Flow Structure in a Precessing Sphere

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Flow structure is simple.

Solid-body rotation

It can be proved from NS equation.
Flow in a Precessing Container

Flow structure is non-trivial.
Flow can be turbulent.
Flow in a Precessing Sphere

- Spin rotation

- Precession rotation

Flow structure is non-trivial.
Flow can be turbulent.
Outline

1. Introduction
2. Experiment – State Diagram
3. Numerical Simulation – Stability boundary
   Flow Structure
4. Asymptotic Analysis – Flow Structure
5. Summary
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Precessing Sphere

We consider the motion of an incompressible viscous fluid in a precessing sphere, where the spin angular velocity $\Omega_s$ and the precession angular velocity $\Omega_p$ are perpendicular to each other.

\[ \Omega = \Omega_s + \Omega_p \]
Research on Precessing Sphere/Spheroid/Spherical Shell

Precession of the Earth
Spin rotation; one day
Precession rotation: 25800 year
Poincare number; $10^{-7}$
Precession angle; $-23.5^\circ$
Ekman number; $10^{-15}<E<10^{-7}$
Research on Precessing Sphere/Spheroid/Spherical Shell

related to Geodynamo

Experiment

Vanyo et al. 1995: Experiments on precessing flows in the Earth’s liquid core

Vanyo & Dunn 2000: Core precession: flow structures and energy

Numerical

Lorenzani & Tilgner 2001: Fluid instabilities in precessing spheroidal cavities

Tilgner & Busse 2001: Fluid flows in precessing spherical shells

Tilgner 2005: Precession driven dynamos
Our Motivation

[1] to make a compact turbulence generator

[2] to understand the flow dynamics as one of the standard systems

[3] to contribute to geophysics such as geodynamo
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1. Introduction

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Apparatus

Acrylic Sphere + water (φ100mm)

Spin axis

Precession axis
Control of rotation speed

Pulse motors
(1 pulse = 0.072°)
Rotation speed can be controlled with high accuracy
Visualization / Measurement

Laser sheet (perp. Spin axis) and video camera are on precession frame.

Velocity field on the sheet is measured by PIV.
Governing Equations (in Precession frame)

\[ \frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \mathbf{\omega} - 2 \frac{R_p}{R_s} \hat{z} \times \mathbf{u} - \nabla P + \frac{1}{R_s} \nabla^2 \mathbf{u} \]

Non-dimensional

\[ P = p + \frac{1}{2} |\mathbf{u}|^2 - \left( \frac{R_p}{R_s} \right)^2 \frac{1}{2} (\boldsymbol{r} \times \hat{z})^2 \]

Modified pressure

\[ \nabla \cdot \mathbf{u} = 0 \]

b. c.

\[ \mathbf{u} = \hat{x} \times \mathbf{r} \quad \text{on} \quad |\mathbf{r}| = 1 \]
Control Parameters

Spin Reynolds number

\[ R_s = \frac{a^2 \Omega_s}{\nu} \]  
(Reciprocal of Ekman number)

Precession Reynolds number

\[ R_p = \frac{a^2 \Omega_p}{\nu} \]

When \( a=5\text{cm}, \nu=0.01 \text{cm}^2/\text{s}, \quad \Omega_s=2\pi n \),

\[ R_s = 1.6 \times 10^4 n \]
Control Parameters

Reynolds number

\[ Re = \frac{a^2 \Omega_s}{\nu} \quad (\equiv R_s) \]

Poincare number

\[ \Gamma = \frac{\Omega_p}{\Omega_s} \quad (\equiv \frac{R_p}{R_s}) \]
Visualized Flow

$Re=15,000$

$\Gamma=0.00625$

Solid –body rotation
Flow in a Rotating Container

$Re = 15,000$

$\Gamma = 0.1$

Turbulent state
Stability Boundary

- Steady
- Unsteady

Graph showing the relationship between $R_p/R_s$ and $R_s$.
Velocity Field (animation)

$R_s \approx 4,500$

Steady: $R_p/R_s = 0.01$

Periodic: $R_p/R_s = 0.04$

Aperiodic: $R_p/R_s = 0.1$
Velocity Field (snap) \( R_s \approx 4,500 \)

Steady: \( R_p/R_s = 0.01 \)

Periodic: \( R_p/R_s = 0.04 \)

Aperiodic: \( R_p/R_s = 0.1 \)

Judged from two-time correlation function
Two-Time Correlation of Velocity

\[ C_i(x, \tau) = \frac{\langle [u_i(x, t) - m_i(x)][u_i(x, t+\tau) - m_i(x)] \rangle}{\sigma_i(x)^2} \]

\[(i = 1, 2)\]

\[ m_i(x) = \langle u_i(x, t) \rangle \]

\[ \sigma_i(x)^2 = \langle (u_i(x, t)^2 \rangle - m_i(x)^2 \]
Two-Time Correlation of Velocity

Steady: $R_p/R_s = 0.01$

Periodic: $R_p/R_s = 0.04$

Noise

Aperiodic: $R_p/R_s = 0.1$

$R_s \approx 4,500$
Fluctuation Magnitude

\[ I(x) = \sqrt{\frac{\langle |u(x, t) - \langle u(x, t) \rangle|^2 \rangle}{\langle |u(x, t)|^2 \rangle}} \]
Fluctuation Magnitude

- Stability boundary by DNS
- Steady or periodic
- aperiodic
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Stability of Steady Flows

By DNS
Governing Equations (in Precession frame)

\[ \frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \mathbf{\omega} - 2 \frac{R_p}{R_s} \hat{z} \times \mathbf{u} - \nabla P + \frac{1}{R_s} \nabla^2 \mathbf{u} \]

\[ \nabla \cdot \mathbf{u} = 0 \]

b. c.

\[ \mathbf{u} = \hat{x} \times \mathbf{r} \quad \text{on } |\mathbf{r}| = 1 \]
Poloidal / Toroidal Representation

1. Incompressible (solenoidal)
2. Spherical geometry

\[
\mathbf{u} = \nabla \times (\nabla \times (r\mathbf{U})) + \nabla \times (r\mathbf{W})
\]

\[
\begin{align*}
    u_r &= -\frac{1}{r} \nabla_\perp^2 U, \\
    u_\theta &= \frac{1}{r} \frac{\partial^2 (rU)}{\partial r \partial \theta} + \frac{1}{\sin \theta} \frac{\partial W}{\partial \varphi}, \\
    u_\varphi &= \frac{1}{r \sin \theta} \frac{\partial^2 (rU)}{\partial r \partial \varphi} - \frac{\partial W}{\partial \theta}
\end{align*}
\]
Vorticity

\[ \omega = \nabla \times \nabla \times (rW) + \nabla \times (r(-\nabla^2 U)) \]

\[ \omega_r = -\frac{1}{r} \nabla^2_\perp W \]

\[ \omega_\theta = \frac{1}{r} \frac{\partial^2(rW)}{\partial r \partial \theta} + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} (-\nabla^2 U) \]

\[ \omega_\phi = \frac{1}{r \sin \theta} \frac{\partial^2(rW)}{\partial r \partial \phi} - \frac{\partial}{\partial \theta} (-\nabla^2 U) \]
Governing Equations (in Precession frame)

\[ \frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \mathbf{\omega} - 2 \frac{R_p}{R_s} \hat{\mathbf{z}} \times \mathbf{u} - \nabla P + \frac{1}{R_s} \nabla^2 \mathbf{u} \]

\[ \nabla \cdot \mathbf{u} = 0 \]

b. c.

\[ \mathbf{u} = \hat{\mathbf{x}} \times \mathbf{r} \quad \text{on} \quad |\mathbf{r}| = 1 \]
Poloidal / Toroidal Equations

\[-\nabla^2_\perp \left( \nabla^2 - R_h \frac{\partial}{\partial t} \right) W \]

\[= -2R_v \left[ \frac{\partial W}{\partial \varphi} + \frac{1}{r^2} \left\{ \nabla^2_\perp \left( r \sin \theta \frac{\partial}{\partial \theta} - r \frac{\partial}{\partial r} r \cos \theta \right) - r \frac{\partial}{\partial r} \left( r \sin \theta \frac{\partial}{\partial \theta} + 2r \cos \theta \right) \right\} U \right] \]

\[+ \frac{R_h}{r^2 \sin^2 \theta} \left[ \frac{\partial}{\partial \varphi} \left( r^2 \sin \theta N_\theta - r \sin \theta \frac{\partial}{\partial \theta} \left( r \sin \theta N_\varphi \right) \right) \right], \]

\[N = u \times \omega\]

\[-\nabla^2_\perp \left( \nabla^2 - R_h \frac{\partial}{\partial t} \right) (-\nabla^2 U) \]

\[= -2R_v \left[ \frac{\partial (-\nabla^2 U)}{\partial \varphi} + \frac{1}{r^2} \left\{ \nabla^2_\perp \left( r \sin \theta \frac{\partial}{\partial \theta} - r \frac{\partial}{\partial r} r \cos \theta \right) - r \frac{\partial}{\partial r} \left( r \sin \theta \frac{\partial}{\partial \theta} + 2r \cos \theta \right) \right\} W \right] \]

\[+ \frac{R_h}{r^2} \left[ \nabla^2_\perp (r N_r) - \frac{1}{r^2 \sin^2 \theta} \left( r \frac{\partial}{\partial r} - 2 \right) r \sin \theta \frac{\partial}{\partial \theta} \left( r^2 \sin \theta N_\theta \right) - \frac{1}{\sin^2 \theta} r \frac{\partial}{\partial r} \frac{\partial}{\partial \varphi} \left( r \sin \theta N_\varphi \right) \right] \]

\[b. c. \quad U = 0, \quad \frac{\partial U}{\partial r} = 0, \quad W = \sin \theta \cos \varphi \quad \text{on } r = 1\]
Numerical Scheme (Time Integration)

\[ -\nabla^2_\perp \left( \nabla^2 - R_h \frac{\partial}{\partial t} \right) W = G \]

\[ -\nabla^2_\perp \left( \nabla^2 - R_h \frac{\partial}{\partial t} \right) (-\nabla^2 U) = H \]

\[ U = 0, \quad \frac{\partial U}{\partial r} = 0, \quad W = \sin \theta \cos \varphi \quad (\text{on } r = 1) \]

Adams-Bashforth / Crank-Nicolson Scheme \[ \Delta t = \frac{2\pi}{1000} \]

\[ -\nabla^2_\perp \left( \nabla^2 - \frac{2R_h}{\Delta t} \right) W^{t+\Delta t} = \nabla^2_\perp \left( \nabla^2 + \frac{2R_h}{\Delta t} \right) W^t + 3G^t - G^{t-\Delta t} \]

\[ -\nabla^2_\perp \left( \nabla^2 - \frac{2R_h}{\Delta t} \right) (-\nabla^2) U^{t+\Delta t} = \nabla^2_\perp \left( \nabla^2 + \frac{2R_h}{\Delta t} \right) (-\nabla^2) U^t + 3H^t - H^{t-\Delta t} \]

\[ U^{t+\Delta t} = 0, \quad \frac{\partial}{\partial r} U^{t+\Delta t} = 0, \quad W^{t+\Delta t} = \sin \theta \cos \varphi \quad (\text{on } r = 1) \]
Numerical Scheme (Spatial Differentiation)

Fourier – Legendre – Jacobi Expansion

\[
U^t(r, \theta, \varphi) = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} \sum_{j=1}^{[(N-l+1)/2]} \hat{U}_{jlm}^t \Phi_k^l(r) \hat{P}_l^{|m|}(\cos \theta) e^{im\varphi}
\]

\[
W^t(r, \theta, \varphi) = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} \sum_{j=1}^{[(N-l+1)/2]} \hat{W}_{jlm}^t \Phi_k^l(r) \hat{P}_l^{|m|}(\cos \theta) e^{im\varphi}
\]

\[
\frac{1}{r^2} \frac{d}{dr} \left( (1 - r^2) r^2 \frac{d}{dr} \right) \Phi_k^l - \frac{l(l+1)}{r^2} \Phi_k^l + k(k+3) \Phi_k^l = 0
\]

\[(0 \leq r \leq 1, 0 \leq l \leq k, \; k + l = \text{even})\]

Orthogonal Relation

\[
\int_0^1 \Phi_n^l(r) \Phi_{n'}^l(r) r^2 dr = \delta_{nn'}
\]

Matsushima & Marcus (1995)
Poloidal / Toroidal Equations

\[-\nabla_\perp^2 \left( \nabla^2 - R_h \frac{\partial}{\partial t} \right) W \]

\[= -2R_v \left[ \frac{\partial W}{\partial \varphi} + \frac{1}{r^2} \left\{ \nabla_\perp^2 \left( r \sin \theta \frac{\partial}{\partial \theta} - r \frac{\partial}{\partial r} r \cos \theta \right) - r \frac{\partial}{\partial r} \left( r \sin \theta \frac{\partial}{\partial \theta} + 2r \cos \theta \right) \right\} \right] \]

\[+ \frac{R_h}{r^2 \sin^2 \theta} \left[ \frac{\partial}{\partial \varphi} (r^2 \sin \theta N_\theta - r \sin \theta \frac{\partial}{\partial \theta} (r \sin \theta N_\varphi) \right], \]

\[N = u \times \omega\]

\[-\nabla_\perp^2 \left( \nabla^2 - R_h \frac{\partial}{\partial t} \right) (-\nabla^2 U) \]

\[= -2R_v \left[ \frac{\partial (-\nabla^2 U)}{\partial \varphi} + \frac{1}{r^2} \left\{ \nabla_\perp^2 \left( r \sin \theta \frac{\partial}{\partial \theta} - r \frac{\partial}{\partial r} r \cos \theta \right) - r \frac{\partial}{\partial r} \left( r \sin \theta \frac{\partial}{\partial \theta} + 2r \cos \theta \right) \right\} \right] \]

\[+ \frac{R_h}{r^2} \left[ \nabla_\perp^2 (rN_r) - \frac{1}{r^2 \sin^2 \theta} \left( r \frac{\partial}{\partial r} - 2 \right) r \sin \theta \frac{\partial}{\partial \theta} (r^2 \sin \theta N_\theta) - \frac{1}{\sin^2 \theta} r \frac{\partial}{\partial r} \frac{\partial}{\partial \varphi} (r \sin \theta N_\varphi) \right] \]

\[b. \ c. \quad U = 0, \quad \frac{\partial U}{\partial r} = 0, \quad W = \sin \theta \cos \varphi \quad \text{on} \ r = 1\]
Time-Series of Enstrophy

\[ R_s = 2450 \]
\[ R_p = 147 \]
\[ \gamma = 0.06 \]
\[ \lambda = 0.25 \]
Stability Boundary

Steady

Unsteady

in progress
Stability Boundary

in progress

The graph shows a scatter plot with data points plotted on the plane defined by the variables $R_p/R_s$ and $R_s$. The stability boundary is indicated by a dashed line separating the regions labeled 'Steady' and 'Unsteady'. The points are color-coded: green for 'Unsteady' and red for 'Steady'.
Time-Series of Enstrophy

\[ R_s = 1880 \]
\[ R_p = 150.4 \]
\[ \gamma = 0.08 \]
\[ \lambda = 0.27 \]
Time-Series of Enstrophy

$R_s = 1515$

$R_p = 166.65$

$\gamma = 0.11$

$\lambda = 0.96$
Time-Series of Enstrophy

\[ R_s = 1270 \]
\[ R_p = 177.8 \]
\[ \gamma = 0.14 \]
\[ \lambda = 1.24 \]
Time-Series of Enstrophy

\[ R_s = 1700 \]
\[ R_p = 255 \]
\[ \gamma = 0.15 \]
\[ \lambda = 1.21 \]
Time-Series of Enstrophy

$R_s = 1900$

$R_p = 351.5$

$\gamma = 0.185$

$\lambda = 2.03$
Time-Series of Enstrophy

\[ R_s = 2080 \]

\[ R_p = 395.2 \]

\[ \gamma = 0.19 \]

\[ \lambda = 10.0 \]

\[ \lambda = 0.04 \]
Time-Series of Enstrophy

\[ R_s = 2140 \]
\[ R_p = 428 \]
\[ \gamma = 0.2 \]
\[ \lambda = 5.0 \]
\[ \lambda = 0.10 \]
Time-Series of Enstrophy

\[ R_s = 2950 \]
\[ R_p = 1475 \]
\[ \gamma = 0.5 \]
\[ \lambda = 1.27 \]
Time-Series of Enstrophy

$R_s = 2700$

$R_p = 2160$

$\gamma = 0.8$

$\lambda = 3.94$
Time-Series of Enstrophy

\[ R_s = 2600 \]
\[ R_p = 2600 \]
\[ \gamma = 1 \]
\[ \lambda = 2.79 \]
Time-Series of Enstrophy

\[ R_s = 2800 \]
\[ R_p = 3080 \]
\[ \gamma = 1.1 \]
\[ \lambda = 2.64 \]
Time-Series of Enstrophy

\[ R_s = 3000 \]
\[ R_v = 3600 \]
\[ \gamma = 1.2 \]
\[ \lambda = 2.58 \]
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Flow Structure

Steady States
Solid-Body Rotation

$\Gamma \ll 1$

or

$Re \ll 1$

Spin axis
Streamline Tori

The whole surface of a torus is covered by a single streamline.

$Rs=10$

$Rp=1$
Cross-Section of Streamline Tori

$Rs=10$

$Rp=1$

Spin axis
Separatrix Surfaces

\[ R_s = 10 \]

\[ R_p = 1 \]
Separatrix Surfaces

\[ R_s = 10 \]
\[ R_p = 1 \]
Cross-Section of Streamline Tori

$R_s = 200$

$R_p = 20$

Spin axis
Separatrix Surfaces

\[ R_s = 200 \]
\[ R_p = 20 \]
Cross-Section of Streamline Tori

$R_s = 500$

$R_p = 50$

Spin axis
Separatrix Surfaces

$R_s = 500$

$R_p = 50$
Location of Stagnation Points
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Asymptotic Analysis

In the double limit of small Reynolds numbers and large times
Low-Reynolds-Number Flow

\[ R_s \frac{\partial u}{\partial t} = R_s u \times \omega - 2\Gamma R_s \hat{z} \times u - R_s \nabla P + \nabla^2 u \]

\[ P = p + \frac{1}{2} |u|^2 + \frac{1}{2} \Gamma^2 (r \times \hat{z})^2 \]

\[ \Gamma = \frac{R_p}{R_s} = \frac{\Omega_p}{\Omega_s} = O(1) \]

\[ R_s \ll 1 \quad R_p \ll 1 \]

\[ u = u^{(0)} + R_s u^{(1)} + R_s^2 u^{(2)} + \cdots, \]

\[ \omega = \omega^{(0)} + R_s \omega^{(1)} + R_s^2 \omega^{(2)} + \cdots, \]

\[ P = P^{(0)} + R_s P^{(1)} + R_s^2 P^{(2)} + \cdots. \]
Low-Reynolds-Number Flow

**0th Order**

\[ u_r^{(0)} = 0, \]
\[ u_\theta^{(0)} = 0, \]
\[ u_\varphi^{(0)} = r \sin \theta, \quad \text{Solid-body rotation} \]

**1st Order**

\[ u_r^{(1)} = 0, \]
\[ u_\theta^{(1)} = \frac{\Gamma}{10} (1 - r^2) r \sin \varphi, \]
\[ u_\varphi^{(1)} = \frac{\Gamma}{10} (1 - r^2) r \cos \theta \cos \varphi, \quad \text{Differential rotation around } y \text{-axis} \]
Low-Reynolds-Number Flow

2\textsuperscript{nd} Order

\[ u_r^{(2)} = \frac{\Gamma}{420} r(1 - r^2)^2 \sin \theta \cos \varphi (\Gamma \sin \theta \sin \varphi + \cos \theta), \]

\[ u_\theta^{(2)} = -\frac{\Gamma}{2520} r(7r^2 - 3)(1 - r^2) (\Gamma \sin 2\theta \sin \varphi + \cos 2\theta) \cos \varphi, \]

\[ u_\varphi^{(2)} = -\frac{\Gamma}{2520} r(7r^2 - 3)(1 - r^2) (\Gamma \sin \theta \cos 2\varphi - \cos \theta \sin \varphi) \]

\[ -\frac{\Gamma^2}{1400} r(9 - 5r^2)(1 - r^2) \sin \theta \]
Low-Reynolds-Number Flow

3rd Order

\[ u_r^{(3)} = \left[ \frac{\Gamma^3(58 - 15r^2)}{623700} + \frac{\Gamma(5 - 3r^2)}{249480} \right] r(1 - r^2)^2 \sin 2\theta \sin \varphi \]

\[ + \frac{\Gamma^2(10 - 3r^2)r(1 - r^2)^2}{113400}(3\cos^2 \theta - 1) \]

\[ - \frac{\Gamma^2(148 - 69r^2)r(1 - r^2)^2}{1247400}\sin^2 \theta \cos 2\varphi, \]
Low-Reynolds-Number Flow

\[ u^{(3)}_{\theta} = -\frac{\Gamma^3}{249480} r^3(1 - r^2)(13 - 9r^2) \sin^2 \theta \sin 3\varphi \]
\[ -\frac{\Gamma^3}{12474000} r(1 - r^2)(4884 - 4555r^2 + 1275r^4) \sin \varphi \]
\[ +\frac{\Gamma^3}{2494800} r(1 - r^2)(232 - 663r^2 + 195r^4) \cos 2\theta \sin \varphi \]
\[ -\frac{\Gamma^2}{2494800} r(1 - r^2)(148 - 287r^2 + 87r^4) \sin 2\theta \cos 2\varphi \]
\[ -\frac{\Gamma^2}{226800} r(1 - r^2)(30 - 85r^2 + 27r^4) \sin 2\theta \]
\[ -\frac{\Gamma}{2494800} r(1 - r^2)(99 - 250r^2 + 135r^4) \sin \varphi \]
\[ +\frac{\Gamma}{249480} r(1 - r^2)^2(5 - 3r^2) \cos 2\theta \sin \varphi, \]
Low-Reynolds-Number Flow

3rd Order

\[ u^{(3)}_\varphi = -\frac{\Gamma^3}{249480} r^3 (1 - r^2)(13 - 9r^2) \sin^2 \theta \cos \theta \cos 3 \varphi \]

\[ -\frac{\Gamma^3}{997920} r^3 (1 - r^2)(13 - 9r^2) \cos 3 \theta \cos \varphi \]

\[ -\frac{\Gamma^3}{249480000} r(1 - r^2)(7448 - 2805r^2 + 825r^4) \cos \theta \cos \varphi \]

\[ +\frac{\Gamma^2}{249480} r^3 (1 - r^2)(13 - 9r^2) \sin 3 \theta \sin 2 \varphi \]

\[ +\frac{\Gamma^2}{623700} r(1 - r^2)(74 - 241r^2 + 111r^4) \sin \theta \sin 2 \varphi \]

\[ +\frac{\Gamma}{249480} r^3 (1 - r^2)(13 - 9r^2) \cos 3 \theta \cos \varphi \]

\[ -\frac{\Gamma}{24948000} r(1 - r^2)(49 - 40r^2 + 15r^4) \cos \theta \cos \varphi. \]
Poincare Map of Streamline

\[ \Delta r = -\frac{R_s^3 \pi \Gamma^2}{16200} r_0 (1 - r_0^2)^2 (1 - 3r_0^2)(3 \cos^2 \theta_0 - 1) + O(R_s^4) \]

\[ \Delta \theta = \frac{R_s^3 \pi \Gamma^2}{32400} (1 - r_0^2)(3 - 22r_0^2 + 27r_0^4) \sin 2\theta_0 + O(R_s^4) \]
Cross-Section of a Streamline Torus

\[
\frac{dr}{d\theta} = -\frac{2r(1 - r^2)(1 - 3r^2)(3 \cos^2 \theta - 1)}{(3 - 22r^2 + 27r^4) \sin 2\theta}
\]

\[R_s \ll 1\]
\[R_p \ll 1\]
\[t = O(R_s R_p^2)\]

\[r_0 \to r, \; \theta_0 \to \theta, \; \Delta r/\Delta \theta \to dr/d\theta\]

\[r^3(1 - r^2)^2\left(\frac{1}{3} - r^2\right)(1 - \cos^2 \theta) \cos \theta = \text{const.}\]

\[(1 - x^2 - y^2 - z^2)^2\left(\frac{1}{3} - x^2 - y^2 - z^2\right)(y^2 + z^2)x = \text{const.}\]
Cross-Section of a Streamline Torus

\[
r^3(1 - r^2)^2\left(\frac{1}{3} - r^2\right)(1 - \cos^2 \theta) \cos \theta = \text{const}.
\]

\[
R_s \ll 1
\]

\[
R_p \ll 1
\]

\[
t = O(R_s R_p^2)
\]
Comparison: DNS & Theory

\[ \frac{1}{\sqrt{3}} = 0.58 \]

\[ R_s=10, \ R_p=1 \]

\[ r^3(1-r^2)^2\left(\frac{1}{3} - r^2\right)(1-\cos^2 \theta)\cos \theta = \text{const.} \]
Summary

① The state diagram of flows in a precessing sphere was constructed experimentally.

② The stability curve of steady flow was revealed partially by DNS.

③ The toral structure of streamlines was observed by DNS. So far no chaotic streamline has been found.

④ An analytical expression was obtained for the streamline tori in the double limit of small Reynolds numbers and large times.
Future Problems

① Complete the stability curve

② Clarify the characteristics of critical modes

③ Perform the linear stability analysis

④ Raise the speed of the numerical code

⑤ Examine the turbulence characteristics, such as intensity, mixing, etc.