the polished surfaces.

Figure 10.6-2 a) Arrangement for tuning the coupling strength of a polished fibre coupler, and b) a commercial tunable fibre coupler (photograph courtesy A.Tobin, Sifam Ltd.).

THE FIBRE LOOP MIRROR

Polarization-preserving couplers allow the construction of fibre circuits that operate on interferometric principles. We have already encountered the integrated optic Mach-Zehnder interferometer, and we will describe the use of fibre MZIs in Chapter 14. Here, we will consider an alternative circuit, the Sagnac interferometer, which can be used as an all-fibre mirror. Figure 10.6-3 shows the circuit, which consists of a fibre coupler with two of its outputs connected together in a loop.

To show that this acts as a mirror, we will consider the generation of a reflected output. There are two possible paths between input and output that lead to a reflection. In the first, light traverses the coupler without cross-coupling, travels clockwise round the loop, and is then cross-coupled on meeting the coupler again. For unity input, the resulting amplitude is then:

\[ A_{rt} = \cos(\kappa L) \exp(-j\phi) - j \sin(\kappa L) \]  

10.6-1
Here the first term is the coupler transmission without cross-coupling, the second is the phase-shift incurred in travelling round the loop, and the third is the coupler transmission with cross-coupling.

In the second pathway, light is cross-coupled on the first encounter with the coupler, travels anti-clockwise round the loop, and passes through the coupler without cross-coupling on the second encounter. Since the phase accumulated in the loop is independent of direction (except under the special circumstances discussed in Chapter 14), the resulting amplitude is:

\[ A_{r1} = -j \sin(\kappa L) \exp(-j\phi) \cos(\kappa L) \]  

10.6-2

Clearly, the total reflected amplitude \( A_r \) must be the sum of these contributions, so that:

\[ A_r = -j \sin(2\kappa L) \exp(-j\phi) \]  

10.6-3

The reflected power, \( P_r = |A_r|^2 \), is then given by:

\[ P_r = \sin^2(2\kappa L) \]  

10.6-4

Equation 10.6-4 implies that the reflected power can reach 100% if the normalised coupling length \( \kappa L \) is suitable (for example, when \( \kappa L = \pi/4 \)), independent of the length of the loop. At first sight, this seems curious. To check we have not made a mistake, we can evaluate the transmitted power in a similar way. The total transmitted amplitude \( A_t \) is again the sum of two components:

\[ A_t = [\cos(\kappa L) \exp(-j\phi) \cos(\kappa L)] + [-j \sin(\kappa L) \exp(-j\phi) . -j \sin(\kappa L)] \]

\[ = \cos(2\kappa L) \exp(-j\phi) \]  

10.6-5

Hence, the reflected power \( P_t = |A_t|^2 \) is given by:

\[ P_t = \cos^2(2\kappa L) \]  

10.6-6

Comparing Equations 10.6-4 and 10.6-6, we see that the result is self-consistent; the two outputs sum to unity (to be expected, since there is no loss) and the transmission is indeed zero when the reflectivity reaches 100%. The physical principle in use is one of interference - the transmission falls to zero when the two components in Equation 10.6-5 cancel out.

**10.7 COUPLING BETWEEN DISSIMILAR WAVEGUIDES**

Significant coupling can still occur between two entirely different waveguides, provided they are synchronous and their modal fields overlap suitably. This principle can be exploited in a range of different devices, of which the most important are directional coupler filters and polarizers.

**THE DIRECTIONAL COUPLER FILTER**

We shall begin with the directional coupler filter, which is typically formed from two quite dissimilar, parallel dielectric waveguides in close proximity. For example, these might be structurally similar, but have different core widths and refractive indices. Alternatively, they might be quite different in shape - Figure 10.7-1 shows a section through a coaxial fibre.
coupler, which operates by coupling between a central cylindrical guide and an outer annular guide.

![Cross-section of a coaxial fibre coupler.](image)

We can understand the effect of using two different guides by considering their dispersion characteristics. For example, Figure 10.7-2 shows typical $\omega - \beta$ diagrams for two slab guides. Note that for most of the frequency range, the guides have different $\beta$-values. However, the curves have been arranged to cross (by careful choice of the waveguide parameters) so that at one frequency $\omega_p$ both propagation constants are equal to $\beta_p$. Now, within a narrow range of this frequency, both curves are approximately linear. Similarly, the variations of $\beta_1$ and $\beta_2$ with $\lambda_0$ are roughly linear within a small range of the phase-matching wavelength $\lambda_p$, as in Figure 10.7-3a.

![Typical dispersion curves for two different guides.](image)

We may approximate these curves by:

$$\beta_1 \approx \beta_p + (\lambda_0 - \lambda_p) \frac{\partial \beta_1}{\partial \lambda_0} \quad \text{and} \quad \beta_2 \approx \beta_p + (\lambda_0 - \lambda_p) \frac{\partial \beta_2}{\partial \lambda_0}$$

10.7-1

where $\frac{\partial \beta_1}{\partial \lambda_0}$ and $\frac{\partial \beta_2}{\partial \lambda_0}$ represent the slopes of the dispersion characteristics near $\lambda_p$.

This allows us to write the difference in propagation constant $\Delta \beta = \beta_2 - \beta_1$ between the two guides as:

$$\Delta \beta \approx (\lambda_0 - \lambda_p) \left\{ \frac{\partial \beta_2}{\partial \lambda_0} - \frac{\partial \beta_1}{\partial \lambda_0} \right\}$$

10.7-2

Consequently, $\Delta \beta$ depends linearly on the difference in wavelength from $\lambda_p$. Without doing any more mathematics, we can deduce that the variation of the cross-coupled output $P_2$ with wavelength will be as previously defined for a switched coupler in Equation 10.5-4, but assuming that $\Delta \beta$ is as given above. A typical response is shown plotted in Figure 10.7-3b. It is clear that the device functions as a band-pass filter, cross-coupling significant power only in the region of the synchronous wavelength $\lambda_p$. The filter bandwidth is set by the size of the proportionality constant $\left\{ \frac{\partial \beta_2}{\partial \lambda_0} - \frac{\partial \beta_1}{\partial \lambda_0} \right\}$. For maximum selectivity, the difference in slopes of the two dispersion characteristics should be large. Unfortunately,
with the relatively weak guides in common use, there are restrictions in the possible range of slopes. This in turn limits typical filter bandwidths to \( \approx 100 \text{ Å} \) using Ti:LiNbO\(_3\) devices.

Figure 10.7-3 Directional coupler filter operation: a) dispersion characteristics, and b) filter response.

POLARIZERS AND POLARIZATION SPLITTERS

The principle of codirectional coupling between two dissimilar guided modes can also be adapted to provide polarizing and polarization-splitting functions, if one of the modes concerned is a surface plasma wave (which we previously introduced in Chapter 5). Surface plasmon polarizers represent a useful alternative to the birefringent devices described in Chapter 9.

We recall that a surface plasmon is a particular field structure that may be supported at the interface between a metal and a dielectric. We have already shown that it can only exist for TM-polarized light, and that is an extremely lossy, short-range mode. Let us suppose that by appropriate choice of parameters its propagation constant may be matched to that of a mode in a nearby parallel dielectric waveguide. Synchronous coupling between these modes may then occur. However, energy transferred to the surface plasma wave will be quickly dissipated as heat due to ohmic loss, so the coupling process will be largely one-way. In this way, the TM mode in the dielectric guide will be heavily attenuated, while the TE mode (which has no plasmon mode to couple to) is virtually unaffected. Such a device therefore acts as a TE-pass polarizer.

Figure 10.7-4 shows the construction of a **surface plasmon fibre polarizer**. It is based on a polished fibre block, of the type previously used to fabricate a fibre directional coupler. However, there are two additional overlayers: a thin metal layer, and a dielectric loading layer. In fact, the surface plasmon mode is supported by a three-layer structure, comprising the two layers just mentioned, and an additional layer provided by the polished block.
Consequently, the form of the plasmon mode is rather more complicated than our previous analysis suggests. Furthermore, detailed analysis shows that more than one such mode can exist. However, the principle described above is nonetheless valid: by careful choice of the layer parameters, it is possible to phase-match one of the possible surface plasma waves to the TM fibre mode, which can then be heavily attenuated. Typical results show 50 - 60dB extinction ratios, with excess losses for the transmitted polarization of only $\approx 0.5$ dB.

By placing another uncoated, polished fibre block on top of such a polarizer, a polarization-splitter can be made. This operates by using the surface plasmon as an intermediary between the two fibres. Now, we have already seen how TM light from the original fibre can be coupled into the plasmon mode. If the additional fibre is also synchronous, the plasmon can in turn couple light into this guide, effectively transferring light between the two different fibres. Since the plasmon now acts only as a 'stepping stone' for a short period, little power is dissipated in the process. Naturally, only TM light is transferred in this way; TE light passes through without coupling.

10.8 SIDELOBE SUPPRESSION USING TAPERED COUPLING

We now consider a slight modification to the basic directional coupler design, which can lead to significantly improved performance. Returning to Figures 10.5-1 and 10.7-3b, we notice a characteristic of both responses: the existence of significant sidelobes. This can be unsatisfactory, since switched states must be accurately located in the sidelobe 'nulls' for good crosstalk performance. Figure 10.8-1 shows an alternative coupler design, which avoids this problem. The modification introduced is simply that the interguide separation is made to be slowly varying with distance, reaching a minimum at the device centre and increasing at either end.

What is the effect of this slow tapering? Well, we already know that the coupling coefficient $\kappa$ depends on $g$, so it is reasonable to assume that a spatially-varying $g(z)$ will lead to a varying $\kappa(z)$. Since $\kappa$ depends roughly exponentially on the gap, a parabolic change in $g$ (due to circularly curved guides) would lead to a Gaussian variation in $\kappa$. In this case, we might expect that the coupled mode equations for the device will be as Equations 10.3-13,
but with the constant $\kappa$ replaced by $\kappa(z)$. For simplicity, we shall just consider the equation for $A_2$, which becomes:

$$\frac{dA_2}{dz} + j\kappa(z) A_1 \exp(+j\Delta\beta z) = 0$$

10.8-1

Now, if we knew the form of $A_1(z)$, this equation could be integrated over the device length, to get:

$$A_2(L) - A_2(0) = -j\int_0^L \kappa(z) A_1(z) \exp(+j\Delta\beta z) \, dz$$

10.8-2

Without solving the equations, we do not know $A_1(z)$. However, we can make an approximation, valid for weakly-coupled devices: if little power is transferred, we may take $A_1(z)$ to be virtually constant. For unity input to Guide 1, we then get:

$$A_2(L) \approx -j\int_0^L \kappa(z) \exp(+j\Delta\beta z) \, dz$$

10.8-3

Equation 10.8-3 now shows that there is an integral relationship between the output $A_2(L)$ and $\kappa(z)$. For constant coupling, we may evaluate this very simply, to get:

$$A_2(L) = -jkL \exp(+j\Delta\beta L/2) \text{sinc}(\Delta\beta L/2)$$

10.8-4

This implies that the cross-coupled output power $P_2$ is given by:

$$P_2 \approx \nu^2 \text{sinc}^2(\xi)$$

10.8-5

where $\nu = \kappa L$ and $\xi = \Delta\beta L/2$, as before.

Equation 10.8-5 could of course have been found from the exact solution (10.5-4) merely by assuming a small value of $\nu$. However, this derivation shows an interesting feature: the amplitude response (Equation 10.8-3) looks very like a Fourier transform of the coupling distribution. The sinc function obtained in Equation 10.8-4 therefore arises from having uniform coupling over a finite distance, i.e. because $\kappa(z)$ is a rectangle function. To suppress the sidelobes, we should choose alternative variations for $\kappa(z)$ whose Fourier transforms have lower lobes. One example is the Gaussian variation described earlier, since this transforms to another Gaussian. If the coupling is tapered in this way, the levels of the sidelobes may be greatly reduced. Most usefully, it turns out that this principle is still valid for strongly coupled devices, even though the relation between the switch response and the coupling variation is no longer a simple Fourier transform.

**10.9 THE REFLECTION GRATING FILTER - BASIC PRINCIPLES**

We now move on to consider another extremely useful device, which operates this time by **contradirectional** coupling - the **reflection grating filter**. This is very similar to the planar waveguide gratings described in qualitative terms in Chapter 7. It can be made by etching a periodic corrugation on the surface of a waveguide, as shown in Figure 10.9-1.

The device can be used to convert a mode travelling in the forward direction into a backward-travelling mode, through the Bragg effect. Each groove acts like a weak mirror, which would individually give only a very small reflection. However, if the components all
sum constructively, the result can be a very strong combined reflection. For this to occur, we require the components reflected from adjacent grooves to add in-phase.

Now, the optical path between grooves is $n_{\text{eff}}\Lambda$, where $n_{\text{eff}}$ is the effective index and $\Lambda$ is the grating wavelength. If a component reflected from one groove is to sum in-phase with the contribution of its neighbour, we require that twice this path is a whole number of optical wavelengths (the factor of two arises from the reversal in direction). We can write this as:

$$2n_{\text{eff}}\Lambda = m\lambda_0$$  \hspace{1cm} 10.9-1

Where $m$ is an integer. This is a special case of Bragg's Law, and if we take $m$ to be unity, the guided wave is incident at the first Bragg condition. We shall assume this is the case from now on.

Now, with this assumption we can clearly re-arrange Equation 10.9-1 as:

$$2 (\Lambda/2\pi) = (\lambda_0/2\pi n_{\text{eff}})$$  \hspace{1cm} 10.9-2

As in Chapter 7, we now define a grating vector $K$, whose direction is normal to the corrugations, and whose magnitude is given by:

$$K = 2\pi/\Lambda$$  \hspace{1cm} 10.9-3

Bearing in mind that the propagation constant $\beta$ is equal to $2\pi n_{\text{eff}}f/\lambda_0$, Equation 10.9-2 implies that:

$$K = 2\beta$$  \hspace{1cm} 10.9-4

Thus, when the Bragg condition is satisfied, the magnitude of the grating vector is exactly twice $\beta$. This corresponds to the $K$-vector closure condition discussed in Chapter 7, and could be represented by a $K$-vector diagram. However, an alternative viewpoint is provided by Figure 10.9-2.

This shows the dispersion characteristics of both a forward- and a backward-travelling mode; clearly, the latter merely has a negative value of $\beta$. Here Equation 10.9-4 is represented by the dashed-line construction, which shows that the forward and backward modes are phase-matched by the grating vector for incidence exactly at the phase-matching frequency $\omega_p$. 

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Figure 10.9-1 a) Schematic of a corrugated reflection grating, and b) electron microscope view of a reflection grating on a ridge guide formed in InP (photograph courtesy G.Heise, Siemens AG).
FIGURE 10.9-2 Phase matching by a grating, illustrated on the $\omega-\beta$ plane.

DESIGN EXAMPLE

We can work out the period needed to satisfy the Bragg condition for a typical design, as follows. We take the optical wavelength to be $\lambda_0 = 1.5\,\mu\text{m}$, a common near-infrared wavelength. We assume that the guides are made in titanium-indiffused LiNbO$_3$ (whose refractive index is $\approx 2.2$). Because guides made by indiffusion are very weak, we may take $n_{\text{eff}} \approx n_s$, so $\beta \approx 2\pi n_s/\lambda_0 = 9.2 \times 10^6$. Hence $K = 2\beta = 18.4 \times 10^6$, and $\Lambda = 2\pi/K = 0.34\,\mu\text{m}$.

The small figure for $\Lambda$ found above implies that gratings are hard to make. Generally, a surface pattern is first prepared, either by electron beam lithography or by exposure to a holographic interference pattern. This is then etched into the guide using ion beam milling, reactive ion etching or wet etching. In semiconductor materials, the refractive index is generally higher, which results in an even smaller value of $\Lambda$. The required pattern is then so hard to generate that a different grating with period $2\Lambda$ or $3\Lambda$ is often used instead, with incidence at a higher Bragg condition.

OFF-BRAGG INCIDENCE

Now, following the argument above, we would expect a strong reflection from the grating when Equation 10.9-4 holds, because the scattered components will add constructively. What happens if this is not the case (if we change the optical wavelength, for example)? Well, there will still be scattered components, but these will now sum with different phases. Clearly, the further we are off-Bragg, the greater the differences involved, and there will soon come a point when they are so large that the effective sum of the components is zero. At this point, there is no net reflection. The grating is therefore wavelength-selective, and can be used as a filter.

10.10 THE REFLECTION GRATING FILTER - THEORETICAL ANALYSIS

We can also analyse gratings using scalar coupled mode theory. As with the coupler, we first briefly consider the properties of the unperturbed guide (i.e. the guide without the grating). This is again oriented in the $z$-direction, and is formed by a two-dimensional index distribution $n(x, y)$, so wave propagation will be governed by a scalar wave equation similar to Equation 10.3-1 (ignoring the subscript $i$). As before, we assume that the guide is single-moded, so a forward-travelling mode $E(x, y, z) = E(x, y) \exp(-j\beta z)$ will satisfy a waveguide equation similar to Equation 10.3-3, as does a backward-travelling mode $E(x, y, z) = E(x, y) \exp(+j\beta z)$. We now assume that the guide is perturbed by a periodic index change $\delta n(x,
y, z), which represents the grating. In the simplest model, $\delta n$ could be defined by a periodic function of the form:

$$\delta n(x, y, z) = \Delta n(x, y) \cos(Kz)$$

Here $\Delta n(x, y)$ describes the variation of the index perturbation in the transverse direction, while the cosine accounts for the periodicity of the grating in the $z$-direction. The scalar wave equation for the complete system (i.e. guide plus grating) is then:

$$\nabla^2 E_y(x, y, z) + [n(x, y) + \delta n(x, y, z)]^2 k_0^2 E_y(x, y, z) = 0$$

Given that the perturbation is likely to be small, we can approximate Equation 10.10-2 by:

$$\nabla^2 E_y + [n^2 + 2n\delta n] k_0^2 E_y = 0$$

Now, we know from the qualitative discussion above that a guided mode incident in the $+z$ direction will be reflected as a mode travelling in the $-z$ direction, provided the Bragg condition is nearly satisfied, so that $K \approx 2\beta$. We therefore assume a solution to Equation 10.10-3 as the sum of a forward- and a backward-travelling mode, in the following form:

$$E_y(x, y, z) = A_F(z) E(x, y) \exp(-j\beta z) + A_B(z) E(x, y) \exp[-j(\beta - K)z]$$

Here $A_F$ and $A_B$ are the amplitudes of the forward- and backward-modes, respectively, and their $z$-dependence accounts for the gradual conversion of one mode into the other. Note that the backward mode can be recognised as varying approximately as $\exp(+j\beta z)$, if $K \approx 2\beta$. Doing the necessary differentiation, we can obtain the Laplacian as:

$$\nabla^2 E_y = [A_F \nabla^2_{xy} E + (d^2 A_F/dz^2 - 2j\beta dA_F/dz - \beta^2 A_F) E] \exp(-j\beta z) + [A_B \nabla^2_{xy} E + (d^2 A_B/dz^2 - 2j(\beta - K)dA_B/dz - (\beta - K)^2 A_B) E] \exp[-j(\beta - K)z]$$

We now substitute this and the assumed solution into the wave equation (10.10-3). This produces a very lengthy equation, which can however be simplified using much the same techniques as we have used before. We first eliminate all the $\nabla^2_{xy}$ terms, using the unperturbed waveguide equation, and neglect second derivatives of $A_F$ and $A_B$ (on the grounds that these vary slowly). This gives:

$$\{-2j\beta dA_F/dz \exp(-j\beta z) + [-2j(\beta - K)dA_B/dz - ((\beta - K)^2 - \beta^2)] A_B \} \exp[-j(\beta - K)z] + 2n\Delta n k_0^2 \cos(Kz) [A_F \exp(-j\beta z) + A_B \exp[-j(\beta - K)z]] \} E = 0$$

We then note that the cosine term above can be written as:

$$\cos(Kz) = (1/2) [\exp(jKz) + \exp(-jKz)]$$

With this substitution, we can write:

$$\cos(Kz) \{ A_F \exp(-j\beta z) + A_B \exp[-j(\beta - K)z] \} = (1/2) \{ A_F \exp[-j(\beta - K)z] + A_F \exp[-j(\beta + K)] + A_B \exp[-j(\beta - 2K)] + A_B \exp(-j\beta z) \}$$

Notice that terms containing exponentials like $\exp[-j(\beta + K)]$ and $\exp[-j(\beta - 2K)]$ have not appeared before, and are not contained in our assumed solution (Equation 10.10-4). In fact, these represent higher diffraction orders of the grating, and we can safely ignore them here.
because they will not be phase-matched. With this approximation, Equation 10.10-6 reduces to:

\[-2j\beta \frac{dA_F}{dz} \exp(-j\beta z) - [2j(\beta - K) \frac{dA_B}{dz} + [(\beta - K)^2 - \beta^2] A_B] \exp[-j(\beta - K)z] + n\Delta n k_0^2 [A_F \exp[-j(\beta - K)z] + A_B \exp(-j\beta z)] \} E = 0\]

10.10-9

Now, if Equation 10.10-9 is to be true for all z, the coefficients of \(\exp(-j\beta z)\) and \(\exp[-j(\beta - K)z]\) must vanish separately. We therefore equate them both with zero, to get the following equations:

\[-2j\beta \frac{dA_F}{dz} + n\Delta n k_0^2 A_B \} E = 0\]
\[-2j(\beta - K)dA_B/dz - [(\beta - K)^2 - \beta^2] A_B + n\Delta n k_0^2 A_f \} E = 0\]

10.10-10

Further approximations are also possible. Because the grating will only have a significant effect if we are close to the Bragg condition, we can put:

\[\beta - K/2 = \Delta\beta\]  
10.10-11

Where \(\Delta\beta\) is a small quantity. Consequently, we can write:

\[(\beta - K)^2 - \beta^2 \approx -4\beta\Delta\beta\]  
10.10-12

We will use this in the lower equation in 10.10-10. We also need an approximation for \(\beta - K\); a reasonable substitution is simply to put \(\beta - K \approx -\beta\). If this is done, Equations 10.10-10 become:

\[\{\frac{dA_F}{dz} + j (k_0^2/2\beta) n\Delta n A_B \} E = 0\]
\[\{\frac{dA_B}{dz} - j2\Delta\beta A_B - j (k_0^2/2\beta) n\Delta n A_f \} E = 0\]

10.10-13

These look very like coupled differential equations once again, apart from the \(x\)- and \(y\)-variations contained in the terms \(n(x, y), \Delta n(x, y)\) and \(E(x, y)\). We can get rid of these by multiplying by \(E^*\) and integrating over the waveguide cross-section. This gives:

\[dA_F/dz + j\kappa A_B = 0\]; \[dA_B/dz - j2\Delta\beta A_B - j\kappa A_f = 0\]  
10.10-14

These are now true coupled mode equations; once again, the rate of change of each mode amplitude depends on the amplitude of the other mode, via a coupling coefficient \(\kappa\), given by:

\[\kappa = (k_0^2/2\beta) < n\Delta n E, E > / < E, E >\]  
10.10-15

Notice how \(\kappa\) depends on an overlap between the index perturbation \(\Delta n\) and the transverse field \(E\). If we put the grating in the wrong place, therefore, the field will simply not see it, and there will little reflection. In the Figure 10.9-1, the corrugation is at the upper surface of the guide. Consequently, the grating will only affect the evanescent part of the field, where the amplitude is weak. This will give a low value of \(\kappa\), so a long grating will be needed for high reflectivity.

10.11 SOLUTION OF THE EQUATIONS AT SYNCHRONISM

As with the directional coupler, we will start by investigating the solution of the equations at synchronism (i.e. when \(\Delta\beta = 0\), so that \(K = 2\beta\)). The coupled mode equations are then:
dA_F/dz + jκ A_B = 0 ; dA_B/dz - jκ A_F = 0  

10.11-1

Notice how similar these are to the equations for a synchronous coupler (10.4-1) - the only real difference is a sign change in the lower equation. We can therefore solve them in a very similar way. We start by differentiating the upper equation, and then eliminating A_B using the lower equation. This gives a second-order equation for A_F, in the form:

\[ d^2A_F/dz^2 - \kappa^2 A_F = 0 \]  

10.11-2

Because of the sign change mentioned above, solutions of this equation are not trigonometric functions; instead, they are hyperbolic. We may therefore put:

\[ A_F = D_1 \cosh(\kappa z) + D_2 \sinh(\kappa z) \]  

10.11-3

Here D_1 and D_2 are constants, chosen to satisfy the boundary conditions. With a forward-going input, A_F is clearly specified at z = 0. However, the backward wave must grow from zero at the far end of the device, which is of length L. Suitable boundary conditions are therefore:

\[ A_F = 1 \text{ at } z = 0, \quad A_B = 0 \text{ at } z = L \]  

10.11-4

It is then quite simple to show that the solutions must be:

\[ A_F(z) = \cosh[\kappa(L - z)] / \cosh(\kappa L) ; \quad A_B(z) = -j \sinh[\kappa(L - z)] / \cosh(\kappa L) \]  

10.11-5

Notice that the modal amplitudes at any point inside the grating now depend on the precise value of \( \kappa L \). Now, the power carried in the +z direction by the forward wave is clearly \( P_F = A_F A_F^* \). However, the power carried in the same direction by the backward wave is \(-P_B\), where \( P_B = A_B A_B^* \). Once again, it can be shown that power is conserved by the system of equations, since \( P_F - P_B = \text{constant} \).

Figure 10.11-1 shows the variation of \( P_F \) and \( P_B \) with distance through the grating, for a typical normalised thickness of \( \kappa L = 1.5 \). As can be seen, the incident wave decays as it travels through the grating. However, as it loses energy, the reflected wave grows.

![Figure 10.11-1 Variation of reflected and transmitted powers with distance.](image)

The separation of the two curves is constant, consistent with the power conservation relation above. In general, we will actually be most interested in the two outputs from the grating.
These are the grating transmissivity $T$ and reflectivity $R$, defined as $T = A_f A_f^*$ at $z = L$ and $R = A_i A_i^*$ at $z = 0$. It is easy to show these are given by:

$$T = \frac{1}{\cosh^2(\kappa L)} \quad \text{and} \quad R = \tanh^2(\kappa L)$$

Equation 10.11-6 implies that 100% peak reflectivity is possible, simply provided the normalised coupling length $\kappa L$ is large enough. For example, if $\kappa L = 2$, we obtain $R = 0.93$; similarly, for $\kappa L = 4$, we get $R = 0.999$. Figure 10.11-2 shows a plot of $T$ and $R$ versus $\kappa L$. Power conservation is again demonstrated, since the sum of $T$ and $R$ is unity throughout.

![Figure 10.11-2 Grating reflectivity and transmissivity versus normalised length.](image)

**10.12 ASYNCHRONOUS SOLUTION**

Now we consider the case when the optical wavelength is slightly incorrect, so that $\Delta \beta \neq 0$. Once again, the solutions can be found by quite straightforward methods at each of the outputs as:

$$T = \left| \frac{1}{[\cosh(\Psi) - j(\xi/\Psi) \sinh(\Psi)]} \right|^2$$

$$R = \left| \frac{1}{[(\xi/\mu) + j(\Psi/\mu) \coth(\Psi)]} \right|^2$$

Where $\mu$, $\xi$ and $\Psi$ are defined by:

$$\mu = \kappa L, \quad \xi = -\Delta \beta L \quad \text{and} \quad \Psi = \sqrt{\mu^2 - \xi^2}$$

These solutions can be plotted against $\xi$ (i.e. as a function of the difference in $\beta$ from the design value), for a given normalised thickness. Figure 10.12-1 shows the response for $\kappa L = 3$. As expected, the results are in keeping with our earlier qualitative discussion. For $\xi = 0$, there is high reflectivity. As $|\xi|$ increases, the reflected power falls, with a typical filter response. This is accompanied by a corresponding rise in transmissivity, and for sufficiently large $|\xi|$, there is no significant reflection.

**DESIGN EXAMPLE**

We can estimate the bandwidth of a typical reflection grating filter as follows. For a reasonable range of normalised lengths, the half-power point in curves like Figure 10.12-1 is reached when:
\[ |\xi| \approx 3 \]

So that:
\[ |\Delta \beta| \approx 3/L \]

Since \( \beta = \frac{2\pi n_{\text{eff}}}{\lambda_0} \), we can write:
\[ |\Delta \beta| = \frac{2\pi n_{\text{eff}} |\Delta \lambda_0|}{\lambda_0^2} \]

And therefore that:
\[ |\Delta \lambda_0| \approx \frac{3\lambda_0^2}{2\pi n_{\text{eff}} L} \]

For the typical parameters used before, namely \( n_{\text{eff}} \approx 2.2 \) and \( \lambda_0 = 1.5 \) µm, and a grating length of \( L = 1 \) mm, we obtain \( |\Delta \lambda_0| \approx 0.5 \) nm. Grating filters are therefore highly wavelength-selective.

\[ \text{Figure 10.12-1 Typical filter response of a reflection grating.} \]

\[ \text{10.13 FIBRE GRATINGS} \]

We have already described how gratings can be made in integrated optics. They can also be made in fibre optic form, by etching a corrugation over a fibre that has been polished back to expose its core. Figure 10.13-1 shows how this is done, starting with one half of a polished fibre coupler. A fibre grating might be used as a frequency selective mirror in a fibre laser, or as a channel selector in a wavelength-division multiplexed (WDM) communications system. Notice that the distance from the fibre core to the grating corrugation varies through the device, due to the curvature of the fibre. Consequently, the coupling coefficient must also vary. It can be shown that tapered coupling of this type has much the same effect as was found in the directional coupler, so we might expect the sidelobes of the filter response to be supressed.

\[ \text{Figure 10.13-1 Optical fibre grating based on half a polished fibre coupler.} \]
More recently, an alternative method of grating fabrication has been investigated, based on the inherent photosensitivity of Ge-doped silica to short-wavelength radiation. This allows a periodic change in refractive index to be obtained inside the fibre core itself, rather than in the cladding. The fibre is simply exposed to an optical interference pattern, either derived from two coherent, counter-propagating guided waves or from two external beams. In the latter case, a grating period suitable for reflection of a guided infrared beam may be obtained, allowing the construction of filters for use in fibre lasers. Figure 10.13-2 shows a typical set-up for external recording. The exposure source is normally an Ar$^+$ laser, operating either at $\lambda = 0.488 \ \mu m$ or at the frequency-doubled wavelength of $0.244 \ \mu m$. Although the work is at an early stage, and the index change obtained is rather weak ($\Delta n \approx 6 \times 10^{-5}$), high-efficiency gratings have already been produced by this method.

![Figure 10.13-2](image)


10.14 OTHER COUPLED MODE INTERACTIONS

It turns out that the directional coupler and the reflection grating are merely the simplest examples of a wide range of devices that operate by some combination of co- and contra-directional coupling. We therefore conclude with a few examples illustrating further coupled mode interactions.

CONVERSION BETWEEN GUIDED AND RADIATION MODES (THE GRATING COUPLER)

The first example occurs in the grating coupler, which we previously met in Chapter 7 as an input-output coupling component. There, its operation was described in terms of a K-vector diagram, but an alternative view is provided by the $\omega - \beta$ diagram of Figure 10.14-1.

Essentially, this is the same as for a simple reflection grating (Figure 10.9-2), but the frequency has been raised (or the wavelength lowered) so that the left hand-end of the grating vector no longer lies in the region where backward-travelling, guided modes exist; instead, it falls in the area of radiation modes. Consequently, we would expect the grating exactly to phase-match a forward-travelling guided mode to a backward-travelling radiation mode, and that this interaction could be described by a pair of coupled mode equations. This simple picture does not, unfortunately, represent the whole story. We have already seen that significant power can be coupled to modes that are not exactly phase-matched, provided they are reasonably close to synchronism. Since the radiation modes are a continuum, the guided mode will be coupled to an entire group of such modes, whose
propagation constants lie close to the phase-matched value. The longer the grating, however, the more selective it becomes, and the more restricted the group; eventually, the description above must become a reasonable one. When this occurs, the radiated output travels predominantly in a defined direction, which must of course be wavelength-dependent. This allows a grating coupler to be used as a form of spectrometer.

**Figure 10.14-1** Phase matching by a grating between guided and radiation modes, illustrated on the $\omega-\beta$ plane.

**GRATING-ASSISTED COUPLING**

A grating can also be used to achieve phase-matching between two otherwise asynchronous, codirectional guided modes. These might be the two different polarization modes in a single-moded guide formed in a birefringent substrate (e.g. LiNbO$_3$). Phase-matching is obtained when one mode is excited as the first diffraction order of the grating, so that:

$$|K| = \beta_{\text{TE}} - \beta_{\text{TM}} = 2\pi \frac{\Delta n_{\text{eff}}}{\lambda_0}$$

10.14-1

where $\Delta n_{\text{eff}}$ is the difference in effective indices of the TE and TM modes. Since $\Delta n_{\text{eff}}$ is usually only moderate, $|K|$ is small, and the grating wavelength $\Lambda$ is large ($\approx 7 \mu$m in Ti:LiNbO$_3$ at $\lambda_0 = 0.6 \mu$m). To obtain coupling between orthogonal modes, a periodic perturbation to an off-diagonal element of the dielectric tensor is needed. This cannot be achieved using a simple surface corrugation, but it can be induced electrically through an off-diagonal electrooptic coefficient, by using a periodic electrode as shown in Figure 10.14-2.

**Figure 10.14-2** A TE-TM mode converter filter.

This type of device is known as a **TE-TM mode converter**, and can be used as a polarization rotator or a filter. Filter operation depends on the difference in dispersion of
the two modes, as for the dissimilar-guide directional coupler described earlier. However, greater wavelength selectivity (≈ 10 Å in Ti:LiNbO₃) can be obtained in this case, because larger differences in slope of the two dispersion characteristics are possible with orthogonal modes.

It is not, however, necessary for the two modes concerned to exist in the same guide. If a grating is combined with a directional coupler, it may be used to phase-match codirectional modes of the same polarization in the two guides (if they are dissimilar) or orthogonal polarizations (if they are the same). Equally, the two modes do not have to be codirectional. Figure 10.14-3 shows a device which acts as a track-changing reflector, by using a reflection grating to phase-match contradirectional modes in a fibre coupler formed between entirely different guides.

Without the grating, any codirectional coupling would be asynchronous, as shown in the \( \omega - \beta \) diagram of Figure 10.14-4. Similarly, the grating period is chosen so that it cannot retroreflect light in either Guide 1 or Guide 2. However, at the correct frequency, it can phase-match a forward mode in Guide 1 to a backward mode in Guide 2. This occurs when:

\[
|K| = \beta_{g1} + \beta_{g2}
\]

10.14-2

The result is that the reflected output now emerges from a different fibre to that used for the input - this is particularly convenient for channel-dropping operations.
PROBLEMS

10.1. Discuss the approximations inherent in the derivation of the coupled mode equations for a directional coupler. Are they valid? If not, can you suggest the likely consequences for device performance?

10.2. An alternative view of a synchronous directional coupler can be based on the well-known optical principle that the output from a linear device is given by the sum of all possible routes between input and output.

Figure (i) below shows one path between an input to guide 1 and an output from guide 2. This involves a single scattering at an arbitrary position z, such that \( 0 \leq z \leq L \). If the scattering amplitude per unit length is \(-j \kappa\), and the input amplitude is \( A_{10} \), the net contribution from all such paths is:

\[
A_{2L1} = -j \kappa A_{10} \int_0^L dz = -j \kappa L A_{10}
\]

Figure (ii) below shows another possible route, involving three scatterings.

(a) If the sum of all such paths is denoted by \( A_{2L3} \), evaluate \( A_{2L3} \).
(b) Draw the paths contributing to \( A_{2L5} \), which has five scatterings, and evaluate the integral for this case.
(c) Sum the series \( A_{2L} = A_{2L1} + A_{2L3} + A_{2L5} \ldots \), and show that it corresponds to the normal solution for a synchronous directional coupler.

10.3 A directional coupler is constructed from two identical, symmetric slab waveguides. The guide thickness is \( h \), and the guide and substrate indices are \( n_g \) and \( n_s \), respectively. Using the methods outlined in Chapter 6, derive a solution for the transverse field of the lowest-order, symmetric TE mode of a single such guide. Then, by direct evaluation of the overlap integral concerned, show that the coupling coefficient \( \kappa \) decreases exponentially with the interguide separation \( g \) in the twin-guide device.

10.4 The coupled differential equations for an asynchronous directional coupler are given by:

\[
dA_1/dz + j \kappa A_2 \exp(-j \Delta \beta z) = 0 \quad ; \quad dA_2/dz + j \kappa A_1 \exp(+j \Delta \beta z) = 0.
\]

Assuming that total power is defined as \( P_T = |A_1|^2 + |A_2|^2 \), show (without solving the equations) that power is conserved by the device.
10.5. From a series of experiments with Ti:LiNbO$_3$ directional couplers made with different interwaveguide gaps $g$, the following data were obtained for the length required for complete coupling: $g = 4 \, \mu\text{m}; L_{\pi/2} = 5 \, \text{mm}; g = 6 \, \mu\text{m}; L_{\pi/2} = 7 \, \text{mm}; g = 7.5 \, \mu\text{m}, L_{\pi/2} = 9 \, \text{mm}$. Estimate the length required to achieve 50% coupling efficiency when $g = 7 \, \mu\text{m}$.

[4.14 mm]

10.6 A fibre directional coupler is to be made by polishing two glass blocks, which have fibres embedded in curved grooves. (a) Suggest how the polishing process can be monitored, so it can be terminated when the core is reached. (b) When the coupler is assembled, it is found that the amount of cross-coupled light is very small, despite attempts to design it for 100% coupling. Suggest two possible explanations, and the remedy in each case.

10.7 The output of a synchronous reflection grating can also be found as the sum of all possible routes between the input and output. Figure (i) below shows one path between a forward-going input and a backward-travelling output. This involves a single scattering at an arbitrary position $z$, such that $0 \leq z \leq L$. If the scattering amplitude per unit length is $-j\kappa$, and the input amplitude is $A_{F0}$, the net contribution from all such paths is:

$$A_{B0} = -j\kappa A_{F0} \int_{0}^{L} dz = -j\kappa L A_{F0}$$

Figure (ii) below shows another possible route, involving three scatterings.

(a) If the sum of all such paths is denoted by $A_{B0}^3$, evaluate $A_{B0}^3$.

(b) Draw the paths contributing to $A_{B0}^5$, which has five scatterings, and evaluate the integral for this case.

(c) Sum the series $A_{B0} = A_{B0}^1 + A_{B0}^3 + A_{B0}^5 \ldots$, and show that it corresponds to the normal solution for a reflection grating, when replayed at the Bragg wavelength.

10.8 The coupled differential equations for a reflection grating at Bragg incidence are given by:

$$\frac{dA_F}{dz} + j\kappa A_B = 0 \ ; \ \frac{dA_B}{dz} - j\kappa A_F = 0$$

Verify that the general solutions to these equations, subject to the boundary conditions $A_F = 1$ at $z = 0$, $A_B = 0$ at $z = L$, are indeed:

$$A_F(z) = \frac{\cosh[\kappa(L - z)]}{\cosh(\kappa L)} \ ; \ A_B(z) = -j \frac{\sinh[\kappa(L - z)]}{\cosh(\kappa L)}.$$
10.9 A Bragg reflector is to be used with an integrated optic channel waveguide. Past measurements with this system suggest that the coupling coefficient is $\kappa = 200 \, \text{m}^{-1}$ at the design wavelength. How long should the grating be, for $\geq 99\%$ reflectivity? [15 mm]

10.10 Compare the wavelength selectivity of waveguide filters based on (a) codirectional coupling between dissimilar waveguides, and (b) contradirectional coupling via a grating. What are the factors limiting selectivity in each case?
SUGGESTIONS FOR FURTHER READING


Alferness R.C., Veselka J.J. "Simultaneous modulation and wavelength multiplexing with a tunable Ti:LiNbO₃ directional coupler filter" Elect. Lett. 21, 466-467 (1985)


