Nonlinear Monte Carlo Methods: From American Options to Fully Nonlinear PDEs

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Outline

1. Monte Carlo Methods for American Options
2. Backward SDEs and semilinear PDEs
3. Second order BSDEs
4. Numerical Examples
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1. Monte Carlo Methods for American Options

2. Backward SDEs and semilinear PDEs

3. Second order BSDEs

4. Numerical Examples
Pricing American Options in Complete Markets

In the context of a complete market with a nonrisky asset $S^0$:

$$S^0_t = e^{rt}, \quad t \geq 0,$$

and a risky security $S$ defined by the Black-Scholes model

$$dS_t = S_t (rdt + \sigma dW_t) \quad \text{where} \quad W \text{ is a Brownian motion,}$$

the no-arbitrage price of the American put option with strike $K > 0$ and maturity $T > 0$:

$$P_0 = \sup_{\tau \in \mathcal{T}_T} \mathbb{E} \left[ e^{-r\tau} (K - S^\tau)^+ \right] = \mathbb{E} \left[ e^{-r\tau^*} (K - S^{\tau^*})^+ \right]$$

where $\mathcal{T}_T = \{ \text{stopping times with values in } [0, T] \}$ and

$$\tau^* = \min \{ t \geq 0 : P_t = (K - S_t)^+ \}$$
Discrete-time Approximation

• Let \( t^n_i = i h_n, \ i = 1, \ldots, n \) where \( n \) is integer (to be sent to \( \infty \)) and \( h_n = \frac{iT}{n} \)

• Define the so-called Snell envelope:

\[
Y^n_T = (K - S_T)^+ \quad \text{and} \quad Y^n_{t^n_i} = \max \left\{ (K - S_{t^n_i})^+, E^{t^n_i}_{t^n_i} \left[ e^{-r h_n Y^n_{t^n_{i+1}}} \right] \right\}
\]

• Then, an approximation of the American put price is:

\[
Y^n_0 \rightarrow P_0
\]

and the error is known to be of order \( n^{-1/2} \), i.e.

\[
\limsup_{n \to \infty} \sqrt{n} \left( Y^n_0 - P_0 \right) < \infty
\]
Approximation of Conditional Expectations

**Main observation**: in the present context all conditional expectations are regressions, i.e.

\[
\mathbb{E}_{t_i^n} \left[ Y^n_{t_{i+1}^n} \right] = \mathbb{E} \left[ Y^n_{t_{i+1}^n} \mid S_{t_i^n} \right]
\]

\[\implies\] Classical methods from statistics:
- Kernel regression (<Carrière>)
- Projection on subspaces of \( L^2(\mathbb{P}) \) (<Longstaff-Schwarz, Gobet-Lemor-Warin AAP05>)

from numerical probabilistic methods
- quantization... (<Bally-Pagès SPA03>)

Integration by parts (<Bouchard-Ekeland-Touzi FS04>)
Approximation of the Replicating Strategy

- Put price is $P_t = P(t, S_t)$ a deterministic function of $(t, S_t)$
- The replicating strategy of the American put is:
  $$\Delta_t = \frac{\partial P}{\partial S}(t, S_t), \quad t < \tau^*$$
- An approximation of the replication strategy within a Monte Carlo estimation of the put price is:
  $$\Delta_{t_i}^n = \mathbb{E}_{t_i}^n \left[ Y_{t_{i+1}}^n \Delta W_{t_{i+1}}^n \right]$$
  where $\Delta W_{t_{i+1}}^n = W_{t_{i+1}}^n - W_{t_i}^n$.

- Finally, the Monte Carlo scheme is:
  $$Y_T^n = (K - S_T)^+$$ and
  $$\hat{Y}_{t_i}^n = \max\left\{ (K - S_{t_i}^n)^+, e^{-r_{n,T}} \mathbb{E}_{t_i}^n \left[ Y_{t_{i+1}}^n \right] \right\}$$
  $$\hat{\Delta}_{t_i}^n = \mathbb{E}_{t_i}^n \left[ Y_{t_{i+1}}^n \Delta W_{t_{i+1}}^n \right]$$
Objective: Monte Carlo technique for the approximation of the American option price and hedge extends to solutions of Fully nonlinear PDEs.

- Fully Nonlinear PDEs are encountered in many areas of applied mathematics. In particular,
  - stochastic control problems can be characterized in terms of the Bellman (dynamic programming) equation
    \[
    0 = -\frac{\partial v}{\partial t} - \sup_{u \in U} \left\{ b(x, u) \cdot Dv + \frac{1}{2} \text{Tr} \left[ \sigma \sigma^T(x, u) D^2v \right] + f(x, u)v - k(x, u) \right\}
    \]
  - stopping problems can also be characterized in terms of the corresponding Bellman equation (free boundary)
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The Stochastic Integral Representation Theorem

- \( \xi : \mathcal{F}_T^W \) -measurable random variable, \( \mathbb{E}|\xi|^2 < \infty \), and

\[
Y_t := \mathbb{E} [\xi | \mathcal{F}_t], \quad t \in [0, T]
\]

**Theorem** There exists an \( \mathbb{F}^W \) -adapted process \( Z \) with
\[
\mathbb{E} \int_0^T |Z_t|^2 dt < \infty
\]
such that

\[
Y_t = \mathbb{E} \xi + \int_0^t Z_s \cdot dW_s, \quad t \in [0, T]
\]

**Theorem (Clark-Ocone)** If \( \xi \) is Malliavin-differentiable, then
\[
Z_t = \mathbb{E}[D_t \xi | \mathcal{F}_t], \ i.e.
\]

\[
Y_t = \mathbb{E} \xi + \int_0^t \mathbb{E}[D_s \xi | \mathcal{F}_s] \cdot dW_s, \quad t \in [0, T]
\]

“Taylor formula with integral rest”
Stochastic representation in the Markov case

- Let $\xi = g(X_T)$, where the process $X$ is defined by

$$X_0 \quad \text{given and} \quad dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

with elliptic $\sigma(.)$ (Hörmander condition is sufficient). Then

$$Y_t = \mathbb{E}[g(X_T)|\mathcal{F}_t] = \mathbb{E}[g(X_T)|X_t] =: V(t, X_t)$$

and it follows from Itô’s lemma that

$$Y_t = \mathbb{E}[g(X_T)] + \int_0^t (\sigma^T D V)(s, X_s) \cdot dW_s$$
Backward SDE : Definition

Find an $\mathbb{F}^W$-adapted $(Y, Z)$ satisfying:

$$Y_t = \xi + \int_t^T F_r(Y_r, Z_r)dr - \int_t^T Z_r \cdot dW_r$$

i.e.

$$dY_t = -F_t(Y_t, Z_t)dt + Z_t \cdot dW_t$$ and $Y_T = \xi$

where the generator $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, and

$$\{F_t(y, z), \ t \in [0, T]\} \text{ is } \mathbb{F}^W \text{ - adapted}$$

If $F$ is Lipschitz in $(y, z)$ uniformly in $(\omega, t)$, and $\xi \in L^2(\mathbb{P})$, then there is a unique solution satisfying

$$\sup_{t \leq T} \mathbb{E}|Y_t|^2 + \mathbb{E}\int_0^T |Z_t|^2 dt < \infty$$
Markov BSDE’s

Let $X_t$ be defined by the (forward) SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

and

$$F_t(y, z) = f(t, X_t, y, z), \quad f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

$$\xi = g(X_T) \in L^2(\mathbb{P}), \quad g : \mathbb{R}^d \rightarrow \mathbb{R}$$

If $f$ continuous, Lipschitz in $(x, y, z)$ uniformly in $t$, then there is a unique solution to the BSDE

$$dY_t = -f(t, X_t, Y_t, Z_t)dt + Z_t \cdot \sigma(X_t)dW_t, \quad Y_T = g(X_T)$$

Moreover, there exists a measurable function $V :$

$$Y_t = V(t, X_t), \quad 0 \leq t \leq T$$
By definition,
\[ Y_s - Y_t = V(s, X_s) - V(t, X_t) = -\int_t^s f(X_r, Y_r, Z_r)dr + \int_t^s Z_r \cdot \sigma(X_r) dW_r \]

If \( V(t, x) \) is smooth, it follows from Itô’s lemma that:
\[ \int_t^s \mathcal{L}V(r, X_r)dr + \int_t^s D\mathcal{V}(r, X_r) \cdot \sigma(X_r) dW_r = -\int_t^s f(X_r, Y_r, Z_r)dr + \int_t^s Z_r \cdot \sigma(X_r) dW_r \]

where \( \mathcal{L} \) is the Dynkin operator associated to \( X \):
\[ \mathcal{L}V = V_t + b \cdot D\mathcal{V} + \frac{1}{2} \text{Tr}[\sigma \sigma^T D^2 V] \]
Stochastic representation of solutions of a semilinear PDE

Under some conditions, the semilinear PDE

\[-\frac{\partial V}{\partial t} - \mathcal{L}V(t, x) - f(x, V(t, x), DV(t, x)) = 0 \]

\[V(T, x) = g(x)\]

has a unique solution which can be represented as

\[V(t, x) = Y_{t}^{t,x}\]

where \(Y_{t}^{t,x}\) solves the BSDE

\[Y_s = g(X_T) + \int_s^T f(X_r, Y_r, Z_r)dr - \int_s^T Z_r \cdot \sigma(X_r)dW_r, \quad t \leq s \leq T\]

and \(X_t = x, \quad dX_s = b(X_s)ds + \sigma(X_s)dW_s, \quad t \leq s \leq T\)
Extension of Feynman-Kac’s formula

Let \( f \equiv 0 \), then

\[
V(t, x) = Y^t,x_t = g(X^t,x_T) - \int_t^T Z_r \cdot \sigma(X^t,x_r) \, dW_r
\]

\( \Rightarrow \) take conditional expectations \( V(t, x) = \mathbb{E}[g(X^t,x_T)] \) with:

\[
X^t,x_t = x \quad \text{and} \quad dX^t,x_r = b(X^t,x_r) \, dr + \sigma(X^t,x_r) \, dW_r
\]

\( \Rightarrow \) Numerical solution by Monte Carlo:

\[
\hat{V}(t, x) := \frac{1}{N} \sum_{i=1}^N g(\hat{X}^{(i)}_T) \quad \Rightarrow \quad V(t, x) \quad \text{a.s. (LLN)}
\]

and

\[
\sqrt{N} \left( \hat{V}(t, x) - V(t, x) \right) \quad \Rightarrow \quad \mathcal{N}(0, \mathbb{V}[g(X_T)]) \quad \text{(CLT)}
\]
Discrete-time approximation

Numerical solution of a semi-linear PDE by simulating the associated backward SDE by means of Monte Carlo methods
Start from Euler discretization: \( Y_{t_n}^n = g\left(X_{t_n}^n\right) \) is given, and

\[
Y_{t_{i+1}}^n - Y_{t_i}^n = -f\left(X_{t_i}^n, Y_{t_i}^n, Z_{t_i}^n\right) \Delta t_i + Z_{t_i}^n \cdot \sigma\left(X_{t_i}^n\right) \Delta W_{t_{i+1}}
\]
Discrete-time approximation

\[ \text{Numerical solution of a semi-linear PDE by simulating the associated backward SDE by means of Monte Carlo methods.} \]

\[ \text{Start from Euler discretization: } Y^n_{t_n} = g \left( X^n_{t_n} \right) \text{ is given, and} \]

\[ \mathbb{E}_{i}^{n} \left[ Y^n_{t_{i+1}} - Y^n_{t_{i}} = -f \left( X^n_{t_{i}}, Y^n_{t_{i}}, Z^n_{t_{i}} \right) \Delta t_{i} + Z^n_{t_{i}} \cdot \sigma \left( X^n_{t_{i}} \right) \Delta W_{t_{i+1}} \right] \]

\[ \implies \text{Discrete-time approximation: } Y^n_{t_i} = g \left( X^n_{t_n} \right) \text{ and} \]

\[ Y^n_{t_i} = \mathbb{E}_{i}^{n} \left[ Y^n_{t_{i+1}} \right] + f \left( X^n_{t_{i}}, Y^n_{t_{i}}, Z^n_{t_{i}} \right) \Delta t_{i} , \quad 0 \leq i \leq n - 1 , \]
Discrete-time approximation

Numerical solution of a semi-linear PDE by simulating the associated backward SDE by means of Monte Carlo methods

Start from Euler discretization: \( Y_{tn} = g(X_{tn}) \) is given, and

\[
\mathbb{E}_i^n [\Delta W_{ti+1}] \rightarrow Y_{ti+1}^n - Y_{ti}^n = -f(X_{ti}^n, Y_{ti}^n, Z_{ti}^n) \Delta t_i + Z_{ti}^n \cdot \sigma (X_{ti}^n) \Delta W_{ti+1}
\]

\( \Rightarrow \) Discrete-time approximation: \( Y_{tn} = g(X_{tn}) \) and

\[
Y_{ti}^n = \mathbb{E}_i^n \left[ Y_{ti+1}^n \right] + f(X_{ti}^n, Y_{ti}^n, Z_{ti}^n) \Delta t_i , \ 0 \leq i \leq n - 1
\]

\[
Z_{ti}^n = (\Delta t_i)^{-1} \mathbb{E}_i^n \left[ Y_{ti+1}^n \Delta W_{ti+1} \right]
\]
Discrete-time approximation

Numerical solution of a semi-linear PDE by simulating the associated backward SDE by means of Monte Carlo methods.

Start from Euler discretization:

\[
\mathbb{E}_i^n [\Delta W_{t_{i+1}}] \rightarrow Y_{t_{i+1}}^n - Y_{t_i}^n = -f (X_{t_i}^n, Y_{t_i}^n, Z_{t_i}^n) \Delta t_i + Z_{t_i}^n \cdot \sigma (X_{t_i}^n) \Delta W_{t_{i+1}}
\]

\[\Rightarrow\] Discrete-time approximation:

\[
Y_{t_i}^n = g (X_{t_n}^n) \quad \text{and}
\]

\[
Y_{t_i}^n = \mathbb{E}_i^n \left[ Y_{t_{i+1}}^n \right] + f (X_{t_i}^n, Y_{t_i}^n, Z_{t_i}^n) \Delta t_i, \quad 0 \leq i \leq n - 1
\]

\[
Z_{t_i}^n = (\Delta t_i)^{-1} \mathbb{E}_i^n \left[ Y_{t_{i+1}}^n \Delta W_{t_{i+1}} \right]
\]

\[\Rightarrow\] Similar to numerical computation of American options.
Discrete-time approximation, continued

**Theorem**  Assume $f$ and $g$ are Lipschitz. Then:

\[
\limsup_{n \to 0} n \left\{ \sup_{0 \leq t \leq 1} E \left| Y^n_t - Y_t \right|^2 + E \left[ \int_0^1 |Z^n_t - Z_t|^2 \, dt \right] \right\} < \infty
\]

- Same rate of convergence as for the simulation of (forward) SDEs
- in the present context all conditional expectations are regressions, i.e.

\[
\begin{align*}
\mathbb{E} \left[ Y^n_{t_{i+1}} \mid \mathcal{F}_{t_i} \right] &= \mathbb{E} \left[ Y^n_{t_{i+1}} \mid X_{t_i} \right] \\
\mathbb{E} \left[ Y^n_{t_{i+1}} \Delta W_{t_{i+1}} \mid \mathcal{F}_{t_i} \right] &= \mathbb{E} \left[ Y^n_{t_{i+1}} \Delta W_{t_{i+1}} \mid X_{t_i} \right]
\end{align*}
\]

\[\implies\text{can be approximated as in the case of American options...}\]
Simulation of Backward SDE’s

1. Simulate trajectories of the forward process \( X \) (well understood)

2. Backward algorithm:

\[
\begin{align*}
\hat{Y}_{tn} & = g \left( X_{tn} \right) \\
\hat{Y}_{ti-1} & = \mathbb{E}_{ti-1} \left[ \hat{Y}_{ti} \right] + f \left( X_{ti-1}, \hat{Y}_{ti-1}, \hat{Z}_{ti-1} \right) \Delta t_i, \quad 1 \leq i \leq n, \\
\hat{Z}_{ti-1} & = \frac{1}{\Delta t_i} \mathbb{E}_{ti-1} \left[ \hat{Y}_{ti} \Delta W_{ti} \right]
\end{align*}
\]

(truncation of \( \hat{Y}^n \) and \( \hat{Z}^n \) needed in order to control the \( L^p \) error)
Simulation of BSDEs: bound on the rate of convergence

**Theorem**  For $p > 1$:

$$\limsup_{n \to \infty} \max_{0 \leq i \leq n} n^{-1 - d/(4p)} N^{1/2p} \left\| \hat{Y}_{t_i}^n - Y_{t_i}^n \right\|_{L^p} < \infty$$

For the time step $\frac{1}{n}$, and limit case $p = 1$:

rate of convergence of $\frac{1}{\sqrt{n}}$

if and only if

$$n^{-1 - \frac{d}{4} N^{1/2}} = n^{1/2}, \quad \text{i.e.} \ N = n^{3 + \frac{d}{2}}$$
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Main purpose

• Enlarge the class of BSDE’s in order to obtain a stochastic representation of Fully Nonlinear PDE’s (In particular, representation of general stochastic control problems)

• Gradient is related to the representation of a random variable as a stochastic integral (up to the driver)

• In order to obtain a fully nonlinear PDE, one needs to include “the Hessian” in the driver...

⇒ Requires understanding local behavior of double stochastic integrals...
Second order BSDEs : Definition

\[ \hat{f}(x, y, z, \gamma) := f(x, y, z, \gamma) + \frac{1}{2} \text{Tr}[\sigma \sigma^T(x)\gamma] \] non-decreasing in \( \gamma \)

Consider the 2nd order BSDE:

\[
\begin{align*}
    dX_t &= \sigma(X_t) dW_t \\
    dY_t &= -f(t, X_t, Y_t, Z_t, \Gamma_t)dt + Z_t \cdot \sigma(X_t) dW_t, \quad Y_T = g(X_T) \\
    dZ_t &= \alpha_t dt + \Gamma_t \sigma(X_t) dW_t
\end{align*}
\]

A solution of (2BSDE) is

a process \((Y, Z, \alpha, \Gamma)\) with values in \( \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times S^n \)

Question: existence? uniqueness? in which class?

<Cheridito, Soner, Touzi and Victoir CPAM 2007>
Second order BSDE: Existence and uniqueness

Feynman-Kac formula for fully-nonlinear PDE

Theorem  Let Assumption \((f)\) hold, and suppose \(g\) has linear growth. Suppose further that \((E)\) satisfies the comparison Assumption \(Com\) and has a smooth solution \(v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}\) satisfying

\[
\max \{ |Dv(t, x)|, |D^2v(t, x)|, |\mathcal{L}Dv(t, x)| \} \leq m(1 + |x|^p)
\]
\[
|D^2v(t, x) - D^2v(s, y)| \leq m(1 + |x|^p + |y|^p)(|t - s| + |x - y|)
\]

Then the process \((\tilde{Y}, \tilde{Z}, \tilde{\alpha}, \tilde{\Gamma})\) defined by

\[
\tilde{Y}_t := v(t, X_t), \quad \tilde{Z}_t := Dv(t, X_t), \quad \tilde{\alpha}_t := \mathcal{L}Dv(t, X_t), \quad \tilde{\Gamma}_t := V_{xx}(t, X_t)
\]

is the unique solution of (2BSDE) with \(Z \in \mathcal{A}_{0,x}\)
A probabilistic numerical scheme for fully nonlinear PDEs

By analogy with BSDE, we introduce the following discretization for 2BSDEs:

\[
Y^n_{t_n} = g\left(X^n_{t_n}\right),
\]
\[
Y^\pi_{t_{i-1}} = \mathbb{E}^\pi_{i-1} \left[ Y^\pi_{t_i} \right] + f\left(X^\pi_{t_{i-1}}, Y^\pi_{t_{i-1}}, Z^\pi_{t_{i-1}}, \Gamma^\pi_{t_{i-1}}\right) \Delta t_i, \quad 1 \leq i \leq n,
\]
\[
Z^\pi_{t_{i-1}} = \frac{1}{\Delta t_i} \mathbb{E}^\pi_{i-1} \left[ Y^\pi_{t_i} \Delta W_{t_i} \right]
\]
\[
\Gamma^\pi_{t_{i-1}} = \frac{1}{(\Delta t_i)^2} \mathbb{E}^\pi_{i-1} \left[ Y^\pi_{t_i} \left( (\Delta W_{t_i})^2 - \Delta t_i \right) \right]
\]
Intuition From Greeks Calculation

- First, use the approximation $f''(x) \sim_{h=0} \mathbb{E}[f''(x + W_h)]$
- Then, integration by parts shows that

$$f''(x) \sim \int f''(x + y) \frac{e^{-y^2/(2h)}}{\sqrt{2\pi}} dy$$

$$= \int f'(x + y) \frac{y e^{-y^2/2}}{h \sqrt{2\pi}} dy = \mathbb{E} \left[ f'(x + W_h) \frac{W_h}{h} \right]$$

$$= \int f(x + y) \frac{y^2 - h e^{-y^2/2}}{h^2} \frac{1}{\sqrt{2\pi}} dy = \mathbb{E} \left[ f(x + W_h) \left( \frac{W_h^2 - h}{h^2} \right) \right]$$

- Connection with Finite Differences: $W_h \sim \sqrt{h} \left( \frac{1}{2} \delta_1 + \frac{1}{2} \delta_{-1} \right)$

$$\mathbb{E} \left[ \psi(x + W_h) \frac{W_h}{h} \right] \sim \frac{\psi(x + \sqrt{h}) - \psi(x - \sqrt{h})}{2h} \text{ Centered FD !}$$
The Convergence Result

<Fahim, Soner and Touzi 2007>

**Theorem**  Suppose that $f$ is Lipschitz and $\|f_\gamma\|_{L^\infty} \leq \sigma$. Then

$$Y_0^n \rightarrow v(t, x)$$

where $v$ is the unique viscosity solution of the nonlinear PDE.

- Bounds on the approximation error are available
- This convergence result is weaker than that of (first order) Backward SDEs...
in BSDEs the drift coefficient $\mu$ of the forward SDE can be changed arbitrarily by Girsanov theorem (importance sampling...)

in 2BSDEs both $\mu$ and $\sigma$ can be changed (numerical results however recommend prudence...)

The heat equation $v_t + v_{xx} = 0$ correspond to a BSDE with zero driver. Splitting the Laplacian in two pieces, it can also be viewed as a 2BSDE with driver $f(\gamma) = \frac{1}{2}\gamma$

$\longrightarrow$ numerical experiments show that the 2BSDE algorithm performs better than the pure finite differences scheme
With $U(x) = -e^{-\eta x}$, want to solve:

$$V(t, x) := \sup_{\theta} \mathbb{E} \left[ U \left( x + \int_t^T \theta u \sigma (\lambda du + dW_u) \right) \right]$$

- An explicit solution is available
- $V$ is the characterized by the fully nonlinear PDE

$$-V_t + \frac{1}{2} \lambda^2 \frac{(V_x)^2}{V_{xx}} = 0 \quad \text{and} \quad V(T, .) = U$$
Fig.: Relative Error (Regression), dimension 1
Fig.: Relative Error (Regression), dimension 2
### Varying the drift of the FSDE

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<th>Drift FSDE</th>
<th>Relative error (Regression)</th>
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### Varying the volatility of the FSDE

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</tbody>
</table>