Abstract. The synthetic CDO market has, over the last year, seen a significant increase in liquidity and transparency. The availability of published prices in both TriBoxx and Tracers permit calibration of model parameters in a way that was not previously achievable.

This paper details what we believe has become the market standard approach in CDO valuation. The valuation model is introduced and analyzed in depth to develop a better practical understanding of its use and the implications of parameter selection and calibration. In particular we examine the idea that correlation within a copula model can be seen to be an equivalent measure to volatility in a standard B&S option-framework and correspondingly we seek to calibrate smile and skew.

Key words. Portfolio Credit Risk Models, Copula Functions, Credit Derivatives

We would like to thank our colleagues James McNabb, Richard Whittle, Mustafa Khalid, Igor Vaysburd and Caroline Tan. This paper is based upon the ideas presented in A. Friend’s M.Sc. thesis. Views expressed in the paper are the authors’ own and do not necessarily reflect those of ABN AMRO. All errors remaining are of course our own, disclaimer is included at the end of this document.

1. Introduction. Credit derivatives have, over the last few years moved from a purely OTC based product to liquid transparent actively quoted instruments available in a range of formats to meet investors risk return requirements. One particular area of interest, which has recently been undergoing dramatic evolution, is the Synthetic Collateralized Debt Obligation (CDO). This product was conceived to provide diversified investments having a range or risk return profiles. In the case where investors seek large coupons and have low risk aversion the enhanced returns from equity level tranches were an attractive option. Investors with increasing risk aversion would select tranches higher in the capital structure for correspondingly lower returns and lower risk.

In traditional CDOs the issuer of the product would select a diversified portfolio and seek to place all tranches and as such the correlation was not deemed critical as it would merely effect the distribution returns passed onto the investors for any given selected tranche. In today’s markets, however, placement of the entire capital structures is no longer the norm, indeed increasingly banks are quoting single tranches on standard portfolios such as Tracers and TriBoxx. When a bank makes prices on a single tranche it effectively runs risk on the remainder of the capital structure and hence being able to calibrate correlation becomes critically important to successful risk management and hedging of the product.

As more standardized products on listed indexes are brought to market the correlation increasingly becomes a market observable. With the increased observability of correlation the understanding of correlation develops and it is thus natural for the pricing framework to be adapted to deal with the increasing levels of sophistication of the market. In this paper we seek to provide insight into the ways in which correlation can be manipulated to calibrate the products pricing to be consistent with the observed market quotes in a similar manner to the way in which an options trader would calibrate his volatility parameter.

1.1. Synthetic CDOs and tranches. Synthetic tranches, such as TriBoxx, have a standard reference portfolio and are specified directly by their upper bound and lower bound. The upper bound $L$ and lower bound $L$ are given as a percentage of the portfolio. Loss payments are only made from the tranche when the total loss amount is between these two bounds. The pay off for any total loss of the portfolio $l$
can be written as

\[ \text{PayOff}(l) = \max \{ \min \{ l, L^+ \} - L^-, 0 \} \]

The table below gives a typical example for the bid/ask of a European TriBoxx run and a US TriBoxx run on November 13, 2003. The WAS gives the weighted average spread on the reference portfolios.

<table>
<thead>
<tr>
<th></th>
<th>EUR</th>
<th>WAS</th>
<th>US</th>
<th>WAS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-3%</td>
<td>1200</td>
<td>1350</td>
<td>500</td>
<td>37.3</td>
</tr>
<tr>
<td>3-6%</td>
<td>150</td>
<td>230</td>
<td>310</td>
<td>350</td>
</tr>
<tr>
<td>6-9%</td>
<td>45</td>
<td>76</td>
<td>94</td>
<td>112</td>
</tr>
<tr>
<td>9-12%</td>
<td>22</td>
<td>43</td>
<td>45</td>
<td>63</td>
</tr>
<tr>
<td>12-22%</td>
<td>6</td>
<td>17</td>
<td>8</td>
<td>19</td>
</tr>
</tbody>
</table>

(1.2)

The standard maturity for these tranches is five years, the standard premium uses an amortizing notional schedule\(^1\), paid quarterly in arrears. This means that the notional over which the premium is paid corresponds to the remaining tranche amount. The coupon that is paid for any total loss of the portfolio \(l\) can be written as

\[ \text{Coupon}(l) = \frac{L^+ - L^- - \text{PayOff}(l)}{L^+ - L^-} \cdot \text{Notional} \cdot \text{Premium} \]

(1.3)

In case of no losses for this tranche, the fraction is equal to one. In case the tranche is wiped out, the fraction is equal to zero.

### 1.2. Portfolio loss distribution.

Given the functional form of the payoff on a tranche, a tranche can be seen as an option on the loss distribution of the reference portfolio. The description of the loss distribution is crucial for the valuation of CDOs, it has to deal with default events in the portfolio and, more precise, with the interaction of default events or default dependency within the portfolio. At the same time, it has to be consistent with the single name market. The marginal default probabilities have to correspond to the probabilities implied by the credit default swap market.

One of the most common approaches to modeling the loss distribution (and valuation of basket credit derivatives) is the Gaussian copula approach as described by Gupton et.al. \[GFB1997\] and Li \[L1999\]. That fact that this approach is one of the first models around and relatively straightforward to implement (it requires the simulation of multivariate Gaussians that are transformed into default times) explain most of its popularity. It cleared the way for research into the use of copula functions in credit risk or, if you like, finance in general; for example Bouyé et.al. \[BDN2000\] and Embrechts et.al. \[ELM2001\] provide a good overview of various copula functions and their relevance to finance.

Another approach to describe the interaction between obligors in the portfolio is through jumps in the default spreads. These jumps are either caused by some joint economic factors, as in Duffie and Singleton \[DS1999\], or by default events in the portfolio, such as Davis and Lo \[DL2001\] and Gieseke and Weber \[G2003\]. The advantage of these approaches is that they make the dynamics of the spreads in the portfolio clearly visible, which is “hidden” in the copula approach. The dynamics (and jumps) in the copula setup are, however, there and they are made explicit in Schönbucher and Schubert \[SS2001\] and Elouerkaoui \[E2002\]. The approach is further explored in Rogge and Schönbucher \[RS2003\], where the copula setup can actually be calibrated to jumps conditional on default.

Further development on the copula approach has focused on Archimedean copula’s, such as Schönbucher \[S2002\] and Rogge and Schönbucher \[RS2003\], and on variations of the Gaussian copula, such as the \(t\)-copula (see for example Mashal and Naldi \[MN2001\]) and the “one-factor” Gaussian copula. The latter is described in more detail in for example Gregory and Laurent \[GL2003a\] and \[GL2003b\], Andersen et.al. \[ASB2003\] and later on by Hull and White \[HW2003\]. The major advantage of the one-factor Gaussian

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\(^1\)The equity piece sometimes differs, as in the US TriBoxx example provided. In this case part of the premium is paid upfront.
A copula is that it allows for relatively quick semi-analytical pricing of CDO pieces compared to the classic Gaussian copula approach while keeping some of its advantages.

Some of the more frequently used copula functions that are discussed in the above literature will briefly be repeated in the first section of the paper. We will, however, focus on those that we believe has become market standard (or at least a part of it). In the second half of this paper we will focus in particular on the one-factor Gaussian copula, for which we will derive implied correlation parameters and manipulate these as if they were volatility in B&S option pricing.

1.3. Outline. The paper is organized as follows: the next section describes the default risk framework for single names and the copula functions used to create the framework for portfolio credit risk. Some frequently used copula functions, such as Gaussian copulas (as in the classic paper by Li [L1999]), t-copulas (as described by for example Mashal and Naldi [MN2001]) and Archimedean copulas (see for example Schönbucher [S2002] or Rogge and Schönbucher [RS2003]) and their properties are briefly discussed. These form the basic building tools needed to price any portfolio credit derivatives. The subsequent section describes specifically the valuation of CDO tranches, using various pricing approaches. In particular we recall pricing approaches such as large portfolio approximations (see for example Schönbucher [S2003] and [S2002]), a semi-analytical approach (as described by Gregory and Laurent [GL2003a] and [GL2003b] and Andersen, Sidenius and Basu [ASB2003]) and simulation (such as Li [L1999], but also other simulation approaches as for example described by Schönbucher [S2002}).

After this section all the mathematical tools and pricing methodologies for CDO tranches are set up and enable us to further study the impact of various parameters and the actual behavior of these models in the current (more transparent and liquid) market. The next section examines the impact of the copula parameters on properties of the tranches, such as the impact on sensitivity to correlation and on jumps conditional on a default event. This provides a better understanding of the influence of certain model parameters. Thereafter we specifically make a model choice for the so-called “one-factor” Gaussian copula approach, which corresponds to the semi-analytical approach described in one of the earlier sections. Treating this model as a possible B&S in synthetic tranche valuation, we tweak correlation as an equivalent measure to volatility for B&S-option pricing. In particular, we examine some possible adjustments to the correlation inputs in this framework to create more realistic behavior, for example a correlation smile and a skew.

2. Framework. The framework required here for the valuation of tranches is based upon marginal default probabilities and copula functions to connect them. However, before providing these in more detail, we also need the rate side.

Notation 2.1. The discount factor from today to time $t$ is given by $D(0,t)$.

2.1. Modelling default risk. Individual obligors are subjected to default risk. A default event is the first time that a certain trigger level is reached.

Notation 2.2. Consider obligors $i = 1,...,N$. Each obligor has a default probability $p_i(t)$ for a given time period $[0,t]$. If necessary, this time period will be mention explicitly, i.e. the default probability is given by $p_i(t)$. As a result, the survival probability is given by $1 - p_i(t)$. For each obligor we can also define a trigger level $U_i$, which is a random variable uniformly distributed on $[0,1]$. An obligor would be in default if at time $t$

$$1 - p_i(t) < U_i \quad (2.1)$$

A popular way to “fill in” the default probability would be to use a hazard rate process with, for example, a piecewise constant hazard rate curve fitted to the credit default swap market.

The way to describe the dependency between defaults of the obligors is to specify the joint distribution of the trigger levels $U_i$. 3
Notation 2.3. The $N$-dimensional copula function $C[0,1]^N \rightarrow [0,1]$ specifies the joint distribution of the trigger levels $U_i$.

A much more rigorous introduction to copula functions is given in for example Joe [J1997]. We shall restrict ourselves here to some of the frequently used copula functions for tranche valuation whilst omitting many technical details.

2.2. Frequently used copula functions. The most frequently used copula function is the Gaussian copula function. The Gaussian copula is, for example, used implicitly in CreditMetrics [GFB1997]. This use has been made explicit by Li [L1999] and has been popular ever since. Closely related to the Gaussian copula is another elliptical copula: the $t$-copula. Although of the same family of copula functions, the $t$-copula possesses heavier tails, introducing the concept of tail-dependency (see for example Embrechts et.al. [ELM2001]). Marshall and Naldi [MN2001] provide more insight in the degrees of freedom that could be used in case a link is made with the equity market, while O’Kane and Schlögl [OS2003] provide an application of the $t$-copula to large portfolios.

Definition 2.4. The standard Gaussian copula function is defined as
\[
C_{\Sigma}^{Ga}(\vec{u}) = \Phi_{\Sigma}(\Phi^{-1}(u_1),...,\Phi^{-1}(u_N))
\]  
with the $N \times N$ correlation matrix $\Sigma$. Directly related to that is the student-$t$ copula function
\[
C_{\Sigma,\nu}^{t}(\vec{u}) = t_{\Sigma,\nu}(t_{\nu}^{-1}(u_1),...,t_{\nu}^{-1}(u_N))
\]  
with $\nu$ degrees of freedom.

One undesirable property of the Gaussian is its singularity at $\vec{u} = \vec{0}$ and $\vec{u} = \vec{1}$, see also Schönbucher [S2003], section 10.7.2. As a result, the dependency structure changes between the corners of the copula function and the middle. This has practical implications for the pricing of, for example, first-to-default swaps on the same reference portfolio but for different maturities. These problems are well known (see for example Razak [R2003]), we will come back to these problems in the last section when we study some practical aspects of using these models.

Due to the rich dependency structure, applying these copula functions to pricing credit derivatives can unfortunately result in relatively slow Monte Carlo simulations (especially for high grade reference obligors). A variation is the one-factor Gaussian copula function, as described by for example Laurent and Gregory [GL2003a], which reduces the dependency to only one Gaussian variable. This allows for quick numerical integration techniques through, for example, Gauss Hermite quadrature.

Definition 2.5. The one-factor Gaussian copula function is defined as
\[
C_{\rho}^{Ga}(\vec{u}) = \int_{-\infty}^{\infty} \varphi(x) \prod_{i=1}^{N} \Phi \left( \frac{\Phi^{-1}(u_i) - \rho_{i} x}{\sqrt{1 - \rho_i^2}} \right) dx
\]

Another class of copula functions, that is increasing in popularity, are the Archimedean copulas. They typically have only one or two parameters to describe the dependency and, as will be demonstrated later on, can be described by one generating factor, similar to the one-factor Gaussian copula.

Definition 2.6. Define a generator function $\phi : [0, \infty) \rightarrow [0, 1]$ such that it is the Laplace transform of a positive random variable $Y$
\[
\phi(s) = \mathbb{E} [e^{-sY}]
\]
The random variable $Y$ is often referred to as frailty variable or mixing variable. Archimedean copula functions can be specified using their generator function $\phi(x)$.
\[
C^{Arch}(\vec{u}) = \phi \left( \sum_{i=1}^{N} \phi^{-1}(u_i) \right)
\]
In particular, the Clayton copula is an Archimedean copula function with \( \phi^{-1} (x) = x^{-\theta} - 1 \), corresponding to the mixing variable \( Y \) having a Gamma \( \left( \frac{1}{\theta} \right) \) distribution, which results in

\[
C_{\theta}^\text{Cl} (\mathbf{u}) = \left( 1 - N + \sum_{i=1}^{N} u_i^{-\theta} \right)^{-\frac{1}{\theta}}
\]

The Clayton copula is of particular interest because the relative jump in conditional default intensity after the first default event in the basket is equal to \( \theta \), a relationship described in more detail by Schönbucher and Schubert [SS2001] and further used in a more general setup in Rogge and Schönbucher [RS2003]. The fact that it’s parameter has a direct interpretation and the simplicity makes the Clayton copula relatively attractive. In addition to that, it does not have the problems of singularity that the Gaussian copula has.

2.3. Parameters of the copula functions. In general, the choice for model parameters in the copula functions is not in straightforward way related to the credit markets. For example, the correlation matrix in the Gaussian copula function could be set to the asset (or equity) correlation using the Merton argument; see for example CreditMetrics [GBF1997] and Li [L1999]. In the one-factor Gaussian copula it could be set to correlation between the individual shares and a market index (or the market beta).

We present two concepts that could provide some more insight in the influence of choice of copula parameters: the discrete default correlation and the widening in spreads after a default event in the portfolio. The discrete default correlation, implied by the choice of copula function and its parameters, is effectively the correlation between two correlation binomial events.

**Definition 2.7.** As before, let \( p_1(t), p_2(t) \) denote the probability of a default event for name 1, 2 between now and time \( t \) and \( p_{12}(t) \) the joint default probability between now and time \( t \). The discrete default correlation\(^2\) is given by

\[
\rho^d (t) = \frac{p_{12}(t) - p_1(t)p_2(t)}{\sqrt{p_1(t)(1 - p_1(t))} \sqrt{p_2(t)(1 - p_2(t))}}
\]

Another, usually indirect way, to relate the parameters of the copula functions to a concept from the credit markets is described by Schönbucher and Schubert [SS2001] and Rogge and Schönbucher [RS2003].

**Corollary 2.8.** The relative jump size \( \Delta \) on the conditional default intensity given the first default event can be described by the copula function (and its parameters).

\[
\Delta = \frac{\partial^2}{\partial u_i \partial u_j} C(\mathbf{u}) \cdot C(\mathbf{u}) - \frac{\partial}{\partial u_i} C(\mathbf{u}) \cdot \frac{\partial}{\partial u_j} C(\mathbf{u}) - 1
\]

As mentioned in the previous section, the expression gives a nice result for the Clayton copula (the jump is the parameter \( \theta \)), but is more complex for other copula functions. This complexity and the fact that it is not straightforward to estimate or predict the jumps on default, are potential drawbacks of this approach.

3. Tranche valuation. The valuation of tranches is based upon the mathematical tools given in the previous section. One the one hand there are the marginal default probabilities, that can be obtain from

\(^2\)Note that we have several correlation parameters by now. The Gaussian copula has correlation as a “connection parameter”, we have the discrete default correlation between (binomial) default events and we could go on to define others, for example the correlation between default times.
the credit default swap market. On the other hand is the copula function and its parameter(s), which basically is a model choice. Given these two components one can calculate for example the expected loss of a tranche or the model-fair premium. They all depend on the loss distribution of the portfolio.

**Notation 3.1.** The probability that the loss percentage \(L\) on a portfolio\(^3\) is less than \(l\) is denoted by

\[
P\{L \leq l\}
\]

Given this loss distribution one can derive some simplified general equations for the premium leg and the loss leg. These are detailed in the first subsection. Subsequently we fill in this general setup for specific loss distributions using some frequently used copula functions and large homogenous portfolio approximations.

### 3.1. General setup

A tranche corresponds to an option pay-off on the loss distribution. In its general form, the expected loss over a tranche is given by the next proposition. The choice of loss distribution function is still open.

**Proposition 3.2.** The expected loss for maturity \(T\) of a tranche with upperbound \(L^+\) and lowerbound \(L^-\) is given by

\[
E[L^+, L^-, T] = \int_0^{L_{\text{max}}-1} \max \{ \min \{l, L^+\} - L^-, 0\} \ dP\{L = l\}
\]

\[
\approx \sum_{i=0}^{L_{\text{max}}-1} \max \{ \min \{l_{i+1}, L^+\} - L^-, 0\} \cdot (P\{L \leq l_{i+1}\} - P\{L < l_i\})
\]

where \(L_{\text{max}}\) is the maximum loss amount and \(L_i, i = 0, ..., L_{\text{max}}\) is a grid over the possible loss amounts.

The loss leg of a tranche can be approximated by small pieces in which a loss events can occur (which have to be discounted accordingly). It corresponds to the discrete evaluation of an integral over time from now to maturity over the discounted (expected) losses of the tranche.

**Proposition 3.3.** Define the grid \(T_0, ..., T_J\) where \(T_0\) is the starting date of the tranche and \(T_J\) it’s maturity. The value of the loss leg of the tranche is given by

\[
V^{\text{Loss}}(0) = \sum_{j=0}^{J-1} D \left( 0, \frac{1}{2} (T_j + T_{j+1}) \right) \left( E[L^+, L^-, T_{j+1}] - E[L^+, L^-, T_j] \right)
\]

The premium leg of the tranche, where the premium declines as the notional of the tranche declines, can be approximated by taking beginning and point of each premium period.

**Proposition 3.4.** Define \(T_0, ..., T_M\) to be the payment dates for the protection fee \(\pi\) with \(T_0\) as today and \(T_M\) as the last payment date. The value of the premium leg using the amortizing notional schedule is given by

\[
V^{\text{Fee}}(0) = \pi \sum_{m=0}^{M-1} \frac{1}{2} \left( \frac{L^+ - L^- - E[L^+, L^-, T_{m+1}]}{L^+ - L^-} + \frac{L^+ - L^- - E[L^+, L^-, T_m]}{L^+ - L^-} \right) D(0, T_m)
\]

---

\(^3\)Note that we are working with loss percentage. In practice, a correction need to be made for the recovery rates of the defaulted names.
The value of the tranche is given by the difference between the two legs, i.e.

\[ V(0) = V^{\text{Loss}}(0) - V^{\text{Fee}}(0) \]

and the model-fair premium is the value of \( \pi \) for which \( V(0) = 0 \). This simplified approach relies on the loss distribution, which is described by marginal default probabilities and the copula function. The next subsections describe several straightforward implementations that, in combination with the loss equation (3.2) allow for a quick calculation of the loss leg (3.3) and the fee leg (3.4).

### 3.2. LHP approximation.

A well-known approach for loss distribution is Vasicek’s large homogeneous portfolio approximation. The assumption is that, in a large enough portfolio, the default probability by each individual obligor can be approximated by default probability \( p \) and that each name is correlated with a market variable via correlation \( \rho \). The general default probability can, for example, be derived from the weighted average spread of the names in the portfolio.

**Proposition 3.5.** The cumulative density function for the portfolio loss distribution in a one-factor Gaussian copula is given by

\[
P\{L \leq l\} = \Phi\left(\frac{1}{\rho} \left(\sqrt{1 - \rho^2} \Phi^{-1}(l) - \Phi^{-1}(p)\right)\right)
\]

**Proof.** See Schönbucher [S2003] section 10.4.4 \[ \square \]

This approach is based upon a link with a Gaussian random variable, but can easily be extended. This is studied, in particular, by Schönbucher [S2002]. He provides similar results for various Archimedean copula functions. We mention the Clayton copula function, where the mixing variable has a Gamma distribution.

**Proposition 3.6.** The cumulative density function for the portfolio loss distribution in a Clayton copula is given by

\[
P\{L \leq l\} = 1 - \frac{1}{\Gamma\left(\frac{1}{\theta}\right)} \int_0^{-\ln(l)/\phi(p)} e^{-y} y^{(1-\sigma)} dy
\]

**Proof.** Consider the mixing variable \( Y \) of the Clayton copula. The loss probability conditional on the mixing variable \( Y = p(Y) \). The proof now follows Schönbucher [S2003] section 10.4.4 for the one-factor Gaussian situation where we have a Gaussian factor \( Y \) instead.

\[
P\{L \leq l\} = P\{p(Y) \leq l\}
\]

\[
= P\{\exp(-Y \phi(p)) \leq l\}
\]

\[
= P\left\{Y \geq -\frac{\ln(l)}{\phi(p)}\right\}
\]

The Clayton copula has the properties that the mixing variable \( Y \) has a standard Gamma distribution \( Y \sim \Gamma\left(\frac{1}{\theta}\right) \) and generating function \( \phi(x) = x^{-\theta} - 1 \) yielding the result. \[ \square \]

These two approaches are based upon the assumption that the default probability \( p \) for all names is equal. Although this assumption simplifies calculation, it is not realistic. The next subsection describes a possible approach for non-equal default probabilities.

### 3.3. Using full loss distribution in one-factor setup.

The following approach is due to Laurent and Gregory [GL2003a] and [GL2003b] and is directly linked to the one-factor Gaussian copula. In
contrast to the previous subsection, we do not define an overall default probability but a default probability for each individual name. These default probabilities are independent conditional on the Gaussian variable \( x \).

**Definition 3.7.** As before, let \( p_i(t) \) denote the binomial default probability for name \( i \) up to time \( t \). For each name \( i \) in the portfolio define the conditional binomial probability \( p_i(x,t) \) as

\[
p_i(x,t) = \Phi \left( \frac{\Phi^{-1}(p_i(t)) - \rho_i x}{\sqrt{1 - \rho_i^2}} \right)
\]

where we have conditioned on a Gaussian random variable \( x \). Furthermore we extend the binomial variable to have outcomes \( \{0, L_i\} \) rather than \( \{0, 1\} \) where \( L_i \) denotes the loss amount\(^4\) of \( L_i \).

Now we can compute the characteristic function of the loss distribution using the conditional probabilities for each possible loss amount.

**Definition 3.8.** The characteristic of the loss distribution is given by

\[
\phi_{x,t}(u) = \mathbb{E} \left[ u^L \right] = \prod_i \left( p_i(x,t) u^{L_i} + (1 - p_i(x,t)) \right)
\]

We also define

\[
\phi^k_{x,t}(u) = \prod_{i=1}^k \left( p_i(x,t) u^{L_i} + (1 - p_i(x,t)) \right) = \phi_{x,t}^{k-1}(u) \left( p_k(x,t) u^{L_k} + (1 - p_k(x,t)) \right)
\]

Continuing this expansion (where we introduce \( \omega_{x,t,l} \) for the coefficients of the polynomial expression) gives

\[
\phi_{x,t}(u) = u^{\sum_i L_i \omega_{x,t,L_i}} + \sum_{l=1} \omega_{x,t,0}
\]

\( \Box \)

The polynomial coefficients \( \omega_{x,t,l} \) can be used to determine the actual loss distribution.

**Proposition 3.9.** The probability of a certain loss amount is given by

\[
P \{ L = l \} = \int_{-\infty}^{\infty} \phi(x) \omega_{x,t,l} dx
\]

\( \Box \)

Proof. As the polynomial coefficient \( \omega_{x,t,l} \) corresponds with the probability of loss \( l \) at time \( t \) given \( x \), we only need to integrate over all possible \( x \) to obtain the unconditional distribution. \( \Box \)

An alternative way of implementing this approach is to use a recursive algorithm as described by Andersen, Sidenius and Basu [ASB2003]. The recursive algorithm is more intuitive and does not have the burden of calculating Fourier transforms. According to Andersen, Sidenius and Basu [ASB2003] using a standard implementation of the Fourier transforms is significantly slower than performing the recursive algorithm. For our purpose, however, the end result (i.e. the loss distribution) is of course the same.

\(^4\)This setup can therefore deal with actual loss amounts, i.e. with recovery. By setting \( L_i = 1 \) we would return to our original framework, however, we need to be able to deal with recovery when studying TriBoxx and Trac-X market quotes.
3.4. Simulation. A very different way to implement the copula is to use simulation techniques. We mention two algorithms below for the Gaussian copula and (in general) Archimedean copulas. Although simulation allows for a more flexible implementation for different pay-off types for CDOs, the framework is generally speaking much slower. Especially the calculation of sensitivities can be a time consuming process if a high level of accuracy is required. Of course, various variance reduction techniques (such as described by for example Joshi and Kainth [JK2003]) can be applied to improve the performance.

 Algorithm 3.10.

- Generate the uniform random numbers per for example the Gaussian copula \( U_1, \ldots, U_N \sim C^{\text{Ga}}_{\Sigma} \).
- Determine the individual default times \( \tau_1, \ldots, \tau_N \) via the definition of the default times.

For the Archimedean copulas we use the idea of the frailty model to simulate random variates.

 Algorithm 3.11.

- Draw \( X_1, \ldots, X_N \) independent \([0, 1] - \) uniformly distributed random numbers.
- Draw random number (mixing variable) \( Y \sim G \) independent of all \( X_i \) with Laplace transform \( \phi^{-1} (\cdot) \).
- Calculate all \( U_i = \phi^{-1} \left( -\frac{1}{\lambda} \ln X_i \right) \).
- Determine the individual default times \( \tau_1, \ldots, \tau_N \) via the definition of the default times.

The simulation techniques present a large amount of flexibility, i.e. we can push the generated set of default times through various different loss and premium payment schedules. It can therefore be a useful tool in creating a generic platform for various structured portfolio trades. The remainder of this paper will not make use of these simulation techniques.

4. Impact of copula parameters. When modeling portfolio credit derivatives in this setup, not only the choice of copula function is important, the choice of parameters for that copula function is equally important. In this section we look at the influence of the copula function and in particular its parameters on some of the properties of the reference portfolio.

4.1. Impact on discrete default correlation. Recall equation (2.7) for discrete default correlation. A drawback of this type of correlation is it’s time dependency, i.e. you need the correlation over a number of years to study the joint behavior of obligors. At any time \( t \), using a copula function to determine the joint probability, this can be written as

\[
\rho^d (t) = \frac{C (p_1 (t), p_2 (t)) - p_1 (t) p_2 (t)}{\sqrt{p_1 (t) (1 - p_1 (t))} \sqrt{p_2 (t) (1 - p_2 (t))}}
\]

The discrete correlation with the bivariate Gaussian copula function and two names with the same default intensity \( \lambda = 1,000 \) bp/a is displayed in figure 1 for various (copula) correlation parameters.
Figure 1. Default time correlation using the bivariate Gaussian copula function.

For short time periods, around time zero, the discrete default correlation is almost zero. As time increases the discrete default correlation increases as well; an increase in copula correlation implies an increase in discrete default correlation.

Figure 2 displays the discrete default correlation for various parameters with the Clayton copula function.

Figure 2. Default time correlation using the Clayton copula function.

The behavior for increasing $\theta$, corresponding to increasing correlation in the Gaussian copula, is very different. The discrete default correlation in the short end actually increases, while for constant $\theta$ the discrete default correlation will decline over time.

4.2. Impact on relative jump size. The choice of copula function and parameter(s) does not only imply a discrete default correlation but also a jump in conditional default intensity given a default of another name. Similar to the previous section, consider the simplified case of two names with a default intensity of $\lambda = 1,000 \, bp/a$. 


Proposition 4.1. The conditional jump in the (multivariate) Gaussian copula framework is given by

\[
\Delta = \Phi_{\rho}^{2} \cdot \frac{1}{\sqrt{1-\rho^{2}}} e^{-\frac{1}{2(1-\rho^{2})}(\rho x^{2}-2xy+\rho y^{2})} \\
\Phi \left( \frac{y-\rho x}{\sqrt{1-\rho^{2}}} \right) \Phi \left( \frac{x-\rho y}{\sqrt{1-\rho^{2}}} \right)
\]

(4.2)

where

\[x = \Phi^{-1} (p_1), \ y = \Phi^{-1} (p_2)\]

while in the Clayton copula framework it is given by

\[\Delta = \theta\]

Proof. The proof relies on equation (2.8). For the proof of the Clayton copula we refer to Schönbucher and Schubert [SS2001], the proof for the jump in the Gaussian copula is included in the appendix. 

The next figure 3 displays the jump in the conditional intensity of the remaining name, given a default event at time \(t\) for various (copula) correlation parameters.

![Figure 3. Jump in default intensity using bivariate Gaussian copula function.](image)

The relative jump is zero for all \(t\) in case of no correlation, as the default events are independent. The higher the correlation gets, the higher the jump size. This behavior is also described by Schönbucher [S2003], figure 10.21.

The next figure 4
Figure 4. Jump in default intensity using bivariate Gaussian copula function.

The figure is fairly trivial as the jump size remains constant over time, in contrast to the Gaussian copula.

4.3. Impact on sensitivities. The impact of the copula parameters is not only on pricing of tranches but also on the sensitivities of the tranches. This section provides some insight to the behavior of the sensitivities of the expected loss of a tranche for various parameter inputs. The following proposition provides the basic setup.

**Proposition 4.2.** Let \( \vartheta \) denote the single parameter in the copula function used to calculate the loss distribution (for example, \( \vartheta \) could be \( \rho \) in the one-factor Gaussian copula). The sensitivity of the expected value of a tranche with regards to \( \vartheta \) can be determined by

\[
\frac{\partial}{\partial \vartheta} \mathbb{E} \left[ L^+, L^-, T \right] \approx \sum_{i=0}^{l} \max \left\{ \min \left\{ l_{i+1}, L^+ \right\} - L^-, 0 \right\} \cdot \left( \frac{\partial}{\partial \vartheta} \mathbb{P} \{ L \leq l_{i+1} \} - \frac{\partial}{\partial \vartheta} \mathbb{P} \{ L < l_{i} \} \right)
\]

\( \square \)

When we are able to calculate the derivatives of the loss probabilities, we can determine the sensitivity of the expected value of the tranche. The following two propositions determine these derivatives in the large homogenous portfolio approximation using the one-factor Gaussian and the Clayton copula functions.

**Proposition 4.3.** For the one-factor Gaussian copula (in the large homogenous portfolio approximation) the sensitivity of the loss probability with regards to correlation is given by

\[
\frac{\partial}{\partial \rho} \mathbb{P} \{ L \leq l \}
\]

\[
= \varphi \left( \frac{1}{\rho} \left( \sqrt{1 - \rho^2} \Phi^{-1} \left( l \right) - \Phi^{-1} \left( p \right) \right) \right) \cdot \left( -\left( \frac{1}{\sqrt{1 - \rho^2}} + \frac{\sqrt{1 - \rho^2}}{\rho^2} \right) \Phi^{-1} \left( l \right) + \frac{\Phi^{-1} \left( p \right)}{\rho^2} \right)
\]

\( \square \)

**Proof.** Per equation (3.5) the loss probability is given by

\[
\mathbb{P} \{ L \leq l \} = \Phi \left( \frac{1}{\rho} \left( \sqrt{1 - \rho^2} \Phi^{-1} \left( l \right) - \Phi^{-1} \left( p \right) \right) \right)
\]
Taking the derivative with regards to correlation

\[
\frac{\partial}{\partial \rho} P \{ L \leq l \} = \frac{\partial}{\partial y} \Phi(y) \bigg|_{y=\frac{1}{\rho} \left( \frac{\sqrt{1 - \rho^2 \Phi^{-1}(l)^{-\theta} - \Phi^{-1}(p)^{-\theta}}}{\rho} \right)} \frac{\partial}{\partial \rho} \left( \frac{\sqrt{1 - \rho^2 \Phi^{-1}(l)^{-\theta} - \Phi^{-1}(p)^{-\theta}}}{\rho} \right)
\]

which yields the result. \(\Box\)

**Proposition 4.4.** For the Clayton copula (in the large homogenous portfolio approximation) the sensitivity of the loss probability with regards to its \(\theta\)-parameter is given by

\[
(4.5) \quad \frac{\partial}{\partial \theta} P \{ L \leq l \} = \frac{\ln(l)}{\Gamma \left( \frac{1}{\theta} \right)} \ln \left( \frac{p^{\theta \ln(l)}}{p - \theta - 1} \right) \frac{\partial}{\partial \theta} \left( \frac{-\ln(l)}{p^{\theta \ln(l)}} \right)
\]

\(\Box\)

**Proof.** Per equation (3.6) the loss probability is given by

\[
P \{ L \leq l \} = 1 - \frac{1}{\Gamma \left( \frac{1}{\theta} \right)} \int_0^{-\ln(l)} e^{\frac{-y}{p^{\theta \ln(l)}}} dy
\]

Now take the derivative

\[
\frac{\partial}{\partial \theta} P \{ L \leq l \} = \frac{\partial}{\partial y} \frac{e^{\frac{-y}{p^{\theta \ln(l)}}}}{\Gamma \left( \frac{1}{\theta} \right)} \bigg|_{y=\frac{-\ln(l)}{p^{\theta \ln(l)}}} \frac{\partial}{\partial \theta} \left( \frac{-\ln(l)}{p^{\theta \ln(l)}} \right)
\]

which gives the result. \(\Box\)

One can use both equation (4.4) and (4.5) for the derivative of the loss distribution in (4.3) to determine the sensitivity of the expected loss. This sensitivity can replace the expected loss in tranche valuation, equations (3.3) and (3.4), to determine the sensitivity of a tranche. Note that the LHP approximation uses a percentage loss of the portfolio, therefore one would need to adjust for actual loss payments (including recovery) before applying it to any real world trade.

Consider the one-factor Gaussian copula case, with an average default probability of 4%. The next figure 5 displays the sensitivity of the expected loss of various tranches for different correlation parameters. The tranche levels are set to the same tranche levels as US TriBoxx.

![Figure 5. Expected loss sensitivity to change in correlation.](image)
The sensitivity of the equity piece is negative and keeps decreasing further as correlation increases. This can be compared to the fair first-to-default premium or equity premium that declines for increasing correlation. The super-senior piece becomes very sensitive for high correlation figures, as a high correlation would make the clustering of defaults more likely, thereby increasing the riskiness of this tranche. The mezzanine tranches are less sensitive to correlation.

5. Copula parameters in practice. When attempting to back correlation from the market we can at best only hope to achieve a homogeneous figure for the portfolio as a whole, thus there is little point using the sophisticated simulation framework as this, whilst having the ability to model the rich dependency structure of pairwise correlations, is overkill. The one factor framework presented previously provides a tractable fast method to price portfolios, having sufficient degrees of freedom to permit market prices to be replicated. The correlation can be solved for using the pricing algorithm and any one of the usual solver algorithms. Once we have the ability to solve for market implied correlation we may then begin to solve for a range of tranches and in this way construct a curve of implied correlation with respect to subordination. Subsequently we examine the skew of correlation for short dated products.

5.1. Correlation Smile. We attempt now to draw parallels between the B&S framework and the CDO Tranche valuation. In the case of the CDO we consider the tranche buyer to be equivalent to a buyer of an option on default whose valuation is determined as a function of both the underlying CDS spread level and the correlation. In the example we know have CDS spreads levels and tranche prices directly quoted which permits us to solve for the remaining unknown of correlation.

As an example, consider the Euro TriBoxx quotes as given in the introduction (1.2). We can use a suitable optimization algorithm to fit the correlation parameter to these market quotes. The results are shown in figure 6 below.

![Figure 6. Implied correlation in Euro TriBoxx (bid/ask) on Nov 13, 2003.](image)

Like the volatility smile is a function of the strike, the correlation smile is a function of the subordination levels. The response the equity tranche with regards to correlation is opposite to that of the other tranches, as the sensitivity with regard to correlation is of the opposite sign to higher tranches as discussed previously. Figure 7 below plots the upper and lower correlation as function of subordination.

---

5 Mezzanine tranches typically have two solutions rather than a unique solution, however, the correlation level implied for the equity and super senior can be used when selecting which level is the more desirable.
The result is similar to a confidence level, as we get an idea of the bounds between which we think the actual correlation should be.

When giving consideration to the implied correlation smile we may draw parallels with the implied volatility smile inherent within the equity options domain whereby the trader implies a smile onto the volatility to take account of the inability of the framework to incorporate the stochastic nature of volatility. We may apply the same principal when considering the smile of correlation, when we use the copula framework and make up for the shortfall of non stochastic time variant correlation through use of the correlation smile. Furthermore we can draw additional parallels with the subordination of the tranche being analogous to the strike level within B&S. Alternatively we may say that one correlation parameter is not sufficient to contain all the market information, i.e. a full correlation matrix, as used in the usual Gaussian copula, might be more capable of explaining the smile. A disadvantage of a full matrix is that it would contain to many parameters to fit.

5.2. Correlation Skew. The term structure of a portfolio’s default intensity is determined by two factors, the choice of the marginal default intensities and the choice of copula and its respective parameters. It can be seen, for example, that a first to default constructed from flat marginal intensities will still have a term structure implied from the use of the Gaussian copula. The Gaussian copula is singular around $0$ and that is causing our problems here. The effect is studied in more detail by for example Razak [R2003]. This structural property causes some undesirable effects in pricing portfolio credit derivatives, in particular, risk on the short end will be over estimated. The result can be seen in the comparison of a running first-to-default swap with a short lifetime remaining which would be valued as very risky compared to a new trade on the same basket. Another result is that a forward starting portfolio trade will have a high chance of being knocked out (by a default event in the reference portfolio) before settlement, resulting in an underestimation of the actual risk.

A possible practical solution to this problem is to skew the correlation input. Skewing correlation requires a minor modification in the definition of the (conditional) default probabilities, as the correlation needs to be time dependent.

**Definition 5.1.** Consider the conditional binomial default probabilities as in equation (3.7). However, the correlation $\rho(t)$ now also depends on time $t$

$$(5.1)\quad p_i(x,t) = \Phi \left( \frac{\Phi^{-1}(p_i(t)) - \rho_i(t)x}{\sqrt{1 - \rho^2(t)}} \right)$$
Using the more formal definition of the default intensity, the next definition provides the tools to determine the first-to-default intensity.

**Definition 5.2.** The intensity corresponding to the minimum of the default times of the names in the portfolio \( \min \{ \tau_1, \ldots, \tau_N \} \) is given by

\[
h(t) = -\frac{\partial}{\partial T} \log C(\bar{u}) \bigg|_{T=t}
\]

in particular, for a one-factor Gaussian copula function with the same correlation \( \rho(t) \) across all names, this intensity is given by

\[
h(t) = -\frac{\partial}{\partial T} \log \int_{-\infty}^{\infty} \varphi(x) \prod_{i=1}^{N} \Phi \left( \frac{\Phi^{-1}(1 - p_i(T)) - \rho(T)x}{\sqrt{1 - \rho^2(T)}} \right) dx \bigg|_{T=t}
\]

As an example, consider a 5 name portfolio with all names trading with a default intensity of 1,000bp/a. The following figure 8 provides the default intensity of the minimum of default time over the first 10 years for various values of correlation.

![Figure 8. Default intensity of minimum of default times for one-factor Gaussian copula.](image)

In case the names are independent (i.e. zero correlation), the intensity of the minimum should be equal to the sum of all the intensities (i.e. 5,000bp/a). As displayed in the figure, for short maturities the intensity of the first default time behaves as if correlation is actually zero. The intensity \( h(t) \) is not flat for varying time \( t \) although the individual intensities of the names in the portfolio are. The copula function will create a term structure on the intensity, which is especially present at the short end of the curve.

If we desire to keep the intensity \( h(t) \) constant over time, we need to change the correlation parameter \( \rho(t) \) over time. For shorter maturities, we need to increase the correlation to prevent \( h(t) \) from going up. In other words, using a term structure of correlation \( \rho(t) \) prevents a term structure for \( h(t) \) and ensures that the model has more realistic properties. Figure 9 below is based upon the same 5 name portfolio, where the intensity \( h(t) = 3,000bp/a \) for all \( t \). In the first year, the correlation declines from roughly 85% to 48%.
Figure 9. Correlation for constant default intensity for minimum of default times.

This correlation skew could be used to price (forward starting) tranches.

6. Concluding remarks. Over 2003 the synthetic CDO market has seen a rapid increase in liquidity and marked improvement in transparency of pricing. There are currently a number of liquid correlation products being publicly quoted in the market such as Tracers and TriBoxx. These quoted products present considerably more opportunities to calibrate the models to market. As the number of products quoted increases correlation is increasingly becoming a market observable. In this paper we thus try to fit the correlation parameters to this new market. The increased transparency in pricing represents an evolution of the market and is desirable, however as the market becomes more efficient the bid/offer spread reduces which in turn means there is less margin to be made in trades. This places further importance on correct calibration and valuation of tranche trades.

In this paper we have outlined various techniques for CDO valuation. In particular, we have outlined what we believe has become the standard over the last year and analyzed the implications of the selection of model parameters. The selection of model parameters influences not only the actual valuation but also the discrete default correlation and the spread widening in case of default events in the reference portfolio. We have investigated the notion of perceiving the correlation within a copula model as an equivalent measure to volatility in the standard B&S option-framework. Consequently, the smile as well as the skew for the correlation input have been examined.

The approaches presented here are by no means the only possible techniques. For example, the skew in correlation, which we put on the short end to overcome the mispricing of for example short dated first-to-default swaps, is in practice no more than a fix to overcome the singularity of the Gaussian copula. Not every copula function suffers from singularity and one might find it more beneficial to work with Archimedean copulas rather than tweaking the parameters of the Gaussian copula. Similar, the smile found in the correlation as function of subordination is a result of the fact that we only try to calibrate a single flat correlation parameter to the market prices. Although not as straightforward, a full correlation matrix in a simulation framework could have more explanatory value and can be easily adjusted to new structured portfolio trades. This could be, however, at the cost of speed.
REFERENCES


Appendix A. Derivation of jump in Gaussian copula framework.

This appendix contains the proof for the expression of the jump in the bivariate Gaussian copula framework.

**Proposition A.1.** The conditional jump in the Gaussian copula framework is given by

\[ \Delta = \Phi^2_{\rho} \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{x^2}{2(1-\rho^2)}} \]

where

\[ x = \Phi^{-1}(p_1), \quad y = \Phi^{-1}(p_2) \]

\[ \square \]

**Proof.** Per equation (2.8) we have the jump size

\[ \Delta = \frac{\partial^2}{\partial u_1 \partial u_2} C(\tilde{u}) \cdot C(\tilde{u}) - 1 \]

Using the bivariate Gaussian copula results in

\[ \Delta = \frac{\partial^2}{\partial u_1} \Phi^2_{\rho} (\Phi^{-1}(u_1), \Phi^{-1}(u_2)) \cdot \Phi^2_{\rho} (\Phi^{-1}(u_1), \Phi^{-1}(u_2)) - 1 \]

\[ = \frac{\partial^2}{\partial x \partial y} \Phi^2_{\rho} (x,y) \cdot \frac{\partial^2}{\partial u_1} \Phi^{-1}(u_1) \cdot \frac{\partial^2}{\partial u_2} \Phi^{-1}(u_2) - 1 \]

The separate derivatives are given by

\[ \frac{\partial}{\partial x} \Phi^2_{\rho} (x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Phi \left( \frac{y - \rho x}{\sqrt{1-\rho^2}} \right) = \phi(x) \Phi \left( \frac{y - \rho x}{\sqrt{1-\rho^2}} \right) \]

and similar

\[ \frac{\partial}{\partial y} \Phi^2_{\rho} (x,y) = \phi(y) \Phi \left( \frac{x - \rho y}{\sqrt{1-\rho^2}} \right) \]

for both \( x \) and \( y \) we get

\[ \frac{\partial^2}{\partial x \partial y} \Phi^2_{\rho} (x,y) = \frac{1}{2\pi \sqrt{1-\rho^2}} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}} \]

\[ = \phi(x) \phi(y) \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}} \]

\[ \square \]
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