Morphology and dynamics of strongly stratified flows
Toroidal cascade

First IMS workshop IC, London

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Context: conceptual approach, not geophysics, but ...

- General theme: rotating stratified flows with two external $N$ and $f$ parameters
- *Pure* rotation ($N = 0$): phase-mixing and *weak* wave-turbulence, simple instances
- *Pure* stratification: simple scaling laws, exact Lin-type equations.
- The toroidal cascade as a toy-model: a very complex anisotropic behaviour for *strong* turbulence
- The QG model revisited. Open problems, conclusions
Rotating stratified (unbounded) flows: governing equations

\[
\frac{\partial u_i}{\partial t} + f \varepsilon_{i3j} u_j - b \delta_{i3} + \frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j}, \quad \frac{\partial u_i}{\partial x_i} = 0
\]

\[
\frac{\partial b}{\partial t} + N^2 u_3 = Pr \nu \nabla^2 b - u_j \frac{\partial b}{\partial x_j}
\]

2 external parameters \( N \) and \( f \) (frequencies)

Valid for a liquid or a gas. \( Pr \) characterizes the diffusivity of the stratifying agent (temperature, salt)
Stable stratification in ocean (under the mixed zone) and in atmosphere (temporary inversion in troposphere, stratosphere)
A first cartoon of linear effects

Fluctuating pressure? incompressibility?
Eigenmodes decomposition

- Incompressibility and pressure → 3D Fourier space

\[(u, b)(x, t) = \sum e^{i k \cdot x} \left( \underbrace{a_0 N^{(0)}}_{\text{vortex (QG)}} + \underbrace{a_{+1} N^{(1)} e^{i \sigma_k t}}_{\text{wave (AG)}} + \underbrace{a_{-1} N^{(-1)} e^{-i \sigma_k t}}_{\text{wave (AG)}} \right)\]

- Dispersion law \( \sigma_k = \sqrt{N^2 \sin^2 \theta + f^2 \cos^2 \theta} \)

- Linear dynamics: slow amplitudes \( a_{0, \pm 1} \) are constant. Nonlinear case

- Advantages \( k \cdot \hat{u} = 0 \), five \((u_1, u_2, u_3, b, p)\) → three \((a_0, a_{+1}, a_{-1})\).

Ref. Cambon et al., Bartello, Smith & Waleffe, Morinishi, Kaneda ... etc.
Geometric aspects of eigenmodes

(Craya-Herring, standard) and “Vortex/wave” ($f/N$ — depending)
Wave aspects

Rotation, phase mixing: Linear and nonlinear

- Relevance of linear solution depends on the order and type of statistical correlations
  - doubles: 2 point 1 time: $e^{i\sigma_k t}, e^{-i\sigma_k t}$
  - doubles: 2 point 2 time: $e^{i\sigma_k (t \pm t')}$
  - triples: 3 point 1 time: $e^{i(\pm \sigma_k \pm \sigma_p \pm \sigma_q) t}$ → nonlinear · · ·

\[
< \omega_3^3 > = \sum \int \exp[ift\left(\frac{k_3}{k} + sp_3 p + \frac{q_3}{q}\right)] S(k, p, \epsilon t) d^3 p d^3 k
\]

needs for initial triple correlations at THREE point. Many other correlations.
Reintroducing stratification: ANTI 2D and toroidal (strong) cascade
Geometric aspects of eigenmodes

(Craya-Herring, standard) and “Vortex/wave” ($f/\mathcal{N}$ — depending)
Conical region in which energy concentrates
Angle-dependent toroidal and poloidal modes (Liechtenstein, 2006)
Pure stratification: scaling arguments


- Froude numbers, horizontal and vertical, $Fr_h = \frac{U}{NL_h}$, $Fr_v = \frac{U}{NL_v}$, $L_h \gg L_v$.

- $L_v \sim \frac{U}{N}$ (“zig-zag” instability? Billant & Chomaz 1999, 2002) → $Fr_v \sim 1$ (to contrast with Riley et al. 1981?)

- Proposed scaling (Linborg 2006) $E_{hh}(k_h) = C_1 \epsilon_k^{2/3} k_h^{-5/3}$, $E_p(k_h) \sim C_2 \epsilon_p \epsilon_K^{-1/3} k_h^{-5/3}$, $E_{vv} \sim N^2 k_v^{-3}$.

- $Ri \sim 1/4$ threshold vs. $Fr^2 Re$ scaling?, rediscovery of polo-toro decomposition (Brethouwer, Linborg, Billant, Chomaz.)
Pure stratification: Generalized Lin equations

\[
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) e^{(\text{tor})} = T^{(\text{tor})} \\
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) e^{(w)} = T^{(w)} \\
\left( \frac{\partial}{\partial t} + 2\nu k^2 + 2iN \frac{k_{\perp}}{k} \right) Z' = T^{(z')} 
\]

Energy spectra \( e^{(\text{tor})}, (\text{pol}), (\text{pot}) (k_{\perp}, k_{\parallel}) \), imbalance deviator \( Z' \),
\( \Re Z' = e^{(\text{pol})} - e^{(\text{pot})} \), \( e^{(w)} = e^{(\text{pol})} + e^{(\text{pot})} \).
A lot of information can be generated, vs. second and third-order structure functions.
The toroidal cascade. Why not 2D?

- Why the toroidal component only? \[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{\omega} \times \mathbf{u} + \nabla \left( p + \frac{u^2}{2} \right) = -\mathbf{b}, \]
  \[ \dot{\mathbf{u}}^{(1)} + \mathbf{e}^{(1)} \cdot \sum_{p+q=k} (\mathbf{\hat{\omega}}(p) \times \mathbf{\hat{u}}(q)) = 0, \]
  \[ \mathbf{\hat{u}} = \mathbf{u}^{(1)} e^{(1)} + \mathbf{u}^{(2)} e^{(2)}, \mathbf{\hat{\omega}} = \imath k \left( \mathbf{u}^{(1)} e^{(2)} - \mathbf{u}^{(2)} e^{(1)} \right) \]

  \[ \dot{u}_k^{(1)} = (p_\perp^2 - q_\perp^2) G u_p^{(1)*} u_q^{(1)*}, \]  
  \[ \dot{u}_p^{(1)} = (q_\perp^2 - k_\perp^2) G u_q^{(1)*} u_k^{(1)*}, \]  
  \[ \dot{u}_q^{(1)} = (k_\perp^2 - p_\perp^2) G u_k^{(1)*} u_p^{(1)*}, \]

- quasi 2D or not, reverse or direct cascade? cylinder to cylinder, shell to shell, angle to angle: very rich and various morphology ...
Analogy with an ‘Euler problem’

- The solid in its principle axes of inertia

\[
I_1 \dot{\Omega}_1 = (I_2 - I_3) \Omega_2 \Omega_3, \quad (7)
\]
\[
I_2 \dot{\Omega}_2 = (I_3 - I_1) \Omega_3 \Omega_1 \quad (8)
\]
\[
I_3 \dot{\Omega}_3 = (I_1 - I_2) \Omega_1 \Omega_2, \quad (9)
\]

- Conservations laws, rot. kin. energy \((I_1 \Omega_1^2 + \ldots) \rightarrow\) kin. energy (triad),
  norm of the angular momentum \((I_1 \Omega_1)^2 + \ldots \rightarrow\) vertical enstrophy (triad)
Instabilities, reverse interactions only.
Revisiting the QG cascade

- Detailed conservation of QG energy and potential enstrophy (linear ?)
  \[ \frac{k^2 \sigma_k^2}{N^2} \xi^{(0)} \xi^{(0)*} = k^2 \nabla^2 u^{(1)} u^{(1)*} + \left( \frac{f}{N} k \right)^2 u^{(3)} u^{(3)*} \]

- Reworking on \( N^{000} \) (Bartello 1995)
  \[
  \begin{align*}
  \dot{\xi}_k^{(0)} &= (p^2 \sigma_p^2 - q^2 \sigma_q^2) G' \xi_p^{(0)*} \xi_q^{(0)*}, \quad (10) \\
  \dot{\xi}_p^{(0)} &= (q^2 \sigma_q^2 - k^2 \sigma_k^2) G' \xi_q^{(0)*} \xi_k^{(0)*}, \quad (11) \\
  \dot{\xi}_q^{(0)} &= (k^2 \sigma_k^2 - p^2 \sigma_p^2) G' \xi_k^{(0)*} \xi_p^{(0)*}, \quad (12)
  \end{align*}
  \]

- ‘linear’ (quadratic) limit ? Ertel theorem, flat isopycnes. Dual cascade or not, why ?
Coexistence of weak and strong turbulence: MHD, aeroacoustics

Alfven-wave turbulence competing with strong turbulence with additional Joule dissipation effect (Moffat 1967, etc)
Concluding comments

1. Stability analysis vs. anisotropic statistical theory
   -) Layering in stratified flows (zig-zag, Kelvin-Helmholtz)
   - Cyclonic/anticyclonic asymmetry in rotating flows (centrifugal)

2. Anisotropic multimodal EDQNM incorporating RDT vs. DNS/LES
   -) A systematic way to construct triadic correlations
   -) Need for improved ED for strong turbulence (TFM, LRA ?)

3. Competition between waves and vorticites to organise Lagrangian diffusion (also plasmas ?)

4. A better link between physical and spectral space ?
More on strong anisotropy
Anisotropic description

- **ANISOTROPY/ inhomogeneity/ Intermitency**
- structure functions or correlations, two-point: \( R_{ij}(\mathbf{r}) = \langle u_i(x) u_j(x + \mathbf{r}) \rangle \)
  
  \[
  u(x) \quad u(x + \mathbf{r})
  \]

  \( \mathbf{r} \)

  \( \langle \mathbf{u}(x) \rangle \)

  \( \langle \mathbf{u}(x + \mathbf{r}) \rangle \)

  \( \langle \mathbf{u}(x) \mathbf{u}(x + \mathbf{r}) \rangle \)

- Single-point: componentality only
- Two-point: directional anisotropy

- Low dimension parameterization, SO(3) symmetry group (Arad et al., PRE, 1999)

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Anisotropic description. 3D Fourier space

- Anisotropic scalar (e.g. spherical harmonics) for both ‘physical’ and ‘spectral’

\[
\frac{1}{2} R_{ii}(\mathbf{r}) \rightarrow \frac{1}{2} \hat{R}_{ii}(\mathbf{k}) = e(\mathbf{k})
\]

\[
\sum r_n^m(\mathbf{r}) Y_n^m(\theta_r, \phi_r) \rightarrow \sum \varphi_n^m(\mathbf{k}) Y_n^m(\theta_k, \phi_k)
\]

Avoiding a ‘schizophrenic’ viewpoint! (Cambon & Teissèdre 1985, CRAS Paris)

- A trace-deviator decomposition restricted to solenoidal space

\[
\hat{R}_{ij} = \underbrace{U(\mathbf{k}) P_{ij}}_{\text{isotropic}} + \underbrace{\mathcal{E}(\mathbf{k}) P_{ij}}_{\text{directional}} + \underbrace{\mathcal{R}(Z(\mathbf{k}) N_i N_j)}_{\text{polarization}}.
\]

\[
(Cambon \& Jacquin, JFM, 1989), \quad P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}, \quad N \text{ ‘helical mode’. Helicity?}
\]
Rotating turbulence, MHD simplified case
More on pure rotation
Pure rotation, phase-mixing

- Basic linear operator
  \[ \hat{u}(k, t) = \Re[N_i N_j^* e^{\sigma_k (t-t')}], \sigma_k = f \frac{k||}{k} \]

- Second-order statistics:
  \[ e(k, t) = e(k, 0) \]
  \[ Z(k, t) = e^{2i\sigma t} Z(k, 0) \]
  \[ \int_0^1 Z(x) \exp(\imath f x t) dx \to 0; \quad x = \cos(\langle k, n \rangle) \]
Phase-mixing for two-time second order statistics

Applications to single-particle Lagrangian diffusion (at the end)
Results. NONLINEAR statistical theory

- From classical EDQNM (isotropic, no rotation, Orszag 1970, Bos & Bertoglio, 2006)

... to EDQNM3 \(\rightarrow\) (A) QNM energy equation (Bellet et al., JFM, 2006)

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Angle-dependent spectrum

- Isotropy breaking by spectral transfer $T^{(e)}(k)$: directional anisotropy:

$$4\pi k^2 e(k, t_f) = 4\pi k^2 e(k, \cos \theta, t_f)$$

- Spherical averaging $\rightarrow E(k, t_f)$, prefactor $E \sim \frac{\Omega}{t} k^{-3}$, not 2D!
\[ E(k_n, \theta_m) \]

- \( k^{-2} \)
- \( \cos \theta \approx 0 \)
- \( \cos \theta \approx 1 \)

512\(^3\) DNS by Liechtenstein et al., JOT, 2005
Inertial wave-turbulence, resonant interactions, 2D or not 2D

- **Low dimension** of active manifolds: overestimated in forced？DNS/LES？
  ‘TRUE’ 2D embedded in 3D: a DIRAC singularity

\[
E(k) \sim f^2 k^{-3}, \quad e(k_\perp, k_\parallel) = \frac{E(k_\perp)}{2\pi k_\perp} \delta(k_\parallel) \approx f^2 k_\perp^{-4}
\]

- Integral singularity from theoretical wave-turbulence

\[
e(k_\perp k_\parallel) \sim k_\parallel^{-1/2} k_\perp^{-7/2} = k^{-4} x^{-1/2} \quad \text{Galtier 2002}
\]

\[
e(k_\perp k_\parallel) \sim k_0^{-1/2} k_\parallel^{-1/2} k_\perp^{-3} = k_0^{-1/2} k^{-7/2} x^{-1/2} \quad \text{CRG 2004}
\]

\[
E(k) \sim \frac{f}{t} k^{-3}, \quad e(k, x \sim 0) \sim k^{-4} \quad \text{BGSC-2006}
\]