DEVELOPMENT, IMPLEMENTATION AND APPLICATION OF PARTIALLY SATURATED SOIL MODELS IN FINITE ELEMENT ANALYSIS

by

KONSTANTINOS GEORGIADIS

Dipl.(Eng.), MSc, DIC

FEBRUARY 2003

A thesis submitted to the University of London
(Imperial College of Science, Technology and Medicine)
in partial fulfilment of the requirements for the degree of
Doctor of Philosophy in the Faculty of Engineering
Abstract

The mechanical behaviour of partially saturated soils can be very different to that of fully saturated soils. It has long been established that for such soils, changes in suction do not have the same effect as changes in the applied stresses, and consequently the effective stress principle is not applicable. Conventional constitutive models, which are based on this principle, are therefore of limited use when analysing geotechnical problems that involve the presence of partially saturated soil zones. Although the existing constitutive models for partially saturated soils can reproduce important features of the behaviour of such soils, such as collapse upon wetting, they are less advanced than the conventional fully saturated soil models, and therefore there is much room for improvement. In addition, only a limited number of applications of such models to boundary value problems has been performed in the past. The performance of partially saturated soil models and more importantly the influence of partial soil saturation on the behaviour of geotechnical structures has therefore not been well established.

This thesis presents two new generalised constitutive models for partially and fully saturated soils and their implementation into the Imperial College Finite Element Program (ICFEP). The implementation and performance of the models is validated through a series of single element analyses and the results are compared to analytical solutions and experimental data. Also presented in this thesis is a series of finite element analyses of shallow and deep foundations. These analyses highlight facets of partially saturated behaviour important for engineering problems and more specifically show the effect of partial soil saturation and of fluctuations of the ground water table on the behaviour of footings and piles.
Acknowledgements

First, I would like to thank my supervisors Prof. D. M. Potts and Dr L. Zdravkovic for their guidance, encouragement and assistance throughout the period of this work.

The work described in this thesis was funded by the Greek State Scholarships Foundation. Their support is gratefully acknowledged.

I would also like to express my gratitude to the College staff for their help and support and especially to Dr M. Coop for his assistance and discussions on issues presented in this thesis.

Many thanks are due to all my colleagues in the section for their contribution in numerous ways and particularly to my friends Angeliki, Felix, Niki, Reinaldo and Stuart.

Finally I would like to thank my parents for their support, encouragement and understanding without which I would not have managed to complete this thesis.
Table of contents

Abstract
Acknowledgements
Table of contents
List of figures
List of tables

Chapter 1 Introduction

1.1 General
1.2 Aims of research
1.3 Outline of thesis

Chapter 2 Mechanical Behaviour and Constitutive Models for Partially Saturated Soils

2.1 Introduction
2.2 Mechanical behaviour
   2.2.1 Stress state variables and the effective stress principle
   2.2.2 Volume change behaviour
      2.2.2.1 Volume change due to changes in suction
      2.2.2.2 Volume change due to changes in confining stress
      2.2.2.3 Volume change due to changes in both confining stress and suction
   2.2.3 Shear strength
2.3 Constitutive models
   2.3.1 Introduction
   2.3.2 Critical state models
   2.3.3 Barcelona Basic Model
      2.3.3.1 General
2.3.3.2 Formulation of model for isotropic stress states
2.3.3.3 Formulation of model for triaxial stress states
2.3.4 Josa et al. (1992) model
  2.3.4.1 General
  2.3.4.2 Modifications to the Barcelona Basic model
2.3.5 Wheeler & Sivakumar Model
  2.3.5.1 General
  2.3.5.2 Formulation of model for isotropic stress states
  2.3.5.3 Formulation of model for triaxial stress states
2.3.6 Bolzon et al. (1996) model
  2.3.6.1 General
  2.3.6.2 Use of a single stress variable
  2.3.6.3 Formulation of partially saturated soil model
2.3.7 Expansive soil models
2.4 Summary and discussion
  2.4.1 Mechanical Behaviour
  2.4.2 Constitutive models

Chapter 3 Finite Element Analysis

3.1 Introduction
3.2 Formulation of the finite element method
  3.2.1 Element discretisation
  3.2.2 Displacement approximation
  3.2.3 Element Equations
  3.2.4 Global Equations
  3.2.5 Boundary Conditions
  3.2.6 Solution of global equations
3.3 Non-linear finite element analysis
  3.3.1 Modified Newton-Raphson method
  3.3.2 Stress point algorithm
  3.3.3 Correction for yield surface drift
3.4 Partially saturated finite element analysis
  3.4.1 Constitutive behaviour
Chapter 4 Development of two Constitutive Models

4.1 Introduction 92

4.2 Formulation of the constitutive models 92

4.2.1 Stress invariants 92

4.2.2 Yield Function and Plastic Potential Surface 94

4.2.3 Isotropic compression line and yield stress 100

4.2.3.1 Constitutive Model 1 - Partially saturated isotropic compression line and yield stress 101

4.2.3.2 Constitutive Model 2 - Partially saturated isotropic compression line and yield stress 105

4.2.4 Critical State 111

4.2.5 Hardening/softening rules 114

4.2.6 Elastic behaviour 115

4.3 Overview of model parameters 118

4.3.1 Yield surface and plastic potential parameters 118

4.3.2 Hardening/Softening parameters 121

4.3.3 Initial hardening/softening parameters 122

4.3.4 Other parameters 125

4.3.5 Summary 126

4.4 Summary and discussion 129

Chapter 5 Implementation and Validation

5.1 Introduction 131

5.2 Implementation 131

5.2.1 Calculation of global stiffness matrix 132

5.2.2 Calculation of residual load vector \( \{\psi\} \) 133

5.3 Validation 139

5.3.1 Single finite element analyses of isotropic stress paths 139

5.3.1.1 Primary yield surface 140
Chapter 7 Results

7.2.2 Results

7.2.2.1 Pile loading
7.2.2.2 Rise of groundwater table

7.3 Analysis of canary wharf piles

7.3.1 Ground profile and pore pressure conditions
7.3.2 Pile analysis – Load test and rise of groundwater table

7.3.2.1 Material properties
7.3.2.2 Details of analyses
7.3.2.3 Results

7.3.3 Pile analysis – Influence of model parameters

7.3.3.1 Influence of OCR
7.3.3.2 Influence of parameters r and β

7.3.4 Pile analysis – Comparison of the two constitutive models

7.3.4.1 Model parameters
7.3.4.2 Results

7.4 Conclusions

Chapter 8 Conclusions and Recommendations

8.1 Introduction

8.2 Two generalised constitutive models for fully and partially saturated soils

8.3 Influence of partial soil saturation on the behaviour of shallow and deep foundations

8.4 Recommendations for further research

References

Appendix I Flow chart for the calculation of the stress and plastic strain increments and the changes in hardening parameters

Appendix II Calculation of the angle of shearing resistance for plane strain conditions
List of figures

Figure 2-1: Oedometer curves for air-dry silt soaked at various constant applied pressures (after Jennings & Burland (1962)) 22

Figure 2-2: Void ratio state surface - Volume changes under all-round pressure plotted in a void ratio–stress space (after Bishop & Blight (1963)) 24

Figure 2-3: Results of tests on saturated and partially saturated Selset clay (after Bishop & Blight (1963)) 25

Figure 2-4: Drying path from zero suction 26

Figure 2-5: Volume changes due to drying (after Toll (1995)) 27

Figure 2-6: The relationship between volumetric strains and the increase in soil suction (after Chen et al. (1999)) 28

Figure 2-7: Swelling followed by collapse during wetting under constant load for three different initial values of tensile pore water pressure (after Escario & Saez (1973)) 29

Figure 2-8: a) Stress paths, and b) volumetric response measured in paths FD and IG (after Josa et al. (1987)) 30

Figure 2-9: Relationship between suction and swelling (after Chu & Mou (1973)) 30

Figure 2-10: Relationship between collapse and log. pressure for different moisture contents (after Booth (1975)) 31

Figure 2-11: Relationship between collapse and normal stress for different soil types (after Yudhibir (1982)) 31

Figure 2-12: Typical water retention curve (after Fredlund (1998)) 33

Figure 2-13: a) Stress paths, and b) yield points for compacted kaolin (after Wheeler & Sivakumar (1995)) 34

Figure 2-14: Influence of suction on parameters $\kappa$ and $\lambda$ (after Rampino et al. (2000)) 35

Figure 2-15: Variation of specific volume during isotropic compression at constant suction (after Alonso et al. (1990)) 36

Figure 2-16: Variation of specific volume during isotropic compression (after Wheeler & Sivakumar (1995)) 36
Figure 2-17: Partially and fully saturated isotropic compression lines

Figure 2-18: State surface for void ratio (after Matyas & Radharkrishna (1968))

Figure 2-19: State surface for degree of saturation (after Matyas & Radharkrishna (1968))

Figure 2-20: Representation of failure surface from Eq.2.13

Figure 2-21: Direct shear tests under controlled suction for a) Madrid grey clay, and b) Guadalix de la Sierra red clay (after Escario & Saez (1986))

Figure 2-22: Changes in shear strength with soil suction (data from Nishimura & Toyota (2002)

Figure 2-23: Variation of critical state stress ratios with degree of saturation (after Toll (1990))

Figure 2-24: Intercept of critical state surface at zero mean net stress plane (after Wheeler (1990)

Figure 2-25: Relationship between the effective stress parameter $\chi$ and the suction ratio (after Khalili & Khabbaz (1998))

Figure 2-26: Projection of yield surface onto $p – s$ plane

Figure 2-27: Projection of yield surface onto $J - p \prime$ plane for Cam Clay

Figure 2-28: Projection of yield surface onto $J - p \prime$ plane for modified Cam Clay

Figure 2-29: State boundary surface for modified Cam Clay

Figure 2-30: a) assumed isotropic compression lines, and b) LC yield surface (after Alonso et al. (1990))

Figure 2-31: Definition of the Suction Increase (SI) yield surface (after Alonso et al. (1990))

Figure 2-32: Yield surfaces of the Barcelona Basic model (after Alonso et al. (1990))

Figure 2-33: Loading-Collapse yield surfaces in the $(p – s)$ space for different values of $p_o$ (after Josa et al. (1992))

Figure 2-34: Shape of $m$ for generic values of $\zeta_c$ and $\zeta_y$ (after Josa et al. (1992))

Figure 2-35: Loading-Collapse yield surface inferred from equalisation stage results (after Wheeler & Sivakumar (1995))
Figure 2-36: a) Yield surface, and b) normal compression and critical state lines (after Wheeler & Sivakumar (1995))

Figure 2-37: Yield surfaces in a) $p$, $s$, and b) $q$, $p$ space (after Alonso et al. (1994))

Figure 3-1: 8-noded isoparametric element (after Potts & Zdravkovic (1999))

Figure 3-2: Location of Gauss points (after Potts & Zdravkovic (1999))

Figure 3-3: Application of the modified Newton-Raphson algorithm to the uniaxial loading of a bar of a nonlinear material (after Potts & Zdravkovic (1999))

Figure 3-4: Yield surface drift (after Potts & Zdravkovic (1999))

Figure 3-5: Geometrical significance of curve parameters

Figure 4-1: Examples of yield surface and plastic potential functions reproduced from Lagioia et al. (1996)

Figure 4-2: Yield function and plastic potential surface for partially saturated conditions

Figure 4-3: Examples of yield surface and plastic potential functions reproduced by Equation 4.16

Figure 4-4: Primary yield surface in isotropic stress space

Figure 4-5: Assumed isotropic compression lines – Model 1 (option 1)

Figure 4-6: Assumed isotropic compression lines – Model 1 (option 2)

Figure 4-7: Variation of potential plastic reduction of specific volume due to wetting with isotropic yield stress

Figure 4-8: Fully and partially saturated isotropic compression lines – Model 2

Figure 4-9: Fully and partially saturated isotropic compression lines – relationship between $p_o$ and $p_o'$ for Model 2

Figure 4-10: Critical State point along a loading/unloading line for a given value of $s_{eq}$ and possible shapes of the Yield and Plastic Potential Surfaces

Figure 4-11: Critical State and isotropic compression lines

Figure 4-12: Critical State lines for partially and fully saturated conditions

Figure 4-13: Coupling between primary and secondary yield surfaces
Figure 4-15: Relationship between bulk modulus and net mean stress 116

Figure 4-16: Cross-section of elastic state surface at constant equivalent suction, for a given value of the equivalent fully saturated isotropic compression line 117

Figure 4-17: Calculation of $v_1$ from the initial specific volume, $v$, for Model 2 125

Figure 5-1: Validation stress paths – isotropic compression lines 141

Figure 5-2: Validation analyses – isotropic compression lines 142

Figure 5-3: Validation stress paths – wetting induced collapse 144

Figure 5-4: Validation analyses – wetting induced collapse 145

Figure 5-5: Validation analyses – wetting induced collapse at various constant values of mean net stress 147

Figure 5-6: Validation stress paths – stress path dependency 149

Figure 5-7: Validation analyses – stress path dependency 150

Figure 5-8: Validation stress paths – secondary yield surface 151

Figure 5-9: Validation analyses – secondary yield surface 151

Figure 5-10: Validation stress paths – coupling between the two yield Surfaces 152

Figure 5-11: Validation analyses – coupling between the two yield Surfaces 153

Figure 5-12: Validation stress paths – transition between partially and fully saturated states 154

Figure 5-13: Validation analyses – transition between partially and fully saturated states 155

Figure 5-14: Initial stress states and position of the Primary Yield surface 157

Figure 5-15: Initial and final positions of the Primary Yield surface 158

Figure 5-16: Stress-strain curves 159

Figure 5-17: Relationship between the volumetric and deviatoric strains 159

Figure 5-18: Water retention curve 161

Figure 5-19: Relationship between apparent cohesion and suction 161

Figure 5-20: Initial and final positions of the Primary Yield surface 162
Figure 5-21: Initial Primary Yield and Plastic Potential surfaces

Figure 5-22: Stress-strain curves for associated and non-associated flow Rule

Figure 5-23: Relationship between the volumetric and deviatoric strains, for associated and non-associated flow rule

Figure 5-24: Stress paths for set 1

Figure 5-25: Change of specific volume along stress paths A-C-I and A-K-L-I

Figure 5-26: Change of specific volume along stress path A-C-I – modified parameters

Figure 5-27: Deviatoric stress – deviatoric strain relationship for shearing at constant mean net stress from point I

Figure 5-28: Stress paths for set 2

Figure 5-29: Change of specific volume along stress paths A-B-C-D-A and A-D-C-B-A

Figure 6-1: Assumed pore pressure profiles

Figure 6-2: Finite element mesh

Figure 6-3: Schematic failure mechanism

Figure 6-4: Load-settlement curves for partially saturated drained loading analyses

Figure 6-5: Load-settlement curves for conventional drained loading analyses

Figure 6-6: Variation of ultimate load with the normalised depth of the G.W.T.

Figure 6-7: Load-settlement curves for partially saturated wetting analyses

Figure 6-8: Vectors of incremental displacements at failure for analysis PW3

Figure 6-9: Progression of vertical movement with rise of groundwater table

Figure 6-10: Load-settlement curves for wetting and subsequent loading analyses

Figure 6-11: Vectors of incremental displacements: a) end of wetting, and b) beginning of subsequent loading
Figure 6-12: Isotropic compression lines for different values of the parameter $b$  

Figure 6-13: Load-settlement curves – influence of the parameter $b$  

Figure 6-14: Progression of vertical movement with rise of groundwater table – influence of the parameter $b$  

Figure 7-1: Single pile analysis: pore pressure profiles  

Figure 7-2: Single pile analyses: Finite element mesh  

Figure 7-3: Predicted YSR profile for $D = 25$m from model 1 - option 1  

Figure 7-4: Predicted YSR profile for $D = 25$m from model 1 - option 2  

Figure 7-5: Load-displacement curves for drained loading analyses  

Figure 7-6: Shaft and base resistances for the fully saturated and conventional ($D = 10$m) analyses  

Figure 7-7: Influence of depth of groundwater table on ultimate pile load  

Figure 7-8: Load-displacement curves for $D = 25$m  

Figure 7-9: Progression of vertical movement of pile during rise of groundwater table ($D = 10$m, $L = 4$MN)  

Figure 7-10: Progression of vertical movement of pile during rise of groundwater table for $D = 25$m.  

Figure 7-11: Vectors of incremental displacement at failure  

Figure 7-12: Canary Wharf pile analyses – Ground profile  

Figure 7-13: Canary Wharf pile analyses – Pore pressure profiles  

Figure 7-14: Yield and plastic potential surfaces  

Figure 7-15: Soil-water characteristic curves for Thanet sands and Lambeth sands  

Figure 7-16: Canary Wharf piles: finite element mesh  

Figure 7-17: Load-settlement curves  

Figure 7-18: Progression of vertical movement with rise of groundwater table  

Figure 7-19: Load-settlement curves for different $OCR$ values  

Figure 7-20: Progression of vertical movement for different $OCR$ values
Figure 7-21: Load-settlement curves for different partially saturated parameters 217
Figure 7-22: Loading-collapse surface for different partially saturated parameters 218
Figure 7-23: Progression of vertical movement for different p. sat. parameters 218
Figure 7-24: Isotropic compression lines for different values of the parameter $b$ 220
Figure 7-25: Load-settlement curves for different values of $b$ 221
Figure 7-26: Progression of vertical displacements: comparison of two models 222
Figure 7-27: Progression of vertical displacements for different values of $b$ 223
## List of tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 2.1</td>
<td>Stress variables used by some of the existing constitutive models</td>
<td>24</td>
</tr>
<tr>
<td>Table 4.1</td>
<td>Parameters for comparison between models from literature and the Lagioia et al. (1996) expression for fully saturated soils (after Lagioia et al. (1996))</td>
<td>120</td>
</tr>
<tr>
<td>Table 4.2</td>
<td>Summary of model input parameters</td>
<td>128</td>
</tr>
<tr>
<td>Table 5.1</td>
<td>Material properties</td>
<td>141</td>
</tr>
<tr>
<td>Table 5.2</td>
<td>Material properties for set 1 (after Alonso et al. (1990))</td>
<td>165</td>
</tr>
<tr>
<td>Table 5.3</td>
<td>Material properties for set 2 (after Alonso et al. (1990))</td>
<td>169</td>
</tr>
<tr>
<td>Table 6.1</td>
<td>Soil parameters for constitutive model 1 – option 1</td>
<td>175</td>
</tr>
<tr>
<td>Table 7.1</td>
<td>Soil properties – single pile in uniform soil analyses</td>
<td>195</td>
</tr>
<tr>
<td>Table 7.2</td>
<td>Soil Properties for Lambeth sand</td>
<td>208</td>
</tr>
<tr>
<td>Table 7.3</td>
<td>Soil Properties for Thanet sand</td>
<td>208</td>
</tr>
<tr>
<td>Table 7.4</td>
<td>Soil Properties for Terrace Gravel, Lambeth Clay and Chalk</td>
<td>211</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

1.1 GENERAL

Conventional soil mechanics theory treats soil as either fully saturated (pores filled with water) or dry (pores filled with air). However, a large number of geotechnical problems involve the presence of partially saturated soil zones where the voids between the soil particles are filled with a mixture of air and water. These zones are usually ignored in practice and the soil is assumed to be either fully saturated or completely dry. It has long been established however that the behaviour of partially saturated soils can be very different to that of fully saturated or completely dry soils.

Experimental and theoretical difficulties (e.g. direct measurement of suction, large number of influencing factors) delayed considerably the development of an understanding of the behaviour of partially saturated soils. It is only during the last few years that theoretical frameworks and constitutive models have been proposed to describe the mechanical behaviour of such soils.

Although the existing constitutive models are capable of reproducing important features of the behaviour of partially saturated soils, most models are basic, compared to those available for fully saturated soils, and often soil type specific. There is therefore need for improvement and an increasing number of researchers around the world are working on improving the understanding and constitutive modelling of the mechanical behaviour of partially saturated soils.

Constitutive models describe the mechanical behaviour of soils, but in most cases, especially for advanced models such as those for partially saturated soils, are only useful in practice when used in numerical analysis. Numerical analysis plays an important role in the investigation of the behaviour of partially saturated soils by highlighting aspects which are important in engineering practice and
illustrating the effect of partial soil saturation on the behaviour of geotechnical structures.

1.2 AIMS OF RESEARCH

The first aim of the research was to identify and improve, where possible, those features of the existing constitutive models for partially saturated soils that are most important in the analysis of geotechnical problems. Then, using these improvements, develop generalised constitutive models that cover both partially and fully saturated soils.

The second aim of the research was to implement the developed constitutive models into a finite element program which is capable of analysing any geotechnical problem involving partially saturated, fully saturated and completely dry soil. This program was then used to perform analyses of shallow and deep foundations. The primary purpose of these analyses was to investigate the influence of the presence of partially saturated soil zones on the behaviour of foundations. In addition, those facets of soil behaviour that are important in geotechnical engineering would also be highlighted, leading to further improvement of the constitutive models.

1.3 OUTLINE OF THESIS

Two constitutive soil models were developed during the research and are presented in this thesis. These models were implemented into the Imperial College Finite Element Program (ICFEP) and used to demonstrate the effect of partial saturation on the behaviour of geotechnical structures. ICFEP with the new constitutive models was used to perform finite element analyses of footings and piles in soil including partially saturated zones. The outline of the thesis is given below in more detail.

Chapter 2 of the thesis is subdivided into two main parts. The first part presents a review of the basic features of the mechanical behaviour of partially saturated
soils. Particular emphasis is given to the applicability of the effective stress principle and the appropriate stress state variables, the volume change behaviour and the shear strength of partially saturated soils. The second part reviews the constitutive modelling of the mechanical behaviour of such soils, and presents some of the current constitutive models.

Chapter 3 gives a brief description of the fundamental aspects of the finite element method and the modifications made during this study to account for partially saturated soil conditions.

Chapter 4 presents the derivation and formulation of the two constitutive models developed during this research. Also presented is an overview of the model parameters.

Chapter 5 contains the finite element implementation and the validation of the constitutive models. The validation was performed through single element analyses and comparison of the results with analytical calculations and experimental data. These analyses also highlight the model features.

Chapter 6 presents a series of finite element analyses of a strip footing. The influence of partial soil saturation and of fluctuations of the groundwater table on the behaviour of footings is investigated. Comparison is made between the predictions of partially saturated and conventional (zero pore water pressures in partially saturated soil zone) analysis, between the finite element predictions and the analytically calculated bearing capacity, and also between the predictions of the two constitutive models developed and presented in the thesis.

Chapter 7 investigates the influence of partial soil saturation on the behaviour of piles. Two finite element studies are presented in this chapter. The first involves a single pile in a uniform soil subjected to vertical loading and fluctuations of the groundwater table. The second study is based on a case history concerning the foundation piles of a high rise building in Canary Wharf, London. Comparison is made between the predictions of partially saturated and conventional analysis for both studies, and between the predictions of the two constitutive models for the second study.
Finally, the conclusions from this thesis and the recommendations for further research are given in Chapter 8.
Chapter 2

Mechanical Behaviour and Constitutive Models for Partially Saturated Soils

2.1 INTRODUCTION

This chapter is subdivided into two main parts. The first part reviews the basic features of the mechanical behaviour of partially saturated soils. In particular the volumetric and shear strength behaviour are discussed. In the second part the constitutive modelling of the mechanical behaviour of such soils is reviewed. Some of the existing models are presented and their important features and differences are highlighted.

2.2 MECHANICAL BEHAVIOUR

2.2.1 Stress state variables and the effective stress principle

Early attempts to describe the behaviour of partially saturated soils made the assumption that the effective stress principle is applicable to such soils and that the mechanical behaviour can be fully described in the conventional (q, p') stress space. Generalised effective stress expressions were proposed in order to include partially saturated soils into the conventional soil mechanics framework, the best known being that proposed by Bishop (1959):

\[ \sigma' = \sigma - u_a + \chi(u_a - u_w) \]  

(2.1)

where \( \chi \) is a function of the degree of saturation, \( u_a \) is the pore air pressure, and \( u_w \) is the pore water pressure. This approach proved capable of reproducing some features of the behaviour of partially saturated soils, such as the shear strength increase due to suction, but could not explain others, such as wetting induced
collapse. This was first demonstrated by Jennings & Burland (1962), who performed a series of oedometer, isotropic compression and wetting tests on partially saturated soils ranging from silty sands to silty clays. Figure 2-1 shows the oedometer curves for five air-dry silt samples soaked at different constant applied stresses (0, 192, 384, 768 and 1536kPa) and that of a sample consolidated from slurry. As seen in this figure, with the exception of the initially soaked sample (at 0kPa), all the other air-dry samples collapsed upon wetting. This collapse behaviour was in contrast to the effective stress principle, according to which only swelling would be expected. Subsequent authors (e.g. Burland (1965), Matyas & Radharkrishna (1968)) added to the criticism of the effective stress approach for partially saturated soils.

Figure 2-1: Oedometer curves for air-dry silt soaked at various constant applied pressures (after Jennings & Burland (1962))
The collapse behaviour due to an increase of the pore water pressure is not unique to partially saturated soils. Fully saturated soils may also collapse, but this only occurs when the pore water pressure becomes equal to the total stress and therefore the effective stress becomes equal to zero. Collapse of partially saturated soils can take place even at high values of the generalised effective stress. For any value of the net stress ($\sigma-u_w$) other than zero, an effective stress approach would predict swelling of a partially saturated soil sample subjected to wetting (increase of pore water pressure).

The inability to explain the collapse behaviour of partially saturated soils is not the only problem of the single effective stress approach for such soils. More importantly, as noted by Jennings & Burland (1962) and many subsequent authors (e.g. Gens (1996), Wheeler & Karube (1996)), changes in suction have a very different effect on the soil structure than changes in applied stress.

It is now generally accepted that two independent stress variables are necessary in order to explain the behaviour of partially saturated soils. Bishop & Blight (1963) first used net stress ($\sigma-u_a$) and suction ($s = u_a-u_w$) to investigate the volumetric and strength behaviour of partially saturated soils and produced graphical representations of state surfaces and failure envelopes for such soils, as shown in Figures 2-2 and 2-3. Fredlund & Morgenstern (1977), suggested that any pair of stress state variables among the following ($\sigma-u_a$), ($\sigma-u_w$) and ($s = u_a-u_w$) should be adopted when describing partially saturated soil behaviour. The most commonly used pair is net total stress ($\sigma-u_a$) and suction ($u_a-u_w$). Recently other stress state variables have also been used successfully. Some examples of proposed stress variables are shown in Table 2.1.

Houlsby (1997) presented an analysis of the work input into an unsaturated granular material and suggested as appropriate stress variables Bishop’s effective stress (Eq.2.1) with $\chi = S_r$, and a modified suction $ns$, where $S_r$ is the degree of saturation and $n$ is the porosity, although other combinations are also acceptable.
<table>
<thead>
<tr>
<th>Authors</th>
<th>Stress variable 1</th>
<th>Stress variable 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alonso et al. (1990)</td>
<td>( \sigma - u_a )</td>
<td>( u_a - u_w )</td>
</tr>
<tr>
<td>Cui et al. (1995)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bolzon et al. (1996)</td>
<td>( \sigma - u_a + \chi(u_a - u_w) ), ( \chi = S_r )</td>
<td>( u_a - u_w )</td>
</tr>
<tr>
<td>Modaressi &amp; Abou-Bekr (1994a &amp; 1994b)</td>
<td>( \sigma - \pi_c )</td>
<td>( \pi_c )</td>
</tr>
<tr>
<td></td>
<td>( \pi_c = \text{capillary pressure} )</td>
<td></td>
</tr>
<tr>
<td>Kohgo et al. (1993a &amp; 1993b)</td>
<td>( \sigma - u_{eq} )</td>
<td>( u_a - u_w - s_e )</td>
</tr>
<tr>
<td></td>
<td>( u_{eq} = \text{equivalent pore pressure} )</td>
<td>( s_e = \text{air entry suction} )</td>
</tr>
</tbody>
</table>

Table 2.1: Stress variables used by some of the existing constitutive models

Figure 2-2: Void ratio state surface - Volume changes under all-round pressure plotted in a void ratio–stress space (after Bishop & Blight (1963))
2.2.2 Volume change behaviour

The volumetric behaviour of partially saturated soils will be discussed in three subsections. First the behaviour due to changes of suction at constant mean net stress will be considered, followed by the effect of isotropic loading at constant suction and finally the overall behaviour will be presented.

2.2.2.1 Volume change due to changes in suction

The volumetric behaviour of partially saturated soils due to changes in suction at constant mean net stress can be summarised in the following points:

- Total volume changes due to drying

During the initial stages of drying from zero suction (path A to B in Figure 2-4), the soil remains fully saturated and the total volume change is equal to the pore water volume change. At this stage the increase of suction is equivalent to an increase of isotropic total stress. The value of suction at which desaturation occurs, called the air entry value of suction, $s_{air}$, varies significantly with the soil
type and is largely dependent on the particle size for granular soils and on the pore size for clayey soils.

The reduction in total volume after desaturation has occurred (point B) is smaller than the pore water volume reduction. This is demonstrated in Figure 2-5, which shows the total and water volume changes with increasing suction according to the conceptual model by Toll (1995). The total volume changes are expressed in terms of void ratio, $e$, and the water volume changes in terms of equivalent void ratio, $e_w (=\text{volume of water} / \text{volume of solids})$. Line A to B is equivalent to the fully saturated virgin compression line (VCL) and is followed by both the void ratio line (1) and the equivalent void ratio Line (2) indicating that the total volume changes are equal to the water volume changes. After desaturation has occurred (point B) both the void ratio and the equivalent void ratio lines deviate from the VCL. Void ratio decreases slightly after point B. On the other hand the water volume decreases sharply. Clearly the effects of suction changes are not equivalent to mean net stress changes once the soil has entered the partially saturated state.

Figure 2-4: Drying path from zero suction
It is generally accepted that for low plasticity soils volumetric changes during drying beyond desaturation are small and reversible. At high values of suction however plastic deformations may take place. Alonso et al. (1990) proposed that the yield suction (point C in Figure 2-4), $s_{ys}$, beyond which the soil is elastoplastic, is independent of the confining stress and equal to the maximum previously attained value of suction. Wheeler & Karube (1996) suggested that yielding due to drying is only possible for partially saturated soils containing saturated clay packets. Chen et al. (1999), however, performed drying tests on compacted low plasticity loess which exhibited a distinct yield value of suction. The volumetric strains during drying at different constant values of applied stress are plotted in Figure 2-6. A yield suction of approximately 100kPa can be identified for the tests at mean total stresses of 5, 50 and 100kPa, while the test at $p = 200kPa$ was not extended beyond this suction value. The obtained yield suction was not however equal to the maximum previously attained value of suction (as proposed by Alonso et al. (1990)). They argued that the value of the yield suction depends not only on the drying-wetting history but also on the initial soil density.
For high plasticity expansive soils, the volumetric deformations due to increasing suction (beyond point B in figure 2-4) can be large and irreversible.

![Figure 2-6: The relationship between volumetric strains and the increase in soil suction (after Chen et al. (1999))](image)

- **Total volume changes due to wetting**

As discussed in section 2.2.1, one of the most distinctive features of partially saturated soil behaviour is the potential of collapse upon wetting. Alonso et al. (1987) stated that a partially saturated soil may either expand or collapse upon wetting if the confining stress is sufficiently low (expansion) or high (collapse), and that it is also possible that a soil might experience a reversal in the volumetric behaviour during wetting (initial expansion followed by collapse).

This behaviour has been reported amongst others by Escario & Saez (1973), Josa et al. (1987) and Burland & Ridley (1996). Figure 2-7 shows the initial swelling followed by collapse during wetting at constant load experienced by three samples of remoulded clay with different initial values of moisture content and tensile pore water pressure (Escario & Saez (1973)). Similar behaviour can be seen in Figures 2-8a and 2-8b (Josa et al. (1987)). The stress paths in suction ($\sigma$) – mean net stress ($p$) space followed by two medium plasticity kaolin samples are plotted in Figure 2-8a and the strains experienced during saturation in Figure 2-8b. The sample wetted at higher pressure (path I to G) experienced collapse...
while the sample at lower pressure (path F to D) swelling. The state surfaces proposed by Bishop & Blight (1963) (Figure 2-2), Matyas & Radharkrishna (1968) and other authors illustrate the dependency of the volumetric response during wetting on the value of the confining stress.

![Figure 2-7: Swelling followed by collapse during wetting under constant load for three different initial values of tensile pore water pressure (after Escario & Saez (1973))](image)

In general if the stress state is not high enough to cause collapse upon wetting, the swelling experienced by a low plasticity non-expansive soil will be small and reversible. On the other hand high plasticity expansive clays can experience large irreversible volumetric strains. Chu & Mou (1973) performed wetting-drying cycles on expansive soils which showed large swelling strains on first wetting which were not recovered on subsequent drying, as seen in Figure 2-9.
Figure 2-8: a) Stress paths, and b) volumetric response measured in paths FD and IG (after Josa et al. (1987))

Figure 2-9: Relationship between suction and swelling (after Chu & Mou (1973))
Figure 2-10: Relationship between collapse and log. pressure for different moisture contents (after Booth (1975))

Figure 2-11: Relationship between collapse and normal stress for different soil types (after Yudhbir (1982))
The volumetric deformations experienced by a given soil when the stress state is such that collapse occurs upon wetting depend on the confining stress at which wetting (reduction of suction) takes place. Results presented by Matyas & Radharkrishna (1968), Booth (1975), Yudhibir (1982) and others indicate that for many soils the amount of collapse increases with confining stress at low stress regions, reaches a maximum and then decreases with stress becoming very small at high confining stresses. This is illustrated in Figures 2-10 and 2-11.

Water volume changes due to drying-wetting

Total volume changes for fully saturated soils are equal to the water volume changes since for the stress ranges relevant to engineering practice both water and solid phases are nearly incompressible and the volume changes are caused by inflow or outflow of water. In the case of partially saturated soils the presence of a third phase (air) in the soil means that the water volume changes are no more equivalent to the overall volume change. In order to fully understand the behaviour of partially saturated soils both the overall and the water volume changes due to changes of stress and suction need to be defined.

Water volume changes due to drying and wetting are usually investigated for unconfined conditions and are presented in the form of relations between volumetric water content, $\theta$, degree of saturation, $S_r$, or gravimetric moisture content, $w$, and suction. These relationships are called soil-water characteristic curves or water retention curves. Volumetric water content, $\theta$, is the ratio of the volume of water to the total volume and is related to the other variables through the following relationships:

$$\theta = \frac{S_r e}{1 + e} = S_r n$$  \hspace{1cm} (2.2)

and

$$\theta = w \frac{\rho_d}{\rho_w}$$  \hspace{1cm} (2.3)

where $e$ is the void ratio, $n$ is the porosity, $\rho_d$ is the dry density and $\rho_w$ is the water density.

A typical water retention curve showing the drying and wetting of a soil sample is shown in Figure 2-12. Three stages can be identified during drying. The
capillary saturation or boundary effect stage where the soil remains fully saturated, the desaturation or transition stage and the residual stage. Similar stages can be identified for the wetting phase.

Figure 2-12: Typical water retention curve (after Fredlund (1998))

An important feature of the water retention curve is the hysteresis observed between drying and wetting behaviour. This is influenced by several factors such as pore fluid composition, pore structure and the movement of wetting and drying fronts (Dineen (1997)). Hysteresis means that a soil can be in a very different state for the same value of suction and therefore have different properties depending on the drying-wetting history.

Many researchers have proposed empirical mathematical expressions for the soil-water characteristic curve (e.g. Burdine (1953), Gardner (1958), Maulem (1976), van Genuchten (1980), Fredlund & Xing (1984)). Most of these ignore hysteresis and assume that soil moves along the same curve during drying and wetting. The equation proposed by van Genuchten (1980), is given by:

\[
\theta = \left[ \frac{1}{1 + (\theta_s - \theta_r) \psi} \right]^{n} \left( \theta_s - \theta_r \right) + \theta_r \tag{2.4}
\]
where, $\psi$, $\omega$ and $\zeta$ are fitting parameters, and the subscripts $r$ and $s$ denote residual and saturated conditions, respectively.

![Stress paths and yield points for compacted kaolin](attachment:stress_paths.png)

Figure 2-13: a) Stress paths, and b) yield points for compacted kaolin (after Wheeler & Sivakumar (1995))

### 2.2.2.2 Volume change due to changes in confining stress

The main effects of suction on the volumetric response of partially saturated soils to changes in the confining stress are the following:

- Suction contributes to an increase in the isotropic yield stress, $p_o$. Figure 2-13a shows the stress paths followed by compacted speswhite kaolin samples in the mean net stress ($p$) – suction ($s$) stress space as reported by Wheeler & Sivakumar (1995). The samples were subjected initially to a pore pressure equalization stage from point A to different values of mean net stress and suction (points C₀, C₁, C₂ and C₃) and were subsequently consolidated under constant applied suction. The yield points observed during the consolidation stage are plotted on Figure 2-13b. The increase of the yield stress with suction is evident for the samples consolidated from points C₁, C₂ and C₃. Samples consolidated from a fully saturated state (point C₀) slightly deviated from the observed trend because yield had already occurred during wetting indicating that the fully saturated yield stress was even smaller. The increase of yield stress with suction
has been reported by many other authors (e.g. Dudley (1970), Maatouk et al. (1995), Cui & Delage (1996), Rampino et al. (1999) and (2000), Dineen (1997)).

- Suction influences the compressibility of partially saturated soils. The pre-yield (elastic) compressibility, $\kappa$, is commonly assumed for simplification to be independent of suction. However, experimental data indicate that it may decrease slightly with suction. Figure 2-14 shows the influence of suction on the elastic compressibility coefficient, $\kappa$, and the elastoplastic (post-yield) compressibility coefficient, $\lambda$, reported by Rampino et al. (2000).

![Graph showing the influence of suction on parameters $\kappa$ and $\lambda$](after Rampino et al. (2000))

Parameter, $\kappa$, decreases only slightly with suction (approximately 9%). The parameter, $\lambda$, however, is largely affected by suction (approximately 38%). Josa (1988) reported a similar variation of $\lambda$ with suction. Figure 2-15 shows the isotropic compression lines for two partially saturated samples of compacted kaolin. The compressibility coefficient, $\lambda$, is lower for the sample with higher suction. Other researchers however have found different effects of suction on $\lambda$. Maatouk et al. (1995) report only a slight decrease of $\lambda$ with suction, while Wheeler & Sivakumar (1995) report an increase, as shown in Figure 2-16.
Figure 2-15: Variation of specific volume during isotropic compression at constant suction (after Alonso et al. (1990))

Figure 2-16: Variation of specific volume during isotropic compression (after Wheeler & Sivakumar (1995))
As mentioned previously and illustrated in Figures 2-10 and 2-11, the amount of potential collapse due to wetting initially increases with confining stress, reaches a maximum value and then decreases. The amount of collapse in the e-logp space represents the difference between the partially and fully saturated isotropic compression lines (Figure 2-17).

The value of $\lambda$ and its dependency on suction measured in laboratory experiments probably depends on the range of mean net stress at which the tests were conducted. As noted by Wheeler & Karube (1996) the increase of $\lambda$ with suction seen in Figure 2-16 presumably implies that the stress range investigated by Wheeler & Sivakumar (1995) was above the value of $p$ corresponding to the maximum collapse for the particular soil.

### 2.2.2.3 Volume change due to changes in both confining stress and suction

As mentioned in the previous sections the overall volume change behaviour of partially saturated soils is usually described as a function of net stress and suction. Bishop & Blight (1963) schematically related net stress ($\sigma - u_a$) and suction ($u_a - u_w$) to void ratio, $e$. Matyas & Radharkrishna (1968) proposed a
similar state surface for void ratio, $e$, and also for the degree of saturation, $S_r$ (Figure 2-18 and 2-19).
Subsequently researchers derived analytical expressions for the state surfaces relating void ratio and degree of saturation or water content to net stress and suction. Some of the expressions that have been proposed are the following:

- **Fredlund (1979)**

\[
e = e_o - C_t \log \frac{(\sigma - u_a)_f}{(\sigma - u_a)_o} - C_m \log \frac{(u_a - u_w)_f}{(u_a - u_w)_o} \tag{2.5}
\]

\[
w = w_o - D_t \log \frac{(\sigma - u_a)_f}{(\sigma - u_a)_o} - D_m \log \frac{(u_a - u_w)_f}{(u_a - u_w)_o} \tag{2.6}
\]

where the \( f \) subscript represents the final stress state and \( o \) represents the initial stress state, \( C_t \) is the compressive index with respect to total stress, \( C_m \) is the compressive index with respect to suction, \( w \) is the water content, \( D_t \) is the water content index with respect to total stress, and \( D_m \) is the water content index with respect to suction. When the degree of saturation approaches 100\%, \( C_t \) is equal to the conventional compressive index \( C_c \) and approximately equal to \( C_m \).

- **Lloret & Alonso (1985)**

Void ratio state surface for a limited variation range of total external stress:

\[
e = a + b(\sigma - u_a) + c \log (u_a - u_w) + d \log \frac{(\sigma - u_a)_f}{(\sigma - u_a)_o} \tag{2.7}
\]

Void ratio state surface for a larger variation range of total external stress:

\[
e = a + b \log (\sigma - u_a) + c \log (u_a - u_w) + d \log \frac{(\sigma - u_a)_f}{(\sigma - u_a)_o} \tag{2.8}
\]

Degree of saturation state surface:

\[
S_r = a - \tanh \left[ b (u_a - u_w) \right] \left[ c + d \log (\sigma - u_a) \right] \tag{2.9}
\]

or

\[
S_r = a - \left[ 1 - \exp \left[ -b (u_a - u_w) \right] \right] \left[ c + d \log (\sigma - u_a) \right] \tag{2.10}
\]

where \( a, b, c \) and \( d \) are constants.
2.2.3 Shear strength

At low values of suction, when the soil is still fully saturated, the shear strength is defined from the effective stresses and increases linearly with suction. Once the soil becomes partially saturated, beyond the air entry value, the increase of shear strength with suction is smaller and non-linear.

Bishop, Alpan, Blight & Donald (1960) extended the Mohr-Coulomb failure criterion to partially saturated soil conditions through the use of Bishop’s effective stress expression. They proposed the following expression for the shear strength of soils (partially or fully saturated):

\[ \tau = c' + (\sigma - u_a) \tan \phi' + \chi (u_a - u_w) \tan \phi' \]  

(2.11)

where \( \tau \) is the shear strength of the soil, \( c' \) is the effective cohesion, \( \phi' \) is the angle of shearing resistance and \( \chi \) is a soil parameter which varies between 0 and 1. For a completely dry soil \( \chi = 0 \) and for a fully saturated soil \( \chi = 1 \) in which case the above expression reduces to the fully saturated Mohr-Coulomb failure criterion:

\[ \tau = c' + (\sigma - u_a) \tan \phi' \]  

(2.12)

Fredlund, Morgenstern & Widger (1978) proposed a modified version of expression (2.11):

\[ \tau = c' + (\sigma - u_a) \tan \phi' + (u_a - u_w) \tan \phi^b \]  

(2.13)

where \( \phi^b \) is the slope of the failure surface in the \( \tau - s \) plane. It was assumed that both \( \phi' \) and \( \phi^b \) are independent of suction. The effect of suction on the shear strength is therefore equivalent to an increase of the effective cohesion:

\[ c = c' + (u_a - u_w) \tan \phi^b \]  

(2.14)

Expression (2.13) only reduces to the fully saturated Mohr Coulomb failure criterion when \( u_w = u_a = 0 \) kPa. A schematical representation of the failure surface given by Eq. 2.13 is given in Fig.20.
Escario & Saez (1986) presented the results of direct shear tests on three reconstituted soils. They showed that the assumption that $\phi'$ is constant is not realistic (Fig. 2.21a and 2.21b). This was confirmed by Fredlund, Rahardjo & Gan (1987), who suggested that $\phi'$ is equal to $\phi'$ at low values of suction and then decreases considerably to a constant low value at high values of suction.

![Figure 2-20: Representation of failure surface from Eq.2.13](image1)

![Figure 2-21: Direct shear tests under controlled suction for a) Madrid grey clay, and b) Guadalix de la Sierra red clay (after Escario & Saez (1986))](image2)
Escario & Juca (1989) suggested that at high values of suction $\phi^b$ becomes negative implying that at very high suctions as the soil becomes completely dry the increase of apparent cohesion due to suction tends to zero. This is supported by the data from Gan & Fredlund (1996) and Nishimura & Toyota (2002) (Figure 2-22) although in the latter case only limited information is available for the shear strength at high suctions.

Figure 2-22: Changes in shear strength with soil suction (data from Nishimura & Toyota (2002)

Karube (1988) performed triaxial tests on compacted kaolin. From the test results he derived the following equation for the shear strength:

$$ q_f = M(p - u_a) $$

where $M = M'(1/\alpha) + (-\Delta v/\Delta \varepsilon)_f$ and

$$ \left( 1/\alpha \right) = \left[ 1 + \frac{f(s)}{p} \right] $$

$(\Delta v/\Delta \varepsilon)_f$ in the above equation is the dilatancy index, $\varepsilon$ is the shear strain defined as $\varepsilon = 2(\varepsilon_1 - \varepsilon_3)/3$, $M'$ is the inclination of the failure lines, $f(s)$ is the intercept of the failure lines on the $(p - u_a)$ axis.
Toll (1990) presented a framework for the behaviour of partially saturated soils, which was supported by a number of triaxial test results on gravel. He proposed the following expression for the critical state shear strength

\[
q = M_a (p - u_a) + M_w (u_a - u_w)
\]

(2.16)

where \( q \) is the deviator stress, \( p \) is the mean total stress, and \( M_a \) and \( M_w \) are two soil parameters called total stress ratio and suction ratio, respectively. \( M_a \) and \( M_w \) depend on the degree of saturation, as shown in Figure 2-23, and represent the contributions of the net mean stress, \( p - u_a \), and suction, \( u_a - u_w \), respectively, to the shear strength.

He suggested that suction has no effect on the soil strength at low degrees of saturation, whereas the contribution of the net mean stress increases with decreasing degree of saturation. This is equivalent to an increase of the critical state angle of shearing resistance in terms of total stress with increasing suction. For fully saturated conditions \( M_a = M_w = M_s \) and the above expression reduces to,
\[ q = M_s (p - u_w) \]  

(2.17)

Determination of the values of \( M_a \) and \( M_w \) is not straightforward. Toll (1990) proposed a regression technique for separating the contribution of the two stress components.

Wheeler (1991) proposed a variation of the above expression, which does not include the degree of saturation as an additional variable

\[ q = M (p - u_a) + f (u_a - u_w) \]  

(2.18)

where \( M \) is the critical state stress ratio and is independent of the suction. An expression for \( f(u_a - u_w) \) was also derived (Figure 2-24) based on the data from Toll (1990).

Fredlund et al. (1995) proposed that the component \( (u_a - u_w) \tan \phi \) of Eq.2.13 can be predicted from the shape of the soil-water characteristic curve. Vanapalli et al. (1996) and Oberg & Salfors (1997) have proposed similar approaches.

Khalili & Khabbaz (1998) analysed shear strength data from 14 cases in the literature and proposed the prediction of the shear strength through Eq.2.11 using
the following relationship between $\chi$ and the ratio of suction over the air entry value (as illustrated in Figure 2-25):

$$\chi = \left[ \left( \frac{u_a - u_w}{u_a - u_w}_b \right) \right]^{-0.55}$$

(2.19)

where $(u_a - u_w)_b$ is the air entry value.

Figure 2-25: Relationship between the effective stress parameter $\chi$ and the suction ratio (after Khalili & Khabbaz (1998))

Vaunat et al. (2002) presented the results of direct shear tests on a silty soil during a drying-wetting cycle and proposed a method for the inclusion of hydraulic hysteresis into the prediction of the shear strength of partially saturated soils. This approach seems very promising although future investigation is required since it is based on a number of assumptions and limited experimental data.

Most approaches for the prediction of the shear strength (with the exception of Karube (1988) and Toll (1990) presented above) assume that the parameters $M$ and $\phi$ are constant and independent of suction. There is however evidence (Escario & Saez (1986), Escario & Juca (1989), Maatouk et al. (1995), Cui & Delage (1996)) that this may not be true.
2.3 CONSTITUTIVE MODELS

2.3.1 Introduction

Constitutive models for partially saturated soils fall into two categories; elastic models and elastoplastic models. Elastic models relate strain increments to increments of net stress and suction. Such models have been proposed by Fredlund & Morgenstern (1976) and Lloret et al. (1987). Wheeler & Karube (1996) in the state of the art report on constitutive modelling presented a comprehensive review of the models of this type. They stated that although elastic models have the advantage that it is relatively easy to implement them within numerical analysis and measure the relevant parameters, there are major disadvantages. The most important is that there is no distinction between reversible and irreversible strains, which means that they can only be used in problems that involve only monotonic loading and unloading. Even in this case there are a number of weaknesses. Consequently, elastic models are not considered with in this thesis and will not be mentioned further.

One of the first elastoplastic constitutive models to be developed for partially saturated soils was the Barcelona Basic Model (Alonso et al. (1990)), which was based on the theoretical framework proposed by Alonso et al. (1987). This model was an extension of the Modified Cam-Clay model for fully saturated soils to partially saturated states through the introduction of the concept of the Loading-Collapse yield surface (Fig.2-26). The concept allows the reproduction of many important features of partially saturated soil behaviour, such as collapse upon wetting, and is the basis upon which most other elastoplastic models have been developed.

There are two main categories of elastoplastic models; expansive and non-expansive models. Expansive soil (high plasticity clays) modelling is not discussed in detail in this thesis but a brief overview is given in this chapter. Some of the existing models for expansive soils are the models by Gens & Alonso (1992), Alonso et al. (1994) and Alonso et al. (2000).
Non-expansive models are intended for low plasticity soils and can in turn be separated into two categories; total stress models, which use net mean stress and suction (or some form of equivalent suction) as stress variables, and ‘effective stress’ models, which use some definition of effective stress and suction or equivalent suction as stress variables. As most experimental results and frameworks of behaviour are given in terms of total stress and suction and since there is no apparent major advantage of an effective stress approach, the research presented in this thesis has concentrated on the first type of models. Three of the best known total stress models are presented in this section, namely the Barcelona Basic model by Alonso et al. (1990), the Josa et al. (1992) model and the Wheeler & Sivakumar (1995) model. An effective stress model, the Bolzon et al. (1996) model, is also presented here. Other models include the models by Kohgo et al. (1993a and 1993b), Modaressi & Abou-Bekr (1994a and 1994b), Jommi & Di Prisco (1994), Cui et al. (1995) and Kato et al. (1995).

Most elastoplastic constitutive models are based on critical state models for fully saturated soils. For this reason before presenting some of the existing partially saturated soil models, a brief description of the basic elements of critical state models will first be given.
2.3.2 Critical state models

Critical State soil mechanics theory was developed in the 1950’s through the work by Drucker et al. (1957), Roscoe et al. (1958) and Calladine (1963). The first critical state models were the Cam Clay model (Roscoe & Schofield (1963) and Schofield & Wroth (1968)) and the modified Cam Clay model (Roscoe & Burland (1968)). A large number of constitutive models have been developed since within the Critical State framework. Most of these models are elastoplastic and require the following elements to be defined:

a) A yield function \( F(\{\sigma\}',\{k\}) = 0 \), where \( \{\sigma\}' \) is the stress state and \( \{k\} \) are state parameters. The yield function represents the surface, which separates purely elastic from elastoplastic behaviour. The following functions are assumed for the Cam Clay and modified Cam Clay models:

Cam Clay
\[
F(\{\sigma\}',\{k\}) = \frac{J}{p'M_J} + \ln\left(\frac{p'}{p_o'}\right) = 0
\] (2.20)

Modified Cam Clay
\[
F(\{\sigma\}',\{k\}) = \left(\frac{J}{p'M_J}\right)^2 - \left(\frac{p_o'}{p'} - 1\right) = 0
\] (2.21)

where \( p' \) is the mean effective stress, \( J \) is the deviatoric stress, \( M_J \) is a model parameter, and \( p_o' \) is the isotropic yield stress. The projection of these two yield surfaces is shown in Fig. 2-27 and 2-28.

b) A plastic potential function \( P(\{\sigma\}',\{m\}) = 0 \), where \( \{m\} \) are state parameters. This function determines the relative magnitudes of the plastic strains at each point of the yield surface and also the position of the critical state line in the specific volume, \( v \), mean effective stress, \( p' \), and deviatoric stress, \( J \), space, which is shown in Fig. 2-29. The critical state line is the line on the yield surface, along which the following condition is satisfied:

\[
\frac{\partial P(\{\sigma\}',\{m\})}{\partial p'} = 0
\] (2.22)
giving zero volumetric strains and infinite shear strains. The Cam Clay and modified Cam Clay models assume a plastic potential function identical to the yield function given by Eq. 2.20 and 2.21.

Figure 2-27: Projection of yield surface onto $J - p'$ plane for Cam Clay

Figure 2-28: Projection of yield surface onto $J - p'$ plane for modified Cam Clay

The six components of incremental plastic strain, $d\varepsilon_i^p$, are determined from the plastic potential function through a flow rule, which can be expressed as follows:

$$d\varepsilon_i^p = \lambda \frac{\partial P(\sigma', m)}{\partial \sigma_i'}$$

(2.23)
where $A$ is a scalar multiplier which depends on the hardening/softening rule discussed below.

![Figure 2-29: State boundary surface for modified Cam Clay](image)

c) A hardening/softening rule, which determines the magnitude of the plastic strains. For the Cam Clay and modified Cam Clay models this is given by:

$$\frac{dp'}{p'_0} = d\varepsilon_v^p \frac{\nu}{\lambda - \kappa}$$  \hspace{1cm} (2.24)

where $\varepsilon_v^p$ are the volumetric strains, $\nu$ is the specific volume, $\lambda$ is the coefficient of compressibility, and $\kappa$ is coefficient of compressibility along swelling lines.

d) Definition of the elastic behaviour within the yield surface. The volumetric elastic strains are given from the shape of the swelling lines. For the Cam Clay and modified Cam Clay models this is given by:

$$d\varepsilon_v^e = \frac{\kappa}{\nu} \frac{dp'}{p'}$$  \hspace{1cm} (2.25)

The elastic shear strains are usually computed from the elastic shear modulus, $G$. An infinite shear modulus was assumed in the Cam Clay and modified Cam Clay models.
2.3.3 Barcelona Basic Model

2.3.3.1 General

The Barcelona Basic model (Alonso et al. (1990)) is intended for partially saturated soils which are slightly or moderately expansive, such as partially saturated sands, silts, clayey sands, sandy clays and clays of low plasticity. The model is formulated in the \((q, p, s)\) stress space, where \(q\) is the deviator stress, \(p\) is the net mean total stress, and \(s\) is the suction.

2.3.3.2 Formulation of model for isotropic stress states

The proposed variation of the specific volume, \(\nu = 1 + e\), with the net mean total stress, \(p\), and suction, \(s\), along virgin and unloading-reloading stress paths is shown in Figure 2-30a. The virgin compression line (at constant \(s\)) is given by

\[
\nu = N(s) - \lambda(s) \ln \frac{p}{p^e}
\]  

(2.26)

where \(p^e\) is a reference stress state for which \(\nu = N(s)\). The unloading-reloading paths (at constant \(s\)) are assumed to be elastic

\[
d\nu = -\kappa \frac{dp}{p}
\]

(2.27)

where \(\kappa\) is assumed to be independent of \(s\).

Figure 2-30a shows the response to isotropic loading of a saturated sample (\(s = 0\)) and a partially saturated sample. The saturated sample yields at a stress \(p_o^*\) (point 3), while the partially saturated sample yields at the higher stress \(p_o\) (point 1). If both points, 1 and 3, belong on the same yield curve in the \((p, s)\) space (Figure 2-30b), the relationship between \(p_o\) and \(p_o^*\) can be obtained by relating the specific volumes at points 1 and 3 through a virtual path which involves an initial unloading, at constant \(s\), from point 1 to point 2, and a subsequent reduction in suction, at constant \(p\), from point 2 to point 3:

\[
\nu_1 + \Delta \nu_p + \Delta \nu_s = \nu_3
\]

(2.28)
The suction unloading (wetting) from 2 to 3 occurs in the elastic domain, so:

\[ \Delta v_s = \kappa_s \ln \frac{s + p_{atm}}{p_{atm}} \]  

(2.29)

where \( p_{atm} \) is the atmospheric pressure, and \( \kappa_s \) is the compressibility coefficient for suction changes within the elastic domain. \( p_{atm} \) is included in the above
equation in order to avoid the calculation of infinite strains as suction tends to zero.

The unloading from 1 to 2 also occurs in the elastic domain, so:

\[ \Delta \nu_p = \kappa \ln \frac{P_o}{P_o'} \]  

(2.30)

The above equations are combined giving

\[ N(s) - \lambda(s) \ln \frac{P_o}{P_o'} + \kappa \ln \frac{P_o}{P_o'} + \kappa_s \ln \frac{s + p_{atm}}{p_{atm}} = N(0) - \lambda(0) \ln \frac{P_o^*}{p^*} \]  

(2.31)

The above equation can be simplified if the assumption is made that \( p^* \) is the mean net stress at which a sample may reach the saturated virgin compression line, starting from a partially saturated virgin compression line, through a path involving only (elastic) swelling. In that case \( p_o^* = p^* = p_o \) and the LC yield curve (defined in Figure 2-30b) becomes a straight line so that changes in suction do not result in plastic deformations:

\[ \Delta \nu(p^*) \left|_{s=0}^{s} \right. = N(0) - N(s) = \kappa \ln \frac{s + p_{atm}}{p_{atm}} \]  

(2.32)

Equation 2.31 is now simplified to:

\[ \left( \frac{p_o}{p^*} \right) = \left( \frac{p_o^*}{p^*} \right)^{\frac{\lambda(0)-\lambda(\infty)}{\lambda(\infty)-\lambda(0)}} \]  

(2.33)

The soil stiffness \( \lambda(s) \) can be obtained from the following empirical equation:

\[ \lambda(s) = \lambda(0) \left[ (1-r) e^{-\beta s} + r \right] \]  

(2.34)

where \( r \) is a constant related to the maximum stiffness of the soil (for an infinite suction), \( r = \lambda(s \rightarrow \infty) / \lambda(0) \), and \( \beta \) is a parameter which controls the rate of increase of soil stiffness with suction.

An increase in suction may also induce irrecoverable strains. Another yield condition is introduced to take account of this fact (Figure 2-31):
where $s_o$ is the maximum previously attained value of suction.

\[ s = s_o = \text{constant} \]  \hspace{1cm} (2.35)

The elastic, plastic and total volumetric deformations caused by an increase in $p$ or $s$ can be obtained from the following equations:

- Increase of mean total stress, $p$:

\[
\text{Elastic:} \quad d\varepsilon_{vp}^e = -\frac{dv}{\nu} = \frac{\kappa}{\nu} \frac{dp}{p} \quad (2.36)
\]
Along the isotropic compression line $\frac{dp}{p} = \frac{dp_o}{p_o}$ and the plastic strains are given by,

$$d\varepsilon_{vp} = \frac{\lambda(s) dp}{p_o} \frac{dp}{p_o} = \frac{\lambda(0) - \kappa dp^*_o}{p_o}$$

(2.37)

Plastic:

$$d\varepsilon_{vp}^p = \frac{\lambda(s) - \kappa dp^*_o}{p_o}$$

(2.38)

- Increase of suction, $s$:

Elastic:

$$d\varepsilon_{vs}^e = \frac{\kappa_s ds}{v (s + p_{atm})}$$

(2.39)

Total:

$$d\varepsilon_{vs} = \frac{\lambda_s ds_o}{v (s_o + p_{atm})}$$

(2.40)

When the SI yield surface is active throughout the entire suction increment, $\frac{ds}{(s+p_{atm})} = \frac{ds_o}{(s_o+p_{atm})}$ and the plastic strains are given by,

Plastic:

$$d\varepsilon_{vs}^p = \frac{\lambda_s - \kappa_s ds_o}{s_o + p_{atm}}$$

(2.41)

Both sets of plastic deformations have similar effects. The two yield curves are coupled and their position is controlled by the total plastic volumetric deformation:

$$d\varepsilon_v^p = d\varepsilon_{vs}^p + d\varepsilon_{vp}^p$$

(2.42)

The hardening laws for the two yield curves are the following:

LC yield surface:

$$\frac{dp^*_o}{p^*_o} = \frac{v}{\lambda(0) - \kappa} d\varepsilon_v^p$$

(2.43)

SI yield surface:

$$\frac{ds_o}{s_o + p_{atm}} = \frac{v}{\lambda_s - \kappa_s} d\varepsilon_v^p$$

(2.44)
2.3.3.3 Formulation of model for triaxial stress states

A version of the modified Cam-clay model is adopted in this model to include the effect of shear stresses. The yield surfaces in the \((q, p)\) stress space for a constant suction \(s\) and for \(s = 0\) are shown in Figure 2-32. The yield surfaces are given by the following equation:

\[
q^2 - M^2 (p + p_s)(p_o - p) = 0
\]  

(2.45)

where \(p_s = ks\). \(k\) is a constant which controls the expansion of the yield surface in the tensile stress region (increase of the apparent cohesion with suction).

Figure 2-32: Yield surfaces of the Barcelona Basic model (after Alonso et al. (1990))
A non-associated flow rule is suggested on the constant $s$ planes. To avoid overestimation of $K_o$ values (Gens & Potts, 1982a) the expression for the associated flow rule is modified by introducing a parameter $\alpha$ (Ohmaki, 1982) resulting in the following equation:

\[
\frac{d\varepsilon_s^p}{d\varepsilon_{vp}} = \frac{2q\alpha}{M^2(2p + p_s - p_o)}
\]  

(2.46)

$\alpha$, is a constant which can be derived by requiring that the direction of the plastic strain increment for zero lateral deformation,

\[
d\varepsilon_s^p = \frac{2}{3}d\varepsilon_{vp}, \quad d\varepsilon_{vp} = d\varepsilon_{vp}^p + d\varepsilon_{vp}^e \quad \text{and} \quad \frac{d\varepsilon_{vp}^p}{d\varepsilon_{vp}^e} = \frac{\lambda(s) - \kappa}{\kappa}
\]

\[
\Rightarrow \frac{d\varepsilon_s^p}{d\varepsilon_{vp}} = \frac{2}{3} \left(1 - \frac{\kappa}{\lambda(s)}\right)
\]  

(2.47)

(where for simplicity $d\varepsilon_{vp}^e$ has been assumed to be zero), is found for stress states satisfying $K_o$ conditions:

\[
\frac{q}{p} + p_s = 3 \frac{1 - K_o}{1 + 2K_o}
\]  

(2.48)

The elastic strains induced by changes in $q$ are given by the following equation:

\[
d\varepsilon_s^e = \frac{2}{3}(d\varepsilon_s^e - d\varepsilon_s^p) = \frac{1}{3G}dq
\]  

(2.49)
2.3.4  Josa et al. (1992) model

2.3.4.1 General

Josa et al. (1992) proposed a modified version of the Barcelona Basic model (Alonso et al. (1990)). The main modification is related to the prediction of potential collapse. The Barcelona Basic model assumes linear isotropic compression lines for partially saturated conditions constantly diverging from the fully saturated isotropic compression line. This implies that the amount of potential collapse due to wetting increases indefinitely with confining stress. However, as mentioned in section 2.2, for most partially saturated soils the amount of potential collapse initially increases with confining stress reaches a maximum value and then decreases tending to zero at very high stresses. The modified model addresses this issue. The formulation is similar to the Barcelona Basic model and will not be presented here in full.

2.3.4.2 Modifications to the Barcelona Basic model

The model proposed by Josa et al. (1992) allows the prediction of maximum collapse at some value of confining stress through the introduction of a modified expression for the Loading-Collapse yield surface in the mean net stress (\(p\)) – suction (s) space (which replaces Equation 2.33 of the original model):

\[
p_o = (p_o^* - p_c^e) + p_c^e \left[ (1 - m) e^{-\alpha s} + m \right]
\]

(2.50)

where \(\alpha\) is a parameter that controls the shape of the yield surface and \(m\) is related to the difference between \(p_o\) for high suction values \((p_o^\infty)\) and \(p_o^*\), and is always higher than 1. Figure 2-33 shows the resulting yield surfaces for different values of \(p_o^*\). \(m\) is a function of \(p_o^*\) and is required to satisfy the following conditions:

• when \(p_o^* = p_c^e\), \(m = 1\)
• for large values of \(p_o^*\), \(m = 1\)
• \(m\) presents a peak \((p_o^* = \zeta, m = \zeta)\)
The following equation is given for $m$:

$$m = 1 + \frac{\zeta_y - 1}{\zeta_x - p^\prime} \left( p_o^* - p^\prime \right) e^{\frac{\zeta_y - p_o^*}{p^\prime}}$$

(2.51)

Figure 2-33: Loading-Collapse yield surfaces in the $(p - s)$ space for different values of $p_o^*$ (after Josa et al. (1992))

Figure 2-34 shows the variation of $m$ with $p_o^*$. Josa et al. (1992) suggest that the parameters $\zeta_x$ and $\zeta_y$ can be replaced by the value of $p_o^*$ corresponding to maximum collapse and the maximum plastic volumetric strain, $\varepsilon_{vmax}^p$, respectively. They also note that the range of validity of this expression is limited by the condition that adjacent yield surfaces should not intersect. No specific limits are provided however in their paper.

Figure 2-34: Shape of $m$ for generic values of $\zeta_x$ and $\zeta_y$ (after Josa et al. (1992))
A second modification to the Barcelona Basic model regards the hardening laws. Equations 2.43 and 2.44 of the original model are replaced by:

LC yield curve:
\[
\frac{dp_o^*}{p_o^*} = \frac{d\varepsilon_v^p}{\lambda(0) - \kappa} 
\]  

(SI yield curve):
\[
\frac{ds_o}{s_o + p_{st}} = \frac{d\varepsilon_v^p}{\lambda_s - \kappa_s} 
\]

giving hyperbolic relationships between void ratio and mean net stress for saturated conditions and between void ratio and suction, as opposed to the usual logarithmic variation given by Equations 2.43 and 2.44. This is done in order to avoid negative values of void ratio, \( e \), for high applied stresses or suction.

Similarly hyperbolic relationships are defined for elastic paths:
\[
d\varepsilon_v^e = \kappa \frac{dp}{p} + \kappa_s \frac{ds}{s + p_{st}} 
\]  

### 2.3.5 Wheeler & Sivakumar Model

#### 2.3.5.1 General

Wheeler & Sivakumar (1995) used the data from a series of controlled suction triaxial tests on samples of compacted speswhite kaolin, in the development of an elastoplastic critical state framework for partially saturated soil. The framework is formulated in the \((q, p, s)\) stress space.

#### 2.3.5.2 Formulation of model for isotropic stress states

Figure 2-35 shows the stress paths in the \((p, s)\) space, followed by four samples during the equalisation stage (wetting). The behaviour of the samples during this stage was found to be consistent with the existence of the LC yield surface (as
defined by Alonso et al. (1990)). The yield surface inferred by the results during this stage is also shown in Figure 2-35.

Figure 2-35: Loading-Collapse yield surface inferred from equalisation stage results (after Wheeler & Sivakumar (1995))

The behaviour of the samples during the consolidation stage (isotropic at constant suction) also indicated the existence of this curve (Figure 2-13b). When the yield stress at a particular value of suction, \( s \), was exceeded, the soil state fell on a unique isotropic normal compression line defined by a linear relationship:

\[
v = N_a(s) - \lambda(s) \ln \frac{p}{p_{atm}}
\]

where \( p_{atm} \) is the atmospheric pressure and \( N_a(s) \) is the specific volume at \( p = p_{atm} \). The test data showed that \( N_a(s) \) is larger for larger values of suction. \( \lambda(s) \) showed relatively little variation for suctions between 100 and 300kPa but a significant drop when \( s = 0 \) (see Figure 2-16). This result is in contrast with the predictions of the Barcelona Basic model, which assumes a reduction of both \( \lambda(s) \) and \( N_a(s) \) with increasing suction.
The shape of the LC yield curve is defined in this model in the same way as in the Barcelona Basic model:

\[
\left( \lambda(s) - \kappa \right) \ln \frac{p_o}{p_{atm}} = \left( \lambda(0) - \kappa \right) \ln \frac{p_o(0)}{p_{atm}} + N_a(s) - N_a(0) + \kappa_s \ln \frac{s + p_{atm}}{p_{atm}} \tag{2.56}
\]

where \(p_o(0)\) is the isotropic yield stress for full saturation, \(N_a(0)\) is the intercept for full saturation, and \(\kappa\) and \(\kappa_s\) are the elastic stiffness parameters for changes in net mean effective stress and suction, respectively.

The assumption made in the Barcelona Basic model that a limiting situation exists at which the LC yield curve becomes a straight vertical line at some reference value of \(p_o\), the characteristic pressure \(p^c\), is not adopted in this model, so the above expression is not simplified. Furthermore no assumption is made regarding the variation of \(\lambda(s)\) with suction and the form of the elastic behaviour inside the yield curve. Instead it is assumed that empirical equations are given for \(\lambda(s)\), \(N(s)\) and the form of elastic behaviour in the elastic region. Wheeler & Sivakumar (1995) argued that this approach has the advantage, over the Barcelona Basic model, that it is easier and more direct to measure values of \(N(s)\) at a few different values of suction than it is to measure \(p^c\). Moreover the basis of the simplification introduced in the Barcelona Basic model, has never been validated experimentally.

The experimental results confirmed the assumption, first made by Alonso et al. (1990), that the phenomenon of collapse on wetting is essentially the same process as the plastic compression that occurs on isotropic loading beyond the yield point. This assumption was therefore included in the developed model.

### 2.3.5.3 Formulation of model for triaxial stress states

The existence of a critical state was confirmed by the test results. The critical state lines are given by

\[
q = M(s)p + \mu(s) \tag{2.57}
\]
\[ \nu = \Gamma(s) - \psi(s) \ln \left( \frac{p}{p_{\text{atm}}} \right) \]  

(2.58)

where \( M(s), \mu(s), \Gamma(s) \) and \( \psi(s) \) are functions of suction. The experimental results showed that the assumption made by Alonso et al. (1990) that \( M(s) \) is constant might be realistic. In contrast the value of \( \mu(s) \), which is equivalent to \( k_s \) (Eq. 2.45), varied with suction in a non-linear fashion.

For a given value of suction Eq.2.58 represents the projection of the critical state line on the \( \nu - \ln p \) plane. Alonso et al. (1990) did not explicitly provide a similar equation; the formulation of the model, however, implied a particular relationship, which did not fit well the experimental results reported by Wheeler & Sivakumar (1995).

The proposed form for the state boundary relationship is shown in Figure 2-36, and defined as follows:

\[ q^2 = M^2 \left( p_o - p \right) \left( p + p_o - 2p_s \right) \]  

(2.59)

\[ M_s = \frac{M(s)p_s + \mu(s)}{p_o - p_s} \]  

(2.60)

\[ \frac{p_o}{p} = \exp \left[ \frac{N(s) - \lambda(s) \ln \left( \frac{p}{p_{\text{atm}}} \right) - \nu}{\lambda(s) - \kappa} \right] \]  

(2.61)

\[ \frac{p_s}{p} = \exp \left[ \frac{\Gamma(s) - \psi(s) \ln \left( \frac{p}{p_{\text{atm}}} \right) - \nu}{\psi(s) - \kappa} \right] \]  

(2.62)

An associated flow rule is assumed for the determination of the plastic strain increments, while the value of the elastic shear modulus \( G \) is assumed to be constant.
2.3.6 Bolzon et al. (1996) model

2.3.6.1 General

Bolzon et al. (1996) presented a study in which they investigated the elastoplastic behaviour of partially saturated soils by using Bishop’s stress and suction as variables, and proposed a generalised elastoplastic constitutive model to describe this behaviour. The model is based on the elastoplastic model for fully saturated soils developed by Pastor et al. (1990).

The mean Bishop’s effective stress is given by:

\[ p' = p - u_o - \chi (u_w - u_o) \quad \text{or} \quad p' = \bar{p} + \chi s \]  \hspace{1cm} (2.63)
the soil parameter $\chi$ is assumed by Bolzon et al. (1996) to be equal to the degree of saturation, $S_r$, so:

$$p' = \bar{p} + S_r s$$  \hspace{1cm} (2.64)

The degree of saturation $S_r$ is assumed to be directly related to suction through the following relationship (Alonso et al., 1990):

$$S_r = 1 - m \tanh(ls)$$  \hspace{1cm} (2.65)

where $m$ and $l$ are material constants.

### 2.3.6.2 Use of a single stress variable

As discussed in section 2.2.1 it is generally accepted that two independent stress variables are necessary in order to explain the behaviour of partially saturated soils. Bolzon et al. (1996) added to this consensus by demonstrating the problems associated with the use of a single stress variable. They showed that a simple introduction of Bishop’s stress into a fully saturated model, namely the Pastor et al. (1990) model, allows volume changes induced by suction to be taken into account, but the predictions are not always in accordance with experimental observations. Some of the drawbacks of such an approach are illustrated below.

Under monotonic loading, the plastic volumetric strain, for saturated conditions, is given by:

$$\varepsilon_v^p = \frac{1}{H_o} \ln \frac{p'}{p'_o}$$  \hspace{1cm} (2.66)

where

$$H_o = \frac{v_o}{\lambda - \kappa}$$  \hspace{1cm} (2.67)

and $v_o$ is the initial specific volume. With the introduction of Bishop’s stress into the above equation:
by assuming rigid-plastic behaviour ($\varepsilon_i^p \equiv \varepsilon_i$):

$$\nu = \nu_o - \frac{\nu_o}{H_o} \ln \frac{p + S_s s}{p_o + S_s s}$$ (2.69)

which does not fully take account of the dependence of soil compressibility on suction, as $H_o$ is independent of suction.

Moreover if the following yield function is assumed for fully saturated conditions (Pastor et al., 1990):

$$F = q - 2M_f \left( p' - \frac{P^2}{P_f} \right)$$ (2.70)

a simple introduction of Bishop’s stress leads to the yield function,

$$F = q - 2M_f \left( p + S_s s - \frac{(p + S_s s)^2}{P_f} \right)$$ (2.71)

which can be shown to predict decreasing isotropic yield stress with increasing suction.

### 2.3.6.3 Formulation of partially saturated soil model

Bolzon et al. (1996) stated that the drawbacks, mentioned above, in the use of a single stress variable can be overcome through the following enhancements to this model, which implicitly introduce suction, $s$, as a second stress variable.

First the multiplicative function $\tilde{H}_w (s)$ is introduced to the hardening modulus, $H$, which in its simplest form, linearly relates changes in stiffness with changes in suction:

$$H = H_o \tilde{H}_w (s) = H_o (1 + \alpha s)$$ (2.72)
where $\alpha$ is a material parameter. The stiffness parameter:

$$\hat{\lambda}(s) = \frac{1 + e_o}{H_o H_w(s)} + \kappa = \frac{1 + e_o}{H_o (1 + \alpha s)} + \kappa$$

is therefore predicted to reduce with increasing suction. This is in agreement with some experimental observations and with the predictions of the BBM model but not with the Wheeler & Sivakumar (1995) model. The isotropic compression line is now given by:

$$\nu = \nu_o - \frac{\nu_o}{H_o H_w(s)} \ln \frac{\bar{p} + S_s s}{\bar{p}_o + S_s s} \quad (2.73)$$

The expression for $\tilde{H}_w(s)$ is refined to fit better experimental data as follows:

$$\tilde{H}_w(s) = \left[ 1 + b_1 \left( e^{-b_2 s} - 1 \right) \right]^{-1} \quad (2.74)$$

or

$$\tilde{H}_w(s) = 1 + (\alpha_1 e^{-\bar{p}'} - \alpha_2) s \quad (2.75)$$

where $b_1$, $b_2$, $\alpha_1$ and $\alpha_2$ are material constants. Both expressions give a value of $\tilde{H}_w(s)$ equal to 1 in fully saturated conditions. Use of the second expression, according to Bolzon et al. (1996), allows the prediction of a maximum collapse during wetting at some value of the net mean stress similar to the Josa et al. (1992) model. This value can be proved to be:

$$p' \big|_{s_{\text{max}}} = \ln \frac{\alpha_1}{\alpha_2} \quad (2.76)$$

However, the applicability of Eq. 2.75 appears to be doubtful since the formulation of the model assumes a straight isotropic compression line.

With regard to the yield surface the assumption is made that a reference value, $p_{c}'$, of Bishop’s stress exists defining a limit situation at which no plastic strains develop as a result of suction changes. The yield surface in the $(p', s)$ plane, in this limit situation, is a straight line vertical to the $p'$ axis. This assumption is similar to that made by Alonso et al. (1990), the only difference being that the
reference stress in that case is a total stress. A similar expression to that of the Barcelona Basic model’s is proposed for the relationship between the partially saturated isotropic yield stress, \( p_y'(s) \), and that equivalent fully saturated yield stress, \( p_{yo}'(s) \):

\[
\frac{p_y'(s)}{p_c'} = \left( \frac{p_{yo}'(s)}{p_c'} \right)^{\tilde{H}_w(s)}
\]  

(2.77)

From Eq.2.66 and Eq.2.72 it can be shown that \( \tilde{H}_w(s) \) is equal to:

\[
\tilde{H}_w(s) = \frac{\dot{\lambda}(0) - \kappa}{\dot{\lambda}(s) - \kappa}
\]  

(2.78)

The yield surface in the \((q, p', s)\) stress space is given by:

\[
F = q - 2M_f \left( p' - \frac{p'^2}{p_{yo}'(s)} \right)
\]  

(2.79)

where \( p' \) is Bishop’s effective stress and \( M_f \) is the slope of the critical state line for saturated conditions.

The flow rule and hardening law are the same as in the model by Pastor et al. (1990).

### 2.3.7 Expansive soil models

The models presented so far in this section assume that the behaviour of the soil within the yield surface/surfaces is purely elastic. This is satisfactory for low plasticity soils, however as mentioned in section 2.2 high plasticity expansive soils may experience large irreversible deformations within the ‘elastic’ zone.

Gens & Alonso (1992), Alonso et al. (1994) and Alonso et al. (2000) proposed elastoplastic models for such soils within the general framework of the Barcelona Basic model. These model the expansive nature of high plasticity soils
by considering two levels of soil structure, the microstructural and the macrostructural level.

The microstructural level corresponds to the active clay minerals and their vicinity, where physicochemical interaction phenomena predominate. This level may in most cases be considered as saturated even when the soil as a whole is in an unsaturated state. The macrostructural level accounts for the larger scale structure of the soil.

Plastic deformations within what is conventionally the ‘elastic’ region are modelled by introducing two internal yield surfaces in place of the suction increase yield surface, as shown in Figure 2-37.

The research presented in this thesis focused on the behaviour of non-expansive soils, and therefore expansive soil constitutive models will not be discussed further.

Figure 2-37: Yield surfaces in a) $p, s$, and b) $q, p$ space (after Alonso et al. (1994))
2.4 SUMMARY AND DISCUSSION

2.4.1 Mechanical Behaviour

The mechanical behaviour of partially saturated soils is different to that of fully saturated soils. Some of the most important features of the behaviour of such soils are the following:

– The effective stress principle

The effective stress principle is not applicable to partially saturated soils. Changes in suction have a very different effect on the soil structure than changes in applied stress. Two independent stress variables are necessary in order to explain the behaviour of such soils.

– Volumetric behaviour

Soils desaturate at values of suction other than zero. The value of suction at which desaturation takes place is called the air entry value, $s_{air}$, and varies significantly for different soils.

Drying before desaturation takes place is equivalent to increasing confining stress and the experienced total volume changes are equal to the water volume changes. The effective stress principle is still applicable at this stage.

For low plasticity soils drying beyond the desaturation point generally causes only small reversible volumetric deformations. In the case of high plasticity expansive soils however large irreversible deformations may take place.

Partially saturated soils may either expand or collapse upon wetting if the confining stress is sufficiently low (expansion) or high (collapse), and it is also possible that a soil might experience a reversal in the volumetric behaviour during wetting (initial expansion followed by collapse).

In general if the stress state is not high enough to cause collapse upon wetting, the swelling experienced by a low plasticity non-expansive soil will be small and reversible. On the other hand high plasticity expansive clays can experience high irreversible volumetric strains.
The amount of collapse due to wetting depends on the value of the confining stress at which wetting takes place. The amount of collapse generally increases with confining stress, reaches a maximum value and then decreases to very small values at high stresses.

Hysteresis is observed in the water volume changes during wetting and drying. A soil can exist in a very different state for the same value of suction depending on the drying-wetting history.

Suction contributes to an increase in the yield stress, $p_0$, and influences both the pre-yield and post-yield compressibility. The elastic compressibility coefficient, $\kappa$, slightly decreases with suction. The elastoplastic compressibility coefficient, $\lambda$, decreases with suction at low confining stresses and increases at high confining stresses.

The volumetric behaviour is sometimes described using state surfaces. Such surfaces relate void ratio, degree of saturation or water content to net stress and suction.

– Shear strength

Suction increases the shear strength of soils. Before desaturation this increase is linear and can be explained from the effective stress principle. Beyond desaturation the increase of shear strength with suction is smaller and non-linear.

Many expressions have been proposed to describe the shear strength of partially saturated soils. Most approaches assume that the effect of suction is to increase the apparent soil cohesion. Early attempts assumed a linear increase but this has been found to be incorrect. The apparent cohesion generally increases non-linearly with suction, reaches a maximum value and then decreases slowly, tending to a very low value at very high values of suction.

Lately the increase in apparent cohesion has been related to the shape of the water retention curve. An approach has also been proposed which includes the effects of hydraulic hysteresis on the shear strength.
Experimental results have shown that the assumption that suction contributes only to an increase in apparent cohesion may not be always true. Suction may also influence the angle of shearing resistance.

### 2.4.2 Constitutive models

Elastic models are relatively easy to implement within numerical analysis and to obtain the relevant parameters, but have some major drawbacks. Most importantly no distinction is made between reversible and irreversible strains.

Elastoplastic constitutive models have been developed for both expansive and non-expansive soils. They all fall into two categories depending on the adopted stress variables; total stress models and effective stress models.

Most elastoplastic models are extensions of models for fully saturated soils and are based on the concept of the Loading-Collapse yield surface. The following elements are usually defined:

a) A yield function, which represents the surface that separates fully elastic from elastoplastic behaviour. This surface expands with increasing suction in order to model the increase of shear strength and yield stress with suction. In this way collapse due to wetting can also be reproduced.

The expansion of the yield surface is defined by the increase of the isotropic yield stress and the apparent cohesion. The increase of the yield stress with suction is related to the variation of the position and shape of the isotropic compression line in \( e \)-log\( p \) space with suction. An assumption is therefore required for either of the two. Models like the Barcelona Basic model (Alonso et al. (1990)), the Wheeler & Sivakumar (1995) model and the Bolzon et al. (1996) model make an assumption about the position and shape of the isotropic compression line, while the Josa et al. (1992) model makes an assumption about the variation of the isotropic yield stress with suction.
The Barcelona Basic model defines a second yield function, which in \((J, p, s)\) space is a vertical ‘wall’ perpendicular to the suction axis.

b) A plastic potential function, which determines the relative magnitudes of the plastic strains at each point of the yield surface. The Barcelona Basic model and the Josa et al. model assume a non-associated flow rule such that for \(K_o\) conditions no lateral strains are predicted. The Wheeler & Sivakumar model assumes an associated flow rule, giving plastic strain increments normal to the yield surface.

The plastic potential function also determines the position of the critical state line in the \((v, p, J)\) space for each value of suction.

c) A hardening/softening rule, which determines the magnitude of the plastic volumetric strains. This is defined in terms of the equivalent fully saturated isotropic yield stress since it is assumed that the process of wetting induced collapse is the same as isotropic compression beyond yield.

d) Definition of the elastic behaviour within the yield surface. The elastic compression coefficients, \(\kappa\) and \(\kappa_s\), and the elastic shear modulus, \(G\), are usually assumed to be independent of suction.

An important difference between the models presented in this chapter is in regard to the shape of the assumed/implied isotropic compression line, which has a significant effect on the predictions especially when the problem analysed involves a wide range of stresses. The isotropic compression line is non-linear for most soils; diverging from the fully saturated line at low confining stresses and converging at high stress values. The Josa et al. model does not explicitly define the shape of the isotropic line but the model formulation implies a shape along these lines. The Barcelona Basic, Wheeler & Sivakumar and Bolzon et al. models, however, assume linear isotropic compression lines. The Barcelona Basic and Bolzon et al. models assume constantly diverging lines for fully and partially saturated conditions, which is realistic for the low confining stress range. The Wheeler & Sivakumar model is more flexible since no particular relationship is assumed between either the fully and the partially saturated compressibility coefficients, \(\lambda(0)\) and \(\lambda(s)\), or the specific volume at atmospheric
pressure, $N(0)$ and $N(s)$, but it is also limited to the confining stress range at which the model parameters are obtained. The absence of a relationship between $\lambda(0)$ and $\lambda(s)$ and between $N(0)$ and $N(s)$ restricts the use of this model to constant suction stress paths. In any other case relationships need to be assumed.
Chapter 3

Finite Element Analysis

3.1 INTRODUCTION

A number of methods of analysis exist in geotechnical engineering. These fall into three broad categories; closed form, simple (e.g. limit equilibrium, stress field, limit analysis) and numerical (beam-spring and full numerical) analysis. In order to obtain an exact theoretical solution the requirements of equilibrium, compatibility, material behaviour and boundary conditions must all be satisfied. While all the methods have their respective advantages and disadvantages, only full numerical analysis satisfies all the required conditions and is therefore capable of approximating sufficiently the exact solution to any geotechnical problem (Potts & Zdravkovic (1999)).

One of the most widely used methods of full numerical analysis is the finite element method. All the developments and analyses presented in this thesis were made with the Imperial College Finite Element Program (ICFEP). ICFEP uses the displacement based finite element method and is capable of performing two-dimensional (plane strain, plane stress and axisymmetric), full three-dimensional and Fourier series aided three-dimensional analyses. Only plane strain and axisymmetric conditions are considered in this study.

This Chapter gives a brief description of the fundamental aspects of the finite element method and the modifications made during this study to account for partially saturated soil conditions. A more detailed presentation of the formulation of the method for fully saturated soils can be found in Potts & Zdravkovic (1999).
3.2 FORMULATION OF THE FINITE ELEMENT METHOD

The basic steps in the finite element method are: element discretisation, displacement approximation, formulation of element equations, assemblage of global equations, definition of boundary conditions, and solution of the global equations.

3.2.1 Element discretisation

The first step in a Finite Element analysis is to define and quantify the geometry of the boundary value problem under investigation. This is then replaced by an equivalent finite element mesh, which consists of small regions called finite elements. Finite elements are usually triangular or quadrilateral in shape, for two-dimensional problems, and their geometry is specified in terms of the coordinates of key points called nodes. The nodes in the simplest case are located at the corners of the element. Elements with additional nodes, usually located at the midpoints of the sides, can also be used. This is necessary when problems are analysed, which involve non-linear material properties and/or complex geometries. The finite elements are connected together by the element sides and a number of nodes.

3.2.2 Displacement approximation

In the displacement based finite element method the primary unknown quantity is the displacement field, which varies over the problem domain. Stresses and strains are treated as secondary quantities, which are calculated from the displacement field once it has been determined. In two-dimensional plane strain or axisymmetric analyses the displacement field is characterised by the two global displacements \( u \) and \( v \), in the \( x \) (or \( r \)) and \( y \) (or \( z \)) coordinate directions respectively.

An assumption needs to be made about the form of the variation of the displacements over the domain under investigation. The accuracy of a finite element analysis depends on the size of the elements and the nature of the
displacement approximation. This has to satisfy the compatibility conditions. Such an approximation is to assume that over each element the displacement components have a polynomial form. The displacement components are then expressed in terms of their values at the nodes:

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix} = [N] \begin{bmatrix}
  u_1, v_1, u_2, v_2, \ldots, u_i, v_i
\end{bmatrix}^T = [N] \begin{bmatrix}
  u_i
\end{bmatrix}_{\text{nodes}}
\]  

(3.1)

where \([N]\) is the matrix of shape functions, and the subscript \(i\) denotes the number of nodes of the element. The variation of displacement across the element is linear for three and four noded elements, and quadratic for six and eight noded elements.

By expressing the unknown displacements within an element as a function of the node displacements, the problem of determining the displacement field throughout the finite element mesh is reduced to determining the displacement components at a finite number of nodes. These nodal displacements are referred to as the unknown degrees of freedom.

Figure 3-1: 8-noded isoparametric element (after Potts & Zdravkovic (1999))

The analyses presented in this thesis were performed using eight noded quadrilateral isoparametric finite elements. An element of this type is shown in Figure 3-1. The global element is derived from a parent element, which has the same number of nodes but is defined with respect to a natural coordinate system.
(-1 ≤ S ≤ 1 and -1 ≤ T ≤ 1). The functions, $N_i$, used to describe the variation of the displacements within the element with respect to the values at the nodes (see Equation 3.1), are the same as those used to map the geometry of the element from the global to the natural coordinates.

The global coordinates $(x, y)$ of a point in an element can be expressed in terms of the global coordinates of the nodes $(x_i, y_i)$:

$$x = \sum_{i=1}^{8} N_i x_i \quad \text{and} \quad y = \sum_{i=1}^{8} N_i y_i \quad (3.2)$$

where $N_i$ are called interpolation functions. For isoparametric elements these are expressed in terms of natural coordinates $S$ and $T$, are equal to the shape functions and are given by (Potts & Zdravkovic (1999)):

- **Mid-side nodes:**
  - $N_5 = \frac{1}{2}(1 - S^2)(1 - T)$
  - $N_6 = \frac{1}{2}(1 + S)(1 - T^2)$
  - $N_7 = \frac{1}{2}(1 - S^2)(1 + T)$
  - $N_8 = \frac{1}{2}(1 - S)(1 - T^2)$

- **Corner nodes:**
  - $N_1 = \frac{1}{4}(1 - S)(1 - T) - \frac{1}{2} N_5 - \frac{1}{2} N_8$
  - $N_2 = \frac{1}{4}(1 + S)(1 - T) - \frac{1}{2} N_5 - \frac{1}{2} N_6$
  - $N_3 = \frac{1}{4}(1 + S)(1 + T) - \frac{1}{2} N_6 - \frac{1}{2} N_7$
  - $N_4 = \frac{1}{4}(1 - S)(1 + T) - \frac{1}{2} N_7 - \frac{1}{2} N_8 \quad (3.3)$

The main advantage of the isoparametric formulation is that the element equations need only be evaluated in the parent element coordinate system. Thus, for each element in the mesh the stiffness matrix integrals can be evaluated by a standard procedure.

### 3.2.3 Element Equations

The element equations govern the deformational behaviour of each element. They combine the compatibility, equilibrium and constitutive conditions.
• Calculation of displacements

The displacements are given by:

\[
\begin{bmatrix}
\Delta d
\end{bmatrix} = \begin{bmatrix}
\Delta u \\
\Delta v
\end{bmatrix} = [N]\begin{bmatrix}
\Delta u \\
\Delta v
\end{bmatrix}_n = [N]\begin{bmatrix}
\Delta d
\end{bmatrix}_n \tag{3.4}
\]

• Compatibility

In order to ensure compatibility (no overlapping of material and no generation of holes) the strains corresponding to the above displacements are given for plane strain conditions by:

\[
\begin{aligned}
\Delta \varepsilon_x &= -\frac{\partial \Delta u}{\partial x}, & \Delta \varepsilon_y &= -\frac{\partial \Delta v}{\partial y}, & \Delta \gamma_{xy} &= -\frac{\partial \Delta u}{\partial y} - \frac{\partial \Delta v}{\partial x}, & \Delta \varepsilon_z &= \Delta \gamma_{xz} = \Delta \gamma_{yz} = 0 \tag{3.5}
\end{aligned}
\]

and for axisymmetric conditions by:

\[
\begin{aligned}
\Delta \varepsilon_z &= -\frac{\partial \Delta u}{\partial r}, & \Delta \varepsilon_r &= -\frac{\partial \Delta v}{\partial z}, & \Delta \gamma_{rz} &= -\frac{\partial \Delta u}{\partial z} - \frac{\partial \Delta v}{\partial r}, & \Delta \varepsilon_\theta &= -\frac{\Delta \mu}{r}, \\
\Delta \gamma_{r\theta} &= \Delta \gamma_{z\theta} = 0 \tag{3.6}
\end{aligned}
\]

Combining Equations 3.4 and 3.5 or 3.6 allows the expression of the strains in terms of nodal displacements. For an element with n nodes:

\[
\{\Delta \varepsilon\} = [B]\{\Delta d\}_n \tag{3.7}
\]

where the matrix \([B]\) contains only global derivatives of the shape functions, \(\partial N_i/\partial x\) (or \(\partial N_i/\partial r\)), \(\partial N_i/\partial y\) (or \(\partial N_i/\partial z\)), and \(\{\Delta d\}_n\) contains the nodal displacements for the element. The global derivatives of the shape functions are calculated from the natural derivatives as follows:

\[
\begin{bmatrix}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y}
\end{bmatrix} = \frac{1}{|J|} \begin{bmatrix}
\frac{\partial y}{\partial T} & -\frac{\partial y}{\partial S} \\
-\frac{\partial x}{\partial T} & \frac{\partial x}{\partial S}
\end{bmatrix} \begin{bmatrix}
\frac{\partial N_i}{\partial T} \\
\frac{\partial N_i}{\partial S}
\end{bmatrix} \tag{3.8}
\]
where \( \partial N_i/\partial x \) and \( \partial N_i/\partial y \) are substituted by \( \partial N_i/\partial r \) and \( \partial N_i/\partial z \) for axisymmetric elements, and \( |J| \) is the determinant of the Jacobian matrix, which is given by:

\[
J = \begin{vmatrix}
\frac{\partial x}{\partial S} & \frac{\partial y}{\partial S} \\
\frac{\partial x}{\partial T} & \frac{\partial y}{\partial T}
\end{vmatrix}
\]  

(3.9)

• Constitutive behaviour

For linear elastic isotropic materials the constitutive behaviour can be expressed as:

\[
\{\Delta \sigma\} = [D]\{\Delta \varepsilon\}
\]  

(3.10)

where \([D]\) is the elastic constitutive matrix. For plane strain conditions:

\[
\begin{bmatrix}
\Delta \sigma_x \\
\Delta \sigma_y \\
\Delta \tau_{xy} \\
\Delta \sigma_z
\end{bmatrix} = \frac{E}{(1 + \mu)(1 - 2\mu)} \begin{bmatrix}
(1 - \mu) & \mu & 0 & \mu \\
\mu & (1 - \mu) & 0 & \mu \\
0 & 0 & (1/2 - \mu) & 0 \\
\mu & \mu & 0 & (1 - \mu)
\end{bmatrix} \begin{bmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y \\
\Delta \gamma_{xy} \\
\Delta \varepsilon_z
\end{bmatrix}
\]  

(3.11)

where \( \mu \) is the Poisson’s ratio, and \( E \) is the Young’s modulus. Non-linear elastoplastic behaviour is discussed in section 3.3.

• Element equilibrium

The element equations are determined by employing the principle of minimum potential energy. This principle states that the static equilibrium position of a loaded body is the one that minimises the total potential energy. For equilibrium:

\[
\delta \Delta E = \delta \Delta W - \delta \Delta L = 0
\]  

(3.12)

where \( \Delta W \) is the strain energy, and \( \Delta L \) is the work done by the applied loads. By combining Equations 3.4, 3.7, 3.10 and 3.12 it can be derived that the separate element equilibrium equations take the following form (Potts & Zdravkovic (1999)):
\[
[K_E] \{\Delta d\}_n = \{\Delta R_E\}
\]  

(3.13)

where \([K_E] = \int_{Vol} [B]^T [D] [B] \, dVol\) is the element stiffness matrix,

\[
[\Delta R_E] = \int_{Vol} \{N\}^T \{\Delta F\} \, dVol + \int_{Srf} \{N\}^T \{\Delta T\} \, dSrf
\]

is the right hand side load vector,

\{\Delta F\} is the vector of body forces,

\{\Delta T\} is the vector of surface tractions,

and \(Srf\) is the part of the boundary of the domain over which the surface tractions are applied.

• Numerical integration

In order to evaluate the element stiffness matrix \([K_E]\) and right hand side load vector \(\{\Delta R_E\}\), integrations over element volumes and surfaces must be performed. In most cases these cannot be done explicitly and therefore a numerical integration scheme is employed.

Numerical integration is usually performed by replacing the integral of a function, \(f(x)\), by a weighted sum of the values of the function at a number of integration points. For a one dimensional integral with \(m\) integration points:

\[
\int_{a}^{b} f(x) \, dx = \sum_{i=1}^{m} W_i f(x_i) = \sum_{i=1}^{m} W_i f(x_1) + W_i f(x_2) + \ldots + W_i f(x_m)
\]

(3.14)

where \(W_i\) are weights. The values of the weights, \(W_i\), and the location of the integration points, \(x_i\), depend on the nature of the integration scheme being used. The number of integration points determines the integration order. The accuracy of the integration process increases with the integration order, but so do the number of function evaluations and consequently the computational cost of an analysis.

The most commonly used integration scheme is Gaussian integration and the integration points are often referred to as Gauss points (Figure 3-2). For Gaussian integration the optimum integration order depends on the type of the
element being used and its shape. A 2x2 order (reduced integration) was adopted for the 8 noded isoparametric elements used in this study.

Figure 3-2: Location of Gauss points (after Potts & Zdravkovic (1999))

3.2.4 Global Equations

The next step in the formulation of the finite element equations is to assemble the separate element equilibrium equations into a set of global equations:

\[
[K_G]\{\Delta d\}_{nG} = \{\Delta R_G\}
\] (3.15)

where \([K_G]\) is the global stiffness matrix, \(\{\Delta d\}_{nG}\) is the vector of the unknown degrees of freedom (nodal displacements) for the finite element mesh, and \(\{\Delta R_G\}\) is the global right hand side load vector, which contains body force, surface traction and suction terms. The terms of the global stiffness matrix are
3.2.5 Boundary Conditions

The final stage in setting up the global system of equations is the application of the boundary conditions. These are the load and displacement conditions, which fully define the boundary value problem being analysed.

Loading conditions affect the right hand side \(\{\Delta R_G\}\) of the global system equations. Examples of such conditions are line loads, boundary stresses, pore pressure variations, body forces, and also forces from excavated and constructed elements.

Displacement conditions affect \(\{\Delta d\}_{nG}\). Such conditions must always be applied in order to ensure that no rotations or translations of the whole finite element mesh take place.

3.2.6 Solution of global equations

Having established the global stiffness matrix and the boundary conditions in the previous stage, the last remaining stage is the solution of the global equations. These form a large system of simultaneous equations. There are several different mathematical techniques for solving large systems of equations. Most finite element programs adopt a technique based on Gaussian elimination (see Potts & Zdravkovic (1999)).

3.3 NON-LINEAR FINITE ELEMENT ANALYSIS

For non-linear materials the constitutive matrix, \([D]\), is not constant. In elastoplastic analyses in particular this is replaced by the elastoplastic constitutive matrix, \([D^{ep}]\), which varies with the stresses and the state parameters.
\{k\} (see Section 3.4.1). This leads to a non-constant global stiffness matrix, \([K_G]\).

In order to obtain a solution, in this case, the change in the boundary conditions is applied in a series of increments. Equation 3.15 needs to be solved for each increment:

\[
[K_G]^i \{\Delta d\}^i_{nG} = \{\Delta R_G\}^i
\]

(3.16)

where \([K_G]^i\) is the incremental global stiffness matrix, \({\Delta d}\)\(_{nG}\) is the vector of incremental nodal displacements, \({\Delta R_G}\)\(_i\) is the vector of incremental nodal forces, and \(i\) is the increment number.

The final solution is obtained by summing the results of each increment. As the global stiffness matrix is dependent on the current stress and strain levels it not only varies between increments but also during each increment. Several different solution strategies exist. The strategy adopted for the analyses presented in this thesis was the modified Newton-Raphson method with the error controlled substepping stress point algorithm, which is briefly presented below.

### 3.3.1 Modified Newton-Raphson method

This method uses an iterative technique to solve Equation 3.16. The technique is illustrated in Figure 3-3. Equation 3.16 is solved for the first iteration using the stiffness matrix, \(K_0\), calculated form the stresses and strains at the end of the previous increment, and a first estimate of the incremental displacements, \(\Delta d^1\), is obtained. These are used to calculate the incremental strains at each integration point. The constitutive model is then integrated along the incremental strain paths and an estimate of the stress changes is obtained. These stress changes are added to the stresses at the beginning of the increment and used to evaluate consistent equivalent nodal forces. The difference between these forces and the externally applied loads gives the residual load vector, \(\psi^1\). Equation 3.16 is then solved again in the next iteration with this residual load vector forming the incremental right hand side vector:
\[
\left[K_0\right] \left(\{\Delta d\}_{n=0}\right) = \{\psi\}^{j-1}
\]

(3.17)

where \(j\) is the iteration number. For the first iteration \(\psi^0\) is given by:

\[
\{\psi\}^0 = \{\Delta R\}^i
\]

(3.18)

This process is repeated until sufficient convergence is achieved. The incremental displacements are equal to the sum of the iterative displacements. An acceleration technique is often applied, in which the iterative displacements are increased in order to reduce the number of iterations (Thomas (1984)).

In the finite element code ICFEP, convergence is checked by setting criteria for both the iterative displacements, \(\{\Delta d\}_{n=0}\), and the residual loads, \(\{\psi\}^i\). These are checked against the incremental and accumulated displacements and global right hand side load vectors, respectively. The convergence criteria are set such that the scalar norm of the iterative displacement vector is less than 1% of both.

Figure 3-3: Application of the modified Newton-Raphson algorithm to the uniaxial loading of a bar of a nonlinear material (after Potts & Zdravkovic (1999))
the incremental and accumulated displacement norms, and the norm of the residual load vector is less than 1-2% of both the incremental and accumulated global right hand side load vector norms. When boundary value problems are analysed, which involve only displacement boundary conditions, the incremental and accumulated right hand side load vectors are zero. Checking the residual loads against these vectors is therefore not appropriate. Such a check is therefore often overridden but the absolute value of the norm of the residual load is kept small.

3.3.2 Stress point algorithm

The integration of the constitutive equations, in order to obtain the stress changes during each iteration, is performed using a stress point algorithm. There are many of these algorithms in use. A substepping algorithm was adopted in this study. In this approach the elastic and elastoplastic proportions of the strain increment are first calculated. If linear elasticity is assumed, the stress changes at the end of the elastic proportion of the iteration are directly calculated. The elastoplastic proportion of the incremental strains is then divided into a number of substeps. The constitutive equations are integrated numerically over each substep using a modified Euler integration scheme with error control, in order to obtain the stress changes at the end of the iteration. The procedure is described by Potts & Zdravkovic (1999) and is based on the scheme presented by Sloan (1987). In this study non-linear elasticity was assumed, and a similar substepping scheme was employed for the elastic proportion of the iteration.

3.3.3 Correction for yield surface drift

The situation may arise if the tolerance used to control the size of the substeps is too large compared to the yield function tolerance, that the combination of the stresses and hardening/softening parameters after each substep may not satisfy the yield condition:
This phenomenon (illustrated in Figure 3-4) called the yield surface drift must be corrected because it can lead to cumulative error. Several methods of addressing this problem exist. The one adopted is based on the approach presented by Potts & Gens (1985) and is described by Potts & Zdravkovic (1999). This involves an iterative procedure in which the correct stresses and position of the yield surface are calculated by assuming that the elastic strains arising from the stress path from point B (Figure 3-4) to the corrected stress point are equal and opposite to the plastic strains.

Figure 3-4: Yield surface drift (after Potts & Zdravkovic (1999))

### 3.4 PARTIALLY SATURATED FINITE ELEMENT ANALYSIS

Strain changes are related to both total stress and pore water pressure changes. For partially saturated soils these cannot be dealt with in a unified effective stress approach. The formulation of the finite element method therefore needs to be modified. The necessary modifications are presented in the following sections.
3.4.1 Constitutive behaviour

As discussed in Chapter 2 two independent stress variables are required in order to describe the behaviour of partially saturated soils. The stress variables adopted in this study are net stress, $\sigma = \sigma - u_a$, and equivalent suction, $s_{eq} = u_a - u_w - s_{air}$. $s_{air}$ is the air entry value of suction, which separates fully from partially saturated behaviour. The immediate implication of the adoption of two independent stress variables is that the stress-strain relationship can no longer be expressed in terms of a single stiffness matrix $[D]$ (or $[D^p]$).

The total strains have two components; strains due to changes in net stress and strains due to changes in equivalent suction. In the general elastoplastic case this can be written as follows:

$$
\{\Delta \varepsilon\} = \{\Delta \varepsilon^e\} + \{\Delta \varepsilon^p\} + \{\Delta \varepsilon_s^e\} + \{\Delta \varepsilon_s^p\} \quad (3.20)
$$

where $\{\Delta \varepsilon^e\}$ and $\{\Delta \varepsilon^p\}$ are the elastic and plastic incremental strains due to changes in net stress, and $\{\Delta \varepsilon_s^e\}$ and $\{\Delta \varepsilon_s^p\}$ are the incremental elastic and plastic strains due to changes in equivalent suction.

The relationship between the elastic strain and net stress changes is given by the constitutive model and can be calculated from:

$$
\{\Delta \sigma\} = [D]\{\Delta \varepsilon^e\} \quad (3.21)
$$

where $[D]$ is the elastic constitutive matrix.

Substituting Equation 3.21 into Equation 3.20 gives:

$$
\{\Delta \sigma\} = [D]\left[\{\Delta \varepsilon\} - \{\Delta \varepsilon^p\} - \{\Delta \varepsilon_s^e\} - \{\Delta \varepsilon_s^p\}\right] \quad (3.22)
$$

The incremental plastic strains are equal to:

$$
\{\Delta \varepsilon^p\} = \Lambda \left[ \frac{\partial G}{\partial \sigma} \right] \quad (3.23)
$$
where $\Lambda$ is the plastic strain multiplier and $G$ is the plastic potential function. Equation 3.22 therefore becomes:

$$\{\Delta \sigma\} = [D]\left(\{\Delta \varepsilon\} - \Lambda\left[\frac{\partial G}{\partial \sigma}\right] - \{\Delta \varepsilon_s\} - \{\Delta \varepsilon_p\}\right)$$

(3.24)

or alternatively,

$$\{\Delta \sigma\} = [D]\left(\{\Delta \varepsilon\} - \{\Delta \varepsilon_s\} - \{\Delta \varepsilon_p\}\right) - [D]A\left[\frac{\partial G}{\partial \sigma}\right]$$

(3.25)

The consistency condition for partially saturated soils includes three terms, instead of the two terms required for the conventional analyses of fully saturated soils, since the yield function is now also dependent on suction:

$$dF = \left[\frac{\partial F}{\partial \sigma}\right]^T \{\Delta \sigma\} + \left[\frac{\partial F}{\partial k}\right]^T \{\Delta k\} + \left[\frac{\partial F}{\partial s}\right]^T \{\Delta s\} = 0$$

(3.26)

where $\{\Delta k\}$ is the change in the state parameters, $\{\Delta s\}$ is the change in suction ($=\{\Delta s_{eq}\}$), and $F$ is the yield function.

Combining Equations 3.25 and 3.26 and solving for $\Lambda$ gives:

$$\Lambda = \frac{\left[\frac{\partial F}{\partial \sigma}\right]^T \{\Delta \sigma\} + \left[\frac{\partial F}{\partial k}\right]^T \{\Delta k\} + \left[\frac{\partial F}{\partial s}\right]^T \{\Delta s\}}{\left[\frac{\partial F}{\partial \sigma}\right]^T [D]\left[\frac{\partial G}{\partial \sigma}\right] + A}$$

(3.27)

where

$$A = -\frac{1}{\Lambda} \left[\frac{\partial F}{\partial k}\right]^T \{\Delta k\}$$

(3.28)

Substituting Equation 3.27 into Equation 3.25 gives:

$$\{\Delta \sigma\} = [D^p] \cdot \left(\{\Delta \varepsilon\} - \{\Delta \varepsilon_s\} - \{\Delta \varepsilon_p\}\right) - [W] \cdot \{\Delta s\}$$

(3.29)
where
\[
[D^p] = [D] - [D]\left[\frac{\partial G}{\partial \sigma}\right]^{T}\left[\frac{\partial F}{\partial \sigma}\right] + A
\]

(3.30)

is the elastoplastic matrix in the case of fully saturated soils,

and
\[
[W] = [D]\left[\frac{\partial G}{\partial \sigma}\right]^{T}\left[\frac{\partial F}{\partial \sigma}\right] + A
\]

(3.31)

3.4.2 Formulation and solution of global equations

As discussed in Section 3.3, for non-linear analyses the change in the boundary conditions is applied in a series of increments and the system of global equations is solved for each increment (Equation 3.16). Since the incremental global stiffness matrix, \([K_G]_i\), varies during each increment, the direct solution of Equation 3.16 is in error and an iterative technique needs to be employed. In the modified Newton-Raphson method, it is during this iterative procedure that the constitutive equations are satisfied in order to obtain the correct solution. The global equations can therefore by formulated by assuming elastic behaviour. This is common practice in conventional (fully saturated) analyses as it significantly reduces computational time.

A similar approach was adopted in the study presented in this thesis for the partially saturated case. Equation 3.10 was used instead of Equation 3.22 in order to formulate the global equations. These were therefore of the form given by Equation 3.15. Equation 3.22 was solved during the iterative solution technique, discussed above, to obtain the correct solution. The steps followed during this solution (using the equations presented in the previous section) are shown in the form of a flow chart in Appendix I.
3.5 SUMMARY

The finite element method involves the following steps: element discretisation, displacement approximation, formulation of element equations, assemblage of global equations, definition of boundary conditions, and solution of the global equations.

The analyses presented in this thesis were performed using eight-noded quadrilateral isoparametric elements with reduced integration. The definition of this type of elements was given in this chapter.

The deformational behaviour of each element is defined from the element equilibrium equations. These are then assembled to form the global equations. Modifications need to be made to the formulation of the element and global equations when partially saturated soil analyses are performed. These modifications are due to the necessary use of two independent stress variables for the description of partially saturated soil behaviour and affect the right hand side load vector.

When non-linear soil behaviour is analysed, the boundary conditions are applied in increments and the global equations are solved for each increment. The solution of the incremental global equations is not straightforward and a solution strategy needs to be employed. The modified Newton-Raphson method in conjunction with a substepping stress point algorithm and a modified Euler integration scheme was used in the analyses presented in this thesis.
Chapter 4

Development of two Constitutive Models

4.1 INTRODUCTION

This chapter presents two constitutive models for partially saturated soils developed during this research project. The initial aim was to produce one constitutive model in order to undertake Finite Element analysis of boundary value problems, which involve partial soil saturation. The Barcelona Basic model was selected as a starting point because of its wide acceptance and its relative simplicity. During the course of the work, a number of modifications were made to the model, in order to reproduce better soil behaviour, which were based on the laboratory observations available in the literature and the results of the performed numerical analyses. The final product was two models, which have many similarities but a fundamental difference regarding the assumed partially saturated isotropic compression line. Both models are based on the Loading-Collapse yield surface concept.

4.2 FORMULATION OF THE CONSTITUTIVE MODELS

4.2.1 Stress invariants

The applicability of the effective stress principle was discussed in Chapter 2. Soil behaviour can be explained and modelled using effective stresses until a particular value of suction is reached, called the air entry value of suction, $s_{air}$. Beyond this value the soil becomes partially saturated and two independent stress variables are required. A convenient choice of stress variables is the pair of net total stress, $\sigma - u_a$, and equivalent suction, $s_{eq}$, which can be defined as the value of suction, $s = u_e - u_w$ in excess of the air entry value, $s_{air}$.
\[ s_{eq} = s - s_{air} \quad (4.1) \]

The developed constitutive models are formulated in four-dimensional stress space \( (J, p, \theta, s_{eq}) \):

- \( J \) is the generalised deviatoric stress:
  \[ J = \frac{1}{\sqrt{2}} \left[ (\sigma_x - p)^2 + (\sigma_y - p)^2 + (\sigma_z - p)^2 + 2\tau_{xy}^2 + 2\tau_{xz}^2 + 2\tau_{yz}^2 \right]^{\frac{1}{2}} \quad (4.2) \]

- \( p \) is the mean stress:
  \[ p = (\sigma_x + \sigma_y + \sigma_z) / 3 \quad (4.3) \]

- \( \theta \) is the Lode’s angle:
  \[ \theta = -\frac{1}{3} \sin^{-1} \left[ \frac{3\sqrt{3}}{2} \frac{\text{det} s}{J^3} \right] \quad (4.4) \]

where \( \text{det} s \) is the determinant of the stress matrix:

\[
\text{det} s = \begin{vmatrix}
\sigma_x - p & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y - p & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z - p
\end{vmatrix} \quad (4.5)
\]

- \( s_{eq} \) is the equivalent suction.

The models switch from fully to partially saturated behaviour and vice versa at the air entry value of suction \( s_{air} \).

For fully saturated conditions: \( s_{eq} = 0 \) and \( \sigma_x, \sigma_y \) and \( \sigma_z \) are effective stresses.

For partially saturated conditions: \( s_{eq} = s - s_{air} \) and \( \sigma_x, \sigma_y \) and \( \sigma_z \) are net total stresses.
4.2.2 Yield Function and Plastic Potential Surface

As discussed in Chapter 2, most constitutive models for partially saturated soils are extensions of fully saturated critical state constitutive models. The extension can be done by substituting effective stress terms with total stress terms, and including a suction invariant (suction or equivalent suction). The model must be formulated such that it switches from fully to partially saturated behaviour when the suction invariant becomes equal to zero.

a) Fully saturated conditions

The Lagioia et al. (1996) expression is adopted for the yield function and plastic potential surface:

\[
\begin{align*}
F_1 &= \frac{p'}{p'_c} \left( 1 + \frac{\eta}{K_2} \right)^{\frac{K_2}{\beta_f}} \left( 1 + \frac{\eta}{K_1} \right)^{\frac{K_1}{\beta_f}} = 0 \\
G_1 &= \frac{c'}{\gamma} \\
K_1, \ K_2, \ \beta_f &\text{are given by}
\end{align*}
\]

where the constants \( K_1, K_2 \) and \( \beta_f \) are given by

\[
K_{1,2} = \frac{\mu(1-\alpha)}{2(1-\mu)} \left( 1 \pm \sqrt{1 - \frac{4\alpha(1-\mu)}{\mu(1-\alpha)^2}} \right) \\
\beta_f = (1-\mu)(K_1 - K_2)
\]

\( \alpha \) and \( \mu \) are parameters which control the shape of the surfaces,

\( p'_c \) is the isotropic effective yield stress,

\( \eta \) is the generalised normalised stress ratio,

\[
\eta = \sqrt{\frac{J_{2\eta}}{J_{2\eta'}}} \\
J_{2\eta} \text{ is the square of the stress ratio,}
\]
\[ J_{2\eta} = \left( \frac{J}{p'} \right)^2 \]  

(4.10)

\( J_{2\eta} \) is the failure value of \( J_{2\eta} \) and is obtained by solving the following cubic equation which is based on the Matsuoka-Nakai criterion (Potts & Zdravkovic (1999)):

\[
\frac{2}{\sqrt{27}} C \cdot \sin(3\theta) \cdot J_{2\eta}^{\frac{3}{2}} + (C - 3) \cdot J_{2\eta} - (C - 9) = 0 \tag{4.11}
\]

in which

\[
C = \frac{9 - M^2}{2M^2 + M^2 - \frac{M^2}{3} + 1} \tag{4.12}
\]

where \( M \) is the gradient of the critical state line in the conventional \( q-p \) space, corresponding to triaxial compression (\( \theta = -30^\circ \)).

The parameters \( \alpha, \mu \) and \( M \) are equal to \( \alpha_f, \mu_f \) and \( M_f \) when the yield surface is being calculated and equal to \( \alpha_g, \mu_g \) and \( M_g \) when the plastic potential surface is being calculated. Similarly \( J_{2\eta} \) is equal to \( J_{2\eta f} \) or \( J_{2\eta g} \). It should be noted that for triaxial compression \( J_{2\eta} = M^2/3 \). An associated flow rule can be assumed if the same values are chosen for yield and plastic potential surface parameters.

Equation 4.6 is the product of the integration of the following equation:

\[
\frac{dp'}{p'} = -\frac{d(\sqrt{J_{2\eta}})}{d + \sqrt{J_{2\eta}}} \tag{4.13}
\]

where \( d \) is the dilatancy (\( = \frac{d\varepsilon'}{d\varepsilon_p} \)). The variation of dilatancy with the stress ratio is selected such that:

\[
\begin{align*}
\sqrt{J_{2\eta}} &\rightarrow 0 \quad \Rightarrow \quad d \rightarrow \infty \\
\sqrt{J_{2\eta}} &= \sqrt{J_{2\eta g}} \quad \Rightarrow \quad d = 0
\end{align*}
\]

(4.14)

and given by:
\[ d = \mu \left( \sqrt{J_{2\eta g}} - \sqrt{J_{2n}} \right) \left( \frac{\alpha \sqrt{J_{2\eta g}}}{\sqrt{J_{2n}}} + 1 \right) \] (4.15)

A graphical representation of the above equation and the physical significance of the parameters are shown in Figure 4-1.

A major advantage of this yield surface and plastic potential expression is that by suitable adjustment of the parameters \( \alpha \) and \( \mu \), a wide range of surface shapes, including the shapes given by more commonly used yield surface and plastic potential functions, can be achieved (Figure 4-2). The values of the parameters \( \alpha \) and \( \mu \) can be adjusted to fit experimental data or to model specific material behavior.
and \( \mu \) used to produce the surface shapes plotted in Figure 4.2 are listed in Table 4.1.

![Figure 4-2: Examples of yield surface and plastic potential functions reproduced from Lagioia et al. (1996)](image)

b) Partially saturated conditions

The Lagioia et al. (1996) expression can be extended to the equivalent suction, \( s_{eq} \), space by substituting the isotropic effective yield stress, \( p'_{cs} \), with \( p_o + f(s_{eq}) \) and the mean effective stress, \( p' \), with \( p + f(s_{eq}) \):

\[
\frac{F_1}{G_1} = \frac{p + f(s_{eq})}{p_o + f(s_{eq})} \left(1 + \frac{\eta}{K_2}\right) = 0
\]

\[\text{(4.16)}\]

where \( p_o \) is the isotropic total yield stress at the current value of suction, and \( f(s_{eq}) \) is a measure of the increase in apparent cohesion due to suction (Figure 4-3). The normalised stress ratio is calculated from Eq.4.9 in which the square of the stress ratio, \( J_{2\eta} \), is now given by:
\[ J_{2\eta} = \left( \frac{J}{p + f(s_{eq})} \right)^2 \]  

(4.17)

The slope of the lines corresponding to \( dF/dp = 0 \) and \( dG/dp = 0 \) in the \( J-p \) plane, \( M_{ji} = \sqrt{J_{2\eta}} \), is assumed to be independent of suction.

Figure 4-3: Yield function and plastic potential surface for partially saturated conditions

\[ \sqrt{J_{2\eta}} f(s_{eq}) \]

\[ J_{ci} = M_{ji} f(s_{eq}) \]  

(4.18)

Two options exist for \( f(s_{eq}) \) for both models:

Option 1: \( f(s_{eq}) = k s_{eq} + s_{air} \), where \( k \) is a constant, thus giving a linear increase of apparent cohesion with equivalent suction. This option is similar to the Barcelona Basic model assumption and is realistic only for low values of suction. The critical state line in this case is given by the following equation,
Option 2: \( f(s_{eq}) = S_r \cdot s_{eq} + s_{air} \), where \( S_r \) is the degree of saturation. In this case the apparent cohesion initially increases with suction, reaches a peak at some value of suction, which depends on the shape of the adopted soil-water characteristic curve, and then reduces to a small value at very high values of suction. This option is more realistic but requires knowledge of the relationship of the degree of saturation with suction. The critical state line in this case is given by the following equation,

\[
J = M_{JG} \cdot \left( p + S_r \cdot s_{eq} + s_{air} \right)
\]  
(4.20)

which in the case of \( s_{air} = 0 \) gives a shear strength increase equivalent to that predicted by an effective stress approach using Bishop’s effective stress with the parameter \( \chi \) set equal to \( S_r \).

Figure 4-4 shows examples of yield surface and plastic potential functions reproduced by Equation 4.16.

![Graph showing yield surface and plastic potential functions](image-url)
The Suction Increase yield surface introduced in the Barcelona Basic model is also included in the two models presented here:

\[ F_2 = G_2 = \frac{S_{eq}}{s_o} - 1 = 0 \]  \hspace{1cm} (4.21)

where \( s_o \) (yield suction) is a limit value of suction beyond which plastic strains occur. The existence of such a yield surface is supported by limited experimental data and it may not be applicable to many partially saturated soils. The use of it is therefore optional. If a very large value is given to \( s_o \) the secondary yield surface becomes irrelevant.

### 4.2.3 Isotropic compression line and yield stress

The isotropic compression line for fully saturated conditions is given by:

\[ v = v_l - \lambda(0) \ln p'_c \]  \hspace{1cm} (4.22)

where \( v \) and \( v_l \) are the values of the specific volume at the current state and at mean effective stress, \( p' \), of 1kPa, respectively, and \( \lambda(0) \) is the fully saturated compressibility coefficient.

The isotropic yield stress, \( p'_c \), is the hardening/softening parameter, which controls the size of the yield surface (Eq.4.6) and the magnitude of the volumetric plastic strains associated with expansion (hardening) or contraction (softening) of the yield surface.

For partially saturated conditions the hardening/softening parameter is a net stress defined as the equivalent fully saturated isotropic yield stress at the transition from fully to partially saturated conditions \( (s = s_{air}) \):

\[ p^*_o = p'_c - s_{air} \]  \hspace{1cm} (4.23)

The relationship between the partially saturated yield stress, \( p_o \), and the equivalent fully saturated yield stress, \( p^*_o \), defines the shape of the yield surface.
in the isotropic stress space $p - s$ (Figure 4-5) and depends on the assumed shape of the isotropic compression line.

![Diagram of Primary yield surface in isotropic stress space](image)

Figure 4-5: Primary yield surface in isotropic stress space

The choice of the shape of the isotropic compression line for partially saturated conditions has a very significant influence on the performance of the constitutive models. This is the main difference between the two models presented in this chapter.

4.2.3.1 Constitutive Model 1 - Partially saturated isotropic compression line and yield stress

There are two options for model 1 with regard to the choice of the isotropic compression line. The first option is to assume a linear partially saturated isotropic compression line constantly diverging from the fully saturated isotropic compression line, as shown in Figure 4-6. This option is the same as that proposed by Alonso et al. (1990) in the Barcelona Basic model and has been adopted in many other models, such as the Bolzon et al. (1996), Cui & Delage (1995), Modaressi & Abou-Bekr (1994) models. The isotropic compression line for this option is given by:
\[ \nu = v_i \left( s_{eq} \right) - \lambda \left( s_{eq} \right) \ln p_o \]  

(4.24)

where \( \lambda(s_{eq}) \) is the partially saturated compressibility coefficient. \( \lambda(s_{eq}) \) is given by the following empirical expression (Alonso et al. (1990)),

\[ \lambda \left( s_{eq} \right) = \lambda \left( 0 \right) \left[ \left( 1 - r \right)e^{-\beta s_{eq}} + r \right] \]  

(4.25)

where \( \beta \) and \( r \) are model parameters which control the shape of the primary yield and plastic potential surfaces in the \( p-s_{eq} \) plane.

![Figure 4-6: Assumed isotropic compression lines – Model 1 (option 1)](image)

This assumption for the isotropic compression line leads, through the same calculations as those described in Alonso et al. (1990), to the following expression relating the isotropic yield stress, \( p_o \), to the equivalent fully saturated yield stress, \( p_o^* \):

\[ p_o = p^c \cdot \left( \frac{p_o^*}{p^c} \right)^{\lambda(s_{eq}) / \lambda(0)} \]  

(4.26)
where $p^c$ is a characteristic pressure, $\lambda(0)$ is the fully saturated compressibility coefficient, and $\kappa$ is the compressibility coefficient along elastic paths and is assumed to be independent of suction. The characteristic pressure, $p^c$, is the mean net stress at which it is assumed that a soil initially lying on the isotropic compression line ($p = p_o$) can follow a constant mean net stress wetting path and experience only elastic swelling. This assumption implies that when $p_o^* = p^c$ the projection of the primary yield surface in the isotropic ($p - s_{eq}$) plane, is a straight line perpendicular to the mean net stress, $p$, axis. The equivalent fully saturated yield stress, $p_o^*$, cannot have values lower than $p^c$.

Equation (4.24) implies that the amount of potential collapse due to wetting (vertical distance between the fully and partially saturated lines in the $v$-ln$p$ plane) increases linearly with the increase of the logarithm of the confining stress, $p$. This is a realistic assumption for the low confining stresses at which many laboratory tests on partially saturated soils are performed, but may give unrealistically high values of the yield stress, $p_o$, and the wetting induced volumetric plastic strains, at high confining stresses.

The characteristic pressure $p^c$ is an arbitrary parameter the value of which is selected such that the shape of the Loading-Collapse yield curve matches the experimental data, and is assumed to be constant and unique for a particular soil. However, to avoid inconsistencies at high stress levels it would appear that $p^c$ must be stress level dependent. An alternative approach is to assume that the ratio $p_o^*/p^c$ is constant for confining stress ranges higher than those at which the experiments where performed. Adopting this approach (option 2), the expression for the primary yield surface in the isotropic stress-suction space becomes,

$$p_o = p_o^* \cdot \alpha_c^{(\lambda(0)-\lambda(s_{eq}))/\lambda(s_{eq})}$$

(4.27)

where, $\alpha_c = p_o^*/p^c$ is a model parameter. The partially saturated isotropic compression line for the second option is bi-linear and is shown in Figure 4-7. At low confining stresses expression (4.26) is adopted giving a linear increase of the amount of collapse with stress, while at high confining stresses expression (4.27) is adopted giving a constant amount of collapse.
The switch from expression (4.26) to expression (4.27) takes place when the two expressions are equal. It can be shown that the confining stress, $p_m$, at which this switch takes place, is given by,

$$ p_m = p^c \cdot \alpha_c \left( \frac{\lambda(0) - \kappa}{\lambda(s_{eq})} \right) $$

Expression (4.28)

$p_m$, depends on the parameter $\alpha_c$ as does the amount of maximum potential collapse expressed in terms of specific volume, $\Delta v_{\text{max}}$. This can be calculated as the vertical distance between the partially and fully saturated isotropic compression lines at a mean net stress value equal to $p_m$:

$$ \Delta v_{\text{max}} = v(s_{eq}) - v(0) $$

$$ = N(s_{eq}) - \lambda(s_{eq}) \ln \frac{p_m}{p^c} - \left( N(0) - \lambda(0) \ln \frac{p_m}{p^c} \right) $$

Expression (4.27)

$$ = \left( \frac{\lambda(0) - \kappa}{\lambda(s_{eq}) - \kappa} \right) \ln \alpha_c - \kappa_s \cdot \ln \frac{s_{eq} + p_{\text{atm}}}{p_{\text{atm}}} $$

Expression (4.26)
where $N(0)$ and $N(s_{eq})$ are the fully and partially saturated values of specific volume at $p_o^* = p_o = p_c$, respectively. Their difference is equal to the elastic swelling due to saturation at constant mean net stress, $p$, as discussed in Chapter 2 (Equation 2.32). $p_{atm}$ is the atmospheric pressure and $\kappa_s$ is the compressibility coefficient for suction changes within the elastic domain.

The isotropic compression line for stresses beyond, $p_m$, is given by:

$$\nu = \nu_i + \Delta \nu_{max} - \lambda(0) \ln p_o$$  \hspace{1cm} (4.30)

This approach by no means reproduces accurately soil behaviour, but is much closer than option 1 to experimental results at high confining stresses (see Section 2.2.2.1) which show that the amount of potential collapse increases with stress at low confining stresses, reaches a maximum and then reduces at high stresses as indicated by the dashed line in Figure 4-7.

4.2.3.2 Constitutive Model 2 - Partially saturated isotropic compression line and yield stress

The influence of confining stress on the amount of potential collapse due to wetting was discussed in Chapter 2 and mentioned in the previous section. Experimental results demonstrating this influence were shown in Figures 2-10 and 2-11. Most constitutive models for partially saturated soils ignore this feature of behaviour and assume linear relationships for the isotropic compression lines, giving either linear increase (e.g. Barcelona Basic model) or decrease (e.g. Wheeler & Sivakumar model) of the amount of collapse with the logarithm of confining stress. This can lead to very inaccurate predictions when problems are analysed which involve stress ranges larger than those for which the models are intended. An example of such a case will be given in Chapter 7. The Josa et al. (1992) model assumes a particular relationship between isotropic yield stress and equivalent fully saturated yield stress, which implies a non-linear partially saturated compression line and is capable of modelling the phenomenon of maximum collapse. The form of the isotropic compression line, however, is not given explicitly. The model presented in this section provides such an
expression from which the shape of the yield surface in the \((p, s_{eq})\) space is then derived.

- Derivation of isotropic compression line

The idealised relationship between the amount of potential plastic reduction of the specific volume, \(\Delta v_p\), due to wetting of a partially saturated soil lying on the isotropic compression line, and the mean net stress, \(p\), is given in Figure 4-8. A mathematical expression of this form is the following:

\[
\Delta v_p = \lambda_m \left( \frac{p}{p^c} \right)^{-b} \ln \frac{p}{p^c}
\]

(4.31)

where \(\lambda_m\) and \(b\) are model parameters, and \(p^c\) is the characteristic pressure defining the limiting lower value of the equivalent fully saturated yield stress, \(p_{oc}^*\), for which the projection of the yield surface in the \((p, s_{eq})\) plane is a vertical line.

Figure 4-8: Variation of potential plastic reduction of specific volume due to wetting with isotropic yield stress
The vertical distance in the $v$-$lnp$ plane between the partially and fully saturated isotropic compression lines is equal to the specific volume changes experienced by the soil during wetting under constant confining stress:

$$\Delta v = \Delta v_p - \Delta v_e$$  \hspace{1cm} (4.32)

where $\Delta v_e$ is the specific volume change due to elastic swelling and is assumed to be independent of confining stress.

$$\Delta v_e = \kappa_s \ln \frac{s_{eq} + p_{atm}}{p_{atm}}$$  \hspace{1cm} (4.33)

As in the Barcelona Basic model, the atmospheric pressure, $p_{atm}$, is introduced in order to avoid the calculation of infinite strains as the equivalent suction tends to zero.

The isotropic compression lines are shown in Figure 4-9 and are given as follows:

Fully saturated: 

$$v = N(0) - \lambda(0) \ln \frac{p^*}{p^c}$$  \hspace{1cm} (4.34)

Partially saturated: 

$$v = N(s_{eq}) - \lambda(0) \ln \frac{p^*}{p^c} + \lambda_m \left(\frac{p^*}{p^c}\right)^{-b} \ln \frac{p^*}{p^c}$$  \hspace{1cm} (4.35)

When $p_o$ is equal to the characteristic pressure or has a very high value the partially saturated line lies below the fully saturated line. The vertical distance between the two lines for these limiting situations is given by:

$$\Delta v = \Delta v_e = N(0) - N(s_{eq}) = \kappa_s \ln \frac{s_{eq} + p_{atm}}{p_{atm}}$$  \hspace{1cm} (4.36)
• Calculation of the slope of the isotropic compression line

The slope of the partially saturated compression line at any value of $p_o$ is calculated as follows:

$$\frac{dv}{dp_o} = -\lambda\left(s_{eq}\right) \frac{dp_o}{p_o} \Rightarrow \lambda\left(s_{eq}\right) = -p_o \frac{dv}{dp_o}$$  \hspace{1cm} (4.37)

Differentiation of Equation 4.35 with respect to $p_o$ gives:

$$\frac{dv}{dp_o} = -\frac{1}{p_o} \left[ \lambda(0) + \lambda_m b \left( \frac{p_o}{p_c} \right)^{-b} \ln \frac{p_o}{p_c} - \lambda_m \left( \frac{p_o}{p_c} \right)^{-b} \right]$$  \hspace{1cm} (4.38)

From Equations 4.37 and 4.38:

$$\lambda\left(s_{eq}\right) = \lambda(0) - \lambda_m \left( \frac{p_o}{p_c} \right)^{-b} \left( 1 - b \ln \frac{p_o}{p_c} \right)$$  \hspace{1cm} (4.39)
• Definition of parameter $\lambda_m$

The initial slope of the isotropic compression line, $\lambda_{in(s_{eq})}$, is obtained by setting $p_o = p^*$ in Equation 4.39:

$$\lambda_m(s_{eq}) = \lambda(0) - \lambda_m$$  \hspace{1cm} (4.40)

The parameter $\lambda_m$ is therefore a measure of the soil stiffness at low confining stresses and is dependent on equivalent suction. It is assumed that the initial slope of the isotropic compression line, $\lambda_{in(s_{eq})}$, is given by Equation 4.25. $\lambda_m$ is obtained from combination of Equations 4.25 and 4.40 as follows:

$$\lambda_m = \lambda(0)(1-r)(1-e^{-\beta_{eq}})$$ \hspace{1cm} (4.41)

• Calculation of point and amount of maximum collapse

The point and amount of maximum collapse is obtained by differentiation of Equation 4.31:

$$\frac{d \Delta\nu_p}{d p_o} = \frac{\lambda_m(p_o)}{p_o} \left( \frac{p_o}{p^*} \right)^{-b} \left( 1 - b \ln \frac{p_o}{p^*} \right) = 0$$ \hspace{1cm} (4.42)

The value of $p_o$ that satisfies the above equation is the value at which maximum collapse takes place:

$$p_m = p^* e^{\frac{b}{b}}$$ \hspace{1cm} (4.43)

Substitution of Equation 4.43 into Equation 4.31 gives the value of the maximum potential plastic reduction of specific volume due to wetting:

$$\Delta\nu_{p_{max}} = \frac{\lambda_m}{eb}$$ \hspace{1cm} (4.44)

• Definition of parameter $b$

The parameter $b$ controls the value of the isotropic yield stress that corresponds to the maximum wetting induced potential collapse. From Equation 4.43 it follows that:
\[ b = \left( \ln \frac{p_a}{p^*} \right)^{-1} \]  

(4.45)

- Shape of the yield surface in the isotropic \((p, s)\) stress space

Consider an elastic stress path \(1 \rightarrow 2 \rightarrow 3\) joining two points lying on the same yield surface, as shown in Figure 4-10. The values of the specific volume at points 1 and 3 are given by Equations 4.35 and 4.34, respectively, and are related through the following equation:

\[ v_3 = v_1 + \Delta v_{12} + \Delta v_{23} \]  

(4.46)

Elastic unloading (1→2):

\[ \Delta v_{12} = \kappa \ln \frac{p_o}{p_o^*} \]  

(4.47)

Elastic wetting (2→3):

\[ \Delta v_{23} = \kappa \ln \frac{s_{eq} + p_{atm}}{p_{atm}} \]  

(4.48)

Figure 4-10: Fully and partially saturated isotropic compression lines – relationship between \(p_o\) and \(p_o^*\) for Model 2
The relationship between the partially saturated and the equivalent fully saturated isotropic yield stress is obtained by substituting Equations 4.34, 4.35, 4.47 and 4.48 into Equation 4.46:

\[
N(0) - \lambda(0) \ln \frac{p^*}{p^c} = N(s_{eq}) - \lambda(0) \ln \frac{p_o}{p^c} + \lambda_m \left( \frac{p_o}{p^c} \right)^{-b} \ln \frac{p_o}{p^c} + \kappa \ln \frac{p_o}{p^c} + \kappa \ln \frac{s_{eq} + p_{atmo}}{p_{atmo}}
\]  

(4.49)

\(N(0)\) and \(N(s_{eq})\) are related through Equation 4.36. Equation 4.49 can therefore be simplified to:

\[
p^*_o = p^c x^{\left(1 + \frac{\lambda(0)-\kappa}{\lambda(0)-\kappa} \right)} , \text{ where } x = \frac{p_o}{p^c}
\]  

(4.50)

### 4.2.4 Critical State

For a given value of equivalent suction, \(s_{eq}\), the soil reaches a critical state (\(d\varepsilon_p=0\)) along a single line in the \(v\)-ln\(p\)-\(J\) space, called the critical state line (as shown in Figure 2-29). The position of this line depends on the shape of the plastic potential surface, which is given by Equation 4.16, and varies for different values of the plastic potential parameters and the equivalent suction. The critical state along an elastic loading/unloading line (for a given value of yield stress, \(p_o\)) is represented by a single point, as shown in Figure 4-11.

The mean net stress that corresponds to the critical state point is calculated by setting the normalised stress ratio, \(\eta\), equal to 1 in Equation 4.16:

\[
G_t = \frac{p_{eq} + f(s_{eq})}{p_o + f(s_{eq})} \left( \frac{1 + \frac{1}{K_{2g}}}{p_{eq}} \right) = 0
\]  

(4.51)
The deviatoric stress at critical state is given by:

\[
p_{cs} = \left( p_o + f(s_{eq}) \right) \left( \frac{1 + \frac{1}{K_{2g}}}{\frac{\kappa_{2g}}{\rho_{2g}}} \right) - f(s_{eq})
\]  \hspace{1cm} (4.52)

In order to fully describe the critical state line, the relationship between specific volume and confining stress needs to be defined (Figure 4-12).

From the unloading path 1→2:

\[
v_{cs} = v(p_o) - \kappa \ln \frac{p_{cs}}{p_o}
\]  \hspace{1cm} (4.54)

where \( v(p_o) \) is the specific volume at point 1 given by Equations 4.24 or 4.30 for model 1 and Equation 4.35 for model 2. The slope of the critical state line for
both models tends towards the slope of the fully saturated isotropic compression line, $\lambda(0)$, at high confining stresses (as shown in Figure 4-13).

Figure 4-12: Critical State and isotropic compression lines

Figure 4-13: Critical State lines for partially and fully saturated conditions

\[
\begin{align*}
\ln p & \quad v \\
\text{Isotropic Compression line - Model 1} & \\
\text{Elastic unloading line} & \\
\text{Critical State line} & \\
\text{Isotropic compression line - Model 2} & \\
\text{CSL - Model 1} & \\
\text{CSL saturated} & \\
\text{CSL - Model 2} & \\
\end{align*}
\]
Experimental results by Wheeler & Sivakumar (1995) showed that both the partially saturated isotropic compression and critical state lines converged towards the equivalent fully saturated lines. As discussed previously this is probably due to the fact that the stress range examined was beyond the stress corresponding to maximum potential collapse. As seen in Figures 4-10 and 4-13 the predictions of model 2 are in agreement with these results.

4.2.5 Hardening/softening rules

The magnitude of the plastic volumetric strains when either of the two yield surfaces is activated is related to the change of the hardening softening parameters, \( p_o^* \) and \( s_o \), through the following equations for both models:

Primary yield surface:

\[
\frac{dp_o^*}{p_o^*} = \frac{v}{\lambda(0) - \kappa} \, d\epsilon_v^p
\]  
(4.55)

Secondary yield surface:

\[
\frac{ds_o}{s_o + p_{atm}} = \frac{v}{\lambda_s - \kappa_s} \, d\epsilon_v^p
\]  
(4.56)

where \( \lambda_s \) is the compressibility coefficient for changes in suction.

Equations 4.55 and 4.56 imply that the two yield surfaces are coupled. Activation and movement of either of the two surfaces will cause movement of the other (as shown in Figure 4-14):

\[
\frac{dp_o^*}{p_o^*} = \frac{\lambda_s - \kappa_s}{\lambda(0) - \kappa} \, \frac{ds_o}{s_o + p_{atm}}
\]  
(4.57)
4.2.6 Elastic behaviour

The volumetric changes due to changes in equivalent suction or mean net stress stress, within the elastic region, are given from the elastic loading/unloading and wetting/drying lines:

Elastic loading/unloading:  \[ d \varepsilon_v = -\frac{\kappa}{v \rho} \, dp \]  (4.58)

Elastic wetting/drying:  \[ d \varepsilon_v = -\frac{\kappa_s}{v \left( s_{eq} + p_{atm} \right)} \, ds_{eq} \]  (4.59)

In order to avoid calculation of infinite strains when \( p \) tends to zero, a minimum bulk modulus \((vp/\kappa)\) is introduced, \( K_{min} \), as an additional model parameter (Figure 4-15).
The specific volume at any value of $s_{eq}$ and $p$, within the elastic region is given by the following expressions:

For $K > K_{min}$:

Model 1: 

\[ p \leq p_m: \quad v = v_1(s_{eq}) - \dot{\lambda}(s_{eq}) \ln \rho_o + \kappa \ln \frac{\rho_o}{p} \]  

(4.60)

\[ p > p_m: \quad v = v_1(0) + \Delta v_{max} - \dot{\lambda}(0) \ln \rho_o + \kappa \ln \frac{\rho_o}{p} \]  

(4.61)

Model 2:  

\[ v = v_1(s_{eq}) - \dot{\lambda}(s_{eq}) \ln \rho_o + \left[ \dot{\lambda}(0) - \lambda_m \left( \frac{\rho_o}{p^o} \right)^{-b} \right] \ln \frac{\rho_o}{p^o} + \kappa \ln \frac{\rho_o}{p} \]  

(4.62)

For $K \leq K_{min}$:  

\[ v = v_{min} e^{\frac{p^o p_{min}}{K_{min}}} \]  

(4.63)

where $p_{min}$ and $v_{min}$ are the confining stress and specific volume values corresponding to $K_{min}$.

Equations 4.60, 4.61, 4.62 and 4.63 define a surface in the $v$-ln$p$-$s_{eq}$ space, for a given value of the equivalent fully saturated yield stress, $p_o^*$. A cross-section of this surface at constant equivalent suction is shown in Figure 4-16.
The incremental strains, related to changes in the deviatoric stress invariant, \( J \), are given by:

\[
d E^e_i = \frac{d J}{\sqrt{3G}}
\]  

(4.64)

where

\[
d E^e_i = \frac{2}{\sqrt{6}} \sqrt{(d \varepsilon_1 - d \varepsilon_2)^2 + (d \varepsilon_2 - d \varepsilon_3)^2 + (d \varepsilon_3 - d \varepsilon_1)^2}
\]  

(4.65)

and \( G \) is the elastic shear modulus. This is related to the bulk modulus, \( K \), through the following Equation:

\[
G = \frac{3(1-\mu)}{2(1+\mu)}K
\]  

(4.66)
where $\mu$ is the Poisson’s ratio. The behaviour within the elastic region can be described by defining constant values for either $\mu$ or $G$. As $K$ varies with net mean stress, $p$, such an assumption means that either $G$ or $\mu$ also vary. Another option is to assume that $G$ is proportional to the yield stress $p_o$:

$$G = g \ p_o$$ (4.67)

where $g$ is a constant.

### 4.3 OVERVIEW OF MODEL PARAMETERS

#### 4.3.1 Yield surface and plastic potential parameters

Six parameters are required in order to define the shapes of the primary yield and plastic potential surfaces in the $J$-$p$ plane: $\alpha_g$, $\mu_g$, $M_g$, $\alpha_f$, $\mu_f$ and $M_f$.

The parameters $M_g$ and $M_f$ are equal to $\sqrt{3J_{2ng}}$ and $\sqrt{3J_{2nf}}$, respectively, in triaxial compression, corresponding to a horizontal tangent of the curves in the $J$-$p$ plane ($dG/dp=0$ and $dF/dp=0$). $M_g$ is related to the angle of shearing resistance, $\phi_{cs}$:

$$M_g = \frac{6\sin \phi_{cs}}{3 - \sin \phi_{cs}}$$ (4.68)

The parameters $M_g$ and $M_f$ can have values between 0 and 3.

Parameters $\alpha_g$ and $\mu_g$ determine the variation of dilatancy, $d$, with the stress ratio, as shown in Figure 4-1a. In particular the parameter $\alpha_g$ defines the proportion of $\sqrt{J_{2ng}}$ that the stress ratio $J/(p + f(s_{eq}))$ must attain for the dilatancy to be

$$d = 2\mu_g \sqrt{J_{2ng}} \left(1 - \alpha_g \right)$$ (4.69)

while $\mu_g$ gives the reduction of dilation at high stress ratios. Parameters $\alpha_f$ and $\mu_f$ are determined by fitting the experimental curve. The following restrictions apply to the choice of values for these parameters:
\[
\begin{align*}
\alpha_g &\neq 1 \quad \alpha_f \neq 1 \\
\mu_g &\neq 1 \quad \mu_f \neq 1
\end{align*}
\] (4.70)

If \( \mu_g \) and \( \mu_f \) are lower than 1, then they must satisfy the following condition:

\[
\mu_g > \frac{4\alpha_g}{(1 - \alpha_g)^2 + 4\alpha_g} \quad \text{and} \quad \mu_f > \frac{4\alpha_f}{(1 - \alpha_f)^2 + 4\alpha_f}
\] (4.71)

If \( \alpha_g \) (or \( \alpha_f \)) < 1 and \( \mu_g \) (or \( \mu_f \)) < 1 then the surface is rounded not only for \( p = p_o \) but also for \( p + f(s_{eq}) = 0 \) and therefore the normalised stress ratio at the origin is

\[
\eta\big|_{p+f(s_{eq})=0} = \infty
\] (4.72)

In any other case a finite value of the normalised stress ratio is obtained at the origin. If \( \alpha_g \) (or \( \alpha_f \)) < 1 and \( \mu_g \) (or \( \mu_f \)) > 1:

\[
\eta\big|_{p+f(s_{eq})=0} = -K_{1g} \left( \text{or} -K_{1f} \right)
\] (4.73)

If \( \alpha_g \) (or \( \alpha_f \)) > 1 and \( \mu_g \) (or \( \mu_f \)) > 1:

\[
\eta\big|_{p+f(s_{eq})=0} = -K_{2g} \left( \text{or} -K_{2f} \right)
\] (4.74)

The use of six parameters to define the shape of the yield and plastic potential surfaces instead of a single parameter \( (M_J) \) in the case of the modified Cam-Clay model should not be considered as a disadvantage of the model but rather as an advantage. As demonstrated in Figure 4-2 the yield surface expressions of the modified Cam Clay and many other fully saturated models can be obtained by suitable selection of the model parameters. Table 4.1 summarises the parameters used to obtain these curves.
<table>
<thead>
<tr>
<th>Model name</th>
<th>Parameters</th>
<th>Lagioia et al. (1996) expression parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cam Clay *Roscoe &amp; Schofield (1963)</td>
<td>$M = 1.4$</td>
<td>$M_g = M_f = 1.4$</td>
</tr>
<tr>
<td></td>
<td>$p'c = 100\text{kPa}$</td>
<td>$\mu_g = \mu_f = 1.0000001$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_g = \alpha_f = 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$p_o = 100\text{kPa}$</td>
</tr>
<tr>
<td>Modified Cam Clay *Roscoe &amp; Burland</td>
<td>$M = 1.4$</td>
<td>$M_g = M_f = 1.4$</td>
</tr>
<tr>
<td>(1968)</td>
<td>$p'c = 100\text{kPa}$</td>
<td>$\mu_g = \mu_f = 0.9$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_g = \alpha_f = 0.4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$p_o = 100\text{kPa}$</td>
</tr>
<tr>
<td>Sinfonietta Classica Nova (1988)</td>
<td>$\gamma = 4.4$</td>
<td>$M_g = M_f = 0.91$</td>
</tr>
<tr>
<td></td>
<td>$\beta = 1.5$</td>
<td>$\mu_g = \mu_f = 0.999999$</td>
</tr>
<tr>
<td></td>
<td>$p'c = 100\text{kPa}$</td>
<td>$\alpha_g = \alpha_f = 0.7$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$p_o = 100\text{kPa}$</td>
</tr>
<tr>
<td>Single Hardening Plastic potential *Kim</td>
<td>$\Psi = 0.0289$</td>
<td>$M_g = 1.21$</td>
</tr>
<tr>
<td>&amp; Lade (1988a)</td>
<td>$\Psi_2 = -3.69$</td>
<td>$\mu_g = 0.45$</td>
</tr>
<tr>
<td></td>
<td>$\mu = 2.26$</td>
<td>$\alpha_g = 0.145$</td>
</tr>
<tr>
<td></td>
<td>$p'c = 100\text{kPa}$</td>
<td>$p_o = 100\text{kPa}$</td>
</tr>
<tr>
<td>Single Hardening Yield surface *Kim &amp;</td>
<td>$\Psi_f = 0.0289$</td>
<td>$M_f = 1.01$</td>
</tr>
<tr>
<td>Lade (1988b)</td>
<td>$h = 0.348$</td>
<td>$\mu_f = 1.15$</td>
</tr>
<tr>
<td></td>
<td>$p'c = 100\text{kPa}$</td>
<td>$\alpha_f = 1.00001$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$p_o = 100\text{kPa}$</td>
</tr>
</tbody>
</table>

Table 4.1: Parameters for comparison between models from literature and the Lagioia et al. (1996) expression for fully saturated soils (after Lagioia et al. (1996))

The position of the yield and plastic potential surfaces with respect to the $p = 0$ axis is determined by the value of $f(s_{eq})$. This is controlled by one parameter ($k = \text{const.}$) if option 1 (linear increase of apparent cohesion with suction) is used, or by the soil-water characteristic curve if option 2 is used. In the latter case an
expression relating degree of saturation, $S_r$, to suction is required. One possible choice is the van Genuchten (1980) expression, which requires three additional parameters (four parameters in total). These are obtained by fitting the expression to the experimentally obtained soil-water characteristic curve. However as the soil-water characteristic curve is only used to determine the increase of apparent cohesion, the parameters that control its shape can alternatively be obtained by drained triaxial compression tests at different values of equivalent suction. Similar tests are required in order to determine the value of $k$ for the first option.

The size of the yield and plastic potential surfaces is determined by the isotropic yield stress, $p_o$. The parameters controlling $p_o$ are discussed in the next section as it is related to the hardening/softening parameter, $p_o^*$.  

### 4.3.2 Hardening/Softening parameters

The expansion or contraction of the primary yield surface and the associated plastic strains are controlled by two parameters; the fully saturated compressibility coefficient, $\lambda(0)$, and the elastic compressibility coefficient, $\kappa$, which is independent of suction. These can be obtained by fully saturated isotropic loading and unloading tests. A single test would theoretically be sufficient but accuracy would obviously be improved if more were performed.

The value of the hardening/softening parameter, $p_o^*$, is related to the isotropic yield stress, $p_o$, through Equations 4.26 and 4.27 for model 1 and Equation 4.50 for model 2. Four parameters are required in order to determine this relationship:

**Model 1:** $p^c, \alpha_c, r$ and $\beta$

**Model 2:** $p^c, b, r$ and $\beta$

The characteristic pressure, $p^c$, is the limiting confining stress at which $p_o = p_o^* = p^c$ so that the projection of the yield surface in the $p-s_{eq}$ plane is a straight line perpendicular to the $p$ axis. In any other case, $p_o > p_o^* > p^c$.  

121
The characteristic stress ratio, $\alpha_c (= p_o^*/p^*)$, controls the value of the confining stress, $p_m$, beyond which a constant amount of potential collapse, in the $v$-ln$p$ plane, due to wetting is predicted (for model 1).

Similarly, the parameter $b$, in model 2, controls the confining stress that corresponds to maximum collapse.

The parameters $r$ and $\beta$ control the increase of the compressibility coefficient, $\dot{\lambda}(s_{eq})$, with equivalent suction, at low confining stresses (when $p_o < p_m$ for model 1 and $p_o = p^c$ for model 2). Parameter $r$ is related to the maximum value of the initial compressibility coefficient, while parameter $\beta$ controls its rate of increase with equivalent suction. Both parameters also control the amount of maximum potential collapse. The combination of $r$, $\beta$ and $\dot{\lambda}(0)$ must result in an initial compressibility coefficient $\dot{\lambda}(s_{eq})$ which is always higher than $\kappa$, so that soil states above the isotropic compression line are avoided.

All the above parameters can be obtained from isotropic compression tests at different constant values of suction. A minimum of two partially saturated compression tests in addition to a fully saturated test is required to be able to determine the model parameters. However in order to achieve sufficient accuracy more tests should be performed. The initial yield stress of the soil samples needs to be low in order to determine the values of the parameters $p^c$, $r$ and $\beta$. If this is not possible they will need to be interpolated from the data indirectly.

Two parameters are required in order to determine the plastic strains due to activation of the secondary yield surface; the compressibility coefficient due to changes in suction, $\lambda_s$, and the elastic compressibility coefficient due to changes in suction, $\kappa_s$. These can be determined from a single drying test at constant confining stress.

4.3.3 Initial hardening/softening parameters

The model parameters described above define the shape and expansion or contraction of the yield and plastic potential surfaces. Their initial position at the
beginning of an analysis is defined by the equivalent fully saturated isotropic yield stress, \( p_0^* \), and the yield suction, \( s_o \), which are the hardening/softening parameters.

The initial hardening/softening parameter, \( p_0^* \), is calculated from the current stress state from the stress state ratio, SSR. There are four options in the formulation of the model for the definition of SSR, allowing the calculation of the initial hardening/softening parameter in the most convenient way for the problem analysed:

Option 1:

\[
SSR = \frac{p_0^*}{|p|}
\]  
(4.75)

where \(|p|\) is the absolute value of the mean net stress. The initial value of \( p \) may be positive or negative, but not equal to zero. This option is useful for isotropic conditions but may lead to situations where the initial stress state is outside the defined yield surface, for values of SSR close to unity and high deviatoric stresses.

Option 2:

\[
SSR = \frac{\sigma^*_{y_{mo}}}{|\sigma_y|}
\]  
(4.76)

where \(|\sigma_y|\) is the absolute value of the net vertical stress. The initial value of \( \sigma_y \) may be positive or negative, but not equal to zero. \( \sigma^*_{y_{mo}} \) is the vertical stress corresponding to a stress state on the equivalent fully saturated yield surface. The remainder of the stress components are calculated by assuming that the vertical stress is a principle stress and the stress state lies on the \( K_o \) line:

\[
\sigma^*_{x_{mo}} = \sigma^*_{z_{mo}} = (1 - \sin \phi) \sigma^*_{y_{mo}} \quad \text{and} \quad r^*_{xy_{mo}} = r^*_{xz_{mo}} = r^*_{yz_{mo}} = 0
\]  
(4.77)

The mean net yield stress, \( p^*_{mo} \), and the deviatoric yield stress, \( J_{mo} \), are then calculated from which the hardening/softening parameter, \( p_o^* \), can be obtained (from substitution into Equation 4.16):
This option is very useful when $K_o$ consolidated soils are modelled.

The first two options relate the initial stress state directly to the hardening/softening parameter irrespective of the value of equivalent suction. This is convenient when problems are analysed that involve large variations of the equivalent suction throughout the soil profile. In this case the increase of the yield stress due to suction, at any point within the partially saturated zone, is automatically calculated.

For other problems (e.g. simulation of a triaxial compression test) it might be more useful to relate the initial stress state to the current isotropic yield stress, $p_o$. This can be done by using the following two options. The hardening/softening parameter is calculated from the isotropic yield stress, $p_o$, through Equations 4.26 or 4.27 for model 1, and Equation 4.50 for model 2.

Option 3: $$SSR = \frac{P_o}{p}$$\tag{4.79}$$

Option 4: $$SSR = \frac{\sigma_{ymo}}{\sigma_y}$$\tag{4.80}$$

$\sigma_{ymo}$ is the vertical stress corresponding to a stress state on the current partially saturated yield surface. The calculation of $p_o$ is performed in a similar way to that of $p_o^*$ in option 2.

The initial position of the secondary yield surface, $F_2$, is controlled by a single parameter, $s_o$, the yield suction. This can be obtained by unconfined drying tests, as it is assumed that $s_o$ is independent of the confining stress. As noted previously the presence of such a yield surface may not apply to many soils. A high value may be given to $s_o$ in such a case so that the secondary yield surface becomes irrelevant.
The second initial hardening/softening parameter is the specific volume at unit pressure, \( v_1(0) \), which controls the position of the fully saturated isotropic compression line. This parameter does not represent a possible stress state as the extension of the isotropic compression line to low stresses is limited by the restriction that \( p_o^\ast \geq p^\ell \). The specific volume at unit pressure corresponding to the current value of equivalent suction, \( v_1(s_{eq}) \), is related to \( v_1(0) \) through the following equation (see Figure 4-17):

\[
v_1(s_{eq}) = v_1(0) - \kappa \ln \frac{s_{eq} + p_{atm}}{p_{atm}} - (\lambda(0) - \lambda(s_{eq})) \ln p^\ell
\]

(4.81)

Figure 4-17: Calculation of \( v_1 \) from the initial specific volume, \( v \), for Model 2

### 4.3.4 Other parameters

Four additional parameters are required in order to fully describe soil behaviour with the two proposed models. These parameters are the atmospheric pressure, \( p_{atm} \), which is usually set equal to 100 kPa, the air entry value of suction, \( s_{air} \), which can be obtained from an unconfined drying test, the minimum bulk
modulus, $K_{min}$, and the shear modulus, $G$. As noted previously, the Poisson’s ratio or the ratio $g = G/p_o$ can be defined instead of the shear modulus.

The minimum bulk modulus, $K_{min}$, is an arbitrary parameter selected such that finite strains are calculated along unloading paths that cross the $p = 0$ axis. A possible choice would be to calculate $K_{min}$ at unit pressure. This would give a value of the order of a few hundred kPa for most soils.

### 4.3.5 Summary

The two proposed models require twenty-two parameters ($\alpha_g$, $\mu_g$, $M_g$, $\alpha_f$, $\mu_f$, $M_f$, $k$, $\lambda(0)$, $\kappa$, $p^*$, $\alpha_c$ or $b$, $r$, $\beta$, $\lambda_s$, $\kappa_s$, $p_o^*$, $s_o$, $v_1(0)$, $p_{atm}$, $s_{air}$, $K_{min}$, and $G$) as opposed to the fourteen parameters of the Barcelona Basic model ($M$, $k$, $\lambda(0)$, $\kappa$, $p^*$, $r$, $\beta$, $\lambda_s$, $\kappa_s$, $p_o^*$, $s_o$, $v_1(0)$, $p_{atm}$, and $G$). The increased number of parameters allows the reproduction of a very large number of yield and plastic potential surfaces ($\alpha_g$, $\mu_g$, $M_g$, $\alpha_f$, $\mu_f$, $M_f$), and the phenomenon of maximum wetting induced collapse ($\alpha_c$ or $b$). In addition the transition from fully to partially saturated conditions is more accurately modelled (through the introduction of $s_{air}$) and the numerical problems at low compressive or tensile confining stresses are solved (through the introduction of $K_{min}$). The model parameters can be obtained from the same testing program required for the Barcelona Basic model, although a wider stress range is necessary in order to obtain the parameters related to the amount of maximum potential soil collapse and its corresponding confining stress. An adequate testing program could include the following tests:

a) One fully saturated undrained triaxial compression test, if known yield and plastic potential surface shapes are adopted (e.g. modified Cam Clay), in which case $\alpha_g$, $\mu_g$, $\alpha_f$, and $\mu_f$ are derived numerically to match the required shapes. A single compression test will provide the parameters $M_f$, $M_g$ and $G$. In a different case two such tests might be required, one on the wet side and one on the dry side.

b) One partially saturated drained triaxial compression test in order to determine the cohesion increase parameter, $k$. This test may not be required if the increase of apparent cohesion is selected to vary with the degree of saturation.
c) One fully saturated isotropic loading and unloading test in order to obtain the parameters $\lambda(0)$, $\kappa$ and $p_o^*$.

d) Two isotropic loading tests at different suction values in order to obtain the parameters $p^\tau$, $a_c$ or $b$, $r$, $\beta$ and $v_f(0)$. These tests need to be extended to high confining stresses.

e) One wetting/drying cycle at constant confining stress. This test will provide the soil-water characteristic curve if this is needed, and the parameters $\lambda_s$, $\kappa_s$, $s_o$, and $s_{air}$.

Although such a testing program can theoretically provide the required information, a more reliable determination of the model parameters would obviously need more tests.

A summary of the model input parameters is given in Table 4.2.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_g$</td>
<td>Plastic potential parameter</td>
<td>$r$</td>
<td>Maximum soil stiffness parameter</td>
</tr>
<tr>
<td>$\mu_g$</td>
<td>Plastic potential parameter</td>
<td>$\beta$</td>
<td>Soil stiffness increase parameter</td>
</tr>
<tr>
<td>$M_g$</td>
<td>Slope of the critical state line in the $q$-$p$ stress space for triaxial compression</td>
<td>$\lambda_s$</td>
<td>Compressibility coefficient for changes in suction</td>
</tr>
<tr>
<td>$\alpha_f$</td>
<td>Yield surface parameter</td>
<td>$\kappa_s$</td>
<td>Elastic compressibility coefficient for changes in suction</td>
</tr>
<tr>
<td>$\mu_f$</td>
<td>Yield surface parameter</td>
<td>$p_o^*$</td>
<td>Equivalent fully saturated isotropic yield stress</td>
</tr>
<tr>
<td>$M_f$</td>
<td>Yield surface parameter</td>
<td>$s_o$</td>
<td>Yield suction</td>
</tr>
<tr>
<td>$k$</td>
<td>Cohesion increase parameter ($=\text{const. or } S_r$)</td>
<td>$v_1(0)$</td>
<td>Specific volume at unit pressure (fully sat.)</td>
</tr>
<tr>
<td>$\lambda(0)$</td>
<td>Fully saturated compressibility coefficient</td>
<td>$p_{\text{atm}}$</td>
<td>Atmospheric pressure</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Elastic compressibility coefficient</td>
<td>$s_{\text{air}}$</td>
<td>Air entry suction</td>
</tr>
<tr>
<td>$p^\circ$</td>
<td>Characteristic pressure</td>
<td>$K_{\text{min}}$</td>
<td>Minimum bulk modulus</td>
</tr>
<tr>
<td>$\alpha_c$</td>
<td>Characteristic stress ratio (model 1 – option 2)</td>
<td>$G$</td>
<td>Shear modulus</td>
</tr>
<tr>
<td>$b$</td>
<td>Maximum collapse parameter (model 2)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2: Summary of model input parameters
4.4 SUMMARY AND DISCUSSION

Two new elastoplastic constitutive models for partially saturated soils, based on the Loading-Collapse yield surface concept, have been described. The two models have the following main features:

- They are formulated in the four-dimensional net stress ($\sigma$)- equivalent suction ($s_{eq}$) stress space. Equivalent suction is the excess of suction over the air entry value ($s_{air}$). The soil is treated as fully saturated for suctions lower than $s_{air}$ and its behaviour is described in the three-dimensional effective stress space.

- Two yield surfaces are defined; the primary yield surface, $F_1$, and the secondary yield surface, $F_2$. The selected expressions for the primary yield and plastic potential surfaces allow the reproduction of many different shapes including those assumed by some well known constitutive models. Both associated and non-associated flow rules can be adopted depending on the choice of the relevant parameters. The secondary yield surface defines the limiting equivalent suction beyond which plastic strains take place.

- The primary yield and plastic potential surfaces expand with increasing suction. The expansion is controlled by the increase in the isotropic yield stress, and the increase in apparent cohesion. The isotropic yield stress is related to the assumed shape of the partially saturated isotropic compression line. Model 1 adopts a bi-linear expression giving a constant amount of potential collapse beyond a certain value of the confining stress. Model 2 adopts an exponential expression, so that the amount of potential collapse increases with confining stress at low stresses, reaches a maximum value and then decreases to zero at very high confining stresses. Real soil behaviour is better represented by Model 2, but Model 1 is more straightforward, particularly at low confining stresses, and relatively easier to implement into a Finite Element program.
• The increase of apparent cohesion with suction can either be assumed to be linear (which is acceptable for low values of the equivalent suction), or non-linear.

• The plastic volumetric strains induced when the primary yield surface is activated are calculated from the change of the equivalent fully saturated yield stress, $p_0^*$, which is selected as the hardening/softening parameter. This implies that loading and wetting have the same effect on the soil structure, which is in accordance with experimental results. The plastic volumetric strains due to activation of the secondary yield surface are calculated from the change of the value of the yield suction, $s_o$, (secondary hardening/softening parameter). The two yield surfaces are coupled, therefore movement of either of them results in movement of the other.

• Within the elastic region, volumetric strains are calculated from the shape of the elastic loading/unloading and wetting/drying paths. An ‘elastic’ surface is defined in the $v$-ln$p$-$s_{eq}$ space for each equivalent fully saturated yield stress value. This surface is warped close to the $p = 0$ plane and in the tensile confining stress region. Shear strains are calculated by adopting a constant shear modulus, $G$, a constant Poisson’s ratio, $\mu$, or a constant ratio $G/p_o$.

• Twenty-two parameters are required for both models. A testing program from which these parameters can be obtained was described in the previous section. This includes a minimum of three isotropic compression tests, two triaxial compression tests and one drying-wetting test.
Chapter 5

Implementation and Validation

5.1 INTRODUCTION

This chapter presents the implementation of the two constitutive models described in the previous chapter into the Imperial College Finite Element Program (ICFEP) and their validation through a series of single finite element analyses.

5.2 IMPLEMENTATION

The finite element program ICFEP was modified during this study so that partially saturated soil analyses could be undertaken. The main modifications to the conventional fully saturated method lie in the determination of the residual load vector, $\{\psi\}$. As discussed in Chapter 3, this vector is obtained from the constitutive equations during the iterative solution of the global equations (modified Newton-Raphson method with the error controlled substepping algorithm) and represents the difference between the estimated loads and the true solution. It is the accurate calculation of this vector rather than the actual formulation of the global equations that is necessary in order to obtain the correct solution for the applied boundary conditions.

The choice of constitutive model influences both the formulation and the solution of the global equations. More specifically the constitutive model is used to:

a) Determine the incremental elastic or elastoplastic matrix, $[D]$ or $[D^{ep}]$, respectively, from which the incremental element stiffness matrix $[K_e]$ is obtained. The incremental global stiffness matrix $[K_G]$ is then calculated to form the global set of equations.
b) Determine the residual load vector $\{\psi^i\}$ through the stress point algorithm for each iteration. As both models presented in the previous chapter assume nonlinear elasticity this calculation is performed for both elastic and elastoplastic paths.

### 5.2.1 Calculation of global stiffness matrix

In nonlinear finite element analysis the first estimate of the displacements $\{\Delta d\}_nG$, at the beginning of each increment, corresponding to changes of the load vector $\{\Delta R_G\}$ are determined from the stiffness matrix $[K_G]$. As mentioned previously, this is obtained by assembling the element stiffness matrices $[K_E]$, which are calculated from the elastic or elastoplastic matrix, $[D]$ or $[D^{ep}]$, corresponding to the stress state at the beginning of the increment, through Equation 3.16. In order to reduce computational time, in the analyses performed for this study, $[K_E]$ was always calculated from the elastic matrix $[D]$. Since the modified Newton-Raphson method was employed instead of the original Newton-Raphson method, the stiffness matrix $[K_G]$ calculated at the beginning of each increment was used for all the solution iterations (Potts & Zdravkovic (1999)).

The elastic matrix, $[D]$, can be expressed in terms of the elastic shear modulus, $G$, and bulk modulus, $K$, as follows:

$$
[D] = \begin{bmatrix}
K + \frac{4}{3}G & K - \frac{2}{3}G & K - \frac{2}{3}G & 0 & 0 & 0 \\
K + \frac{4}{3}G & K - \frac{2}{3}G & 0 & 0 & 0 & 0 \\
K + \frac{4}{3}G & 0 & 0 & 0 & 0 \\
sym & G & 0 & 0 & 0 \\
 & G & 0 & 0 & 0 \\
 & G & 0 & 0 & 0 \\
\end{bmatrix}
$$

(5.1)

For the two constitutive models presented in Chapter 4, the bulk modulus depends on the current stress state, and is given by:
\[ K = \frac{v p}{\kappa} \tag{5.2} \]

where \( p \) is the mean net stress, \( v (= 1 + e) \) is the specific volume, and \( \kappa \) is the elastic loading/unloading compressibility coefficient.

The elastic shear modulus, \( G \), can be chosen to be constant or vary with either mean net stress, \( p \), (constant Poisson’s ratio, \( \mu \), option) or isotropic yield stress, \( p_0 \), as explained in section 4.2.6.

As mentioned in the previous chapter, a constant minimum bulk modulus, \( K_{\text{min}} \), is used when the mean net stress is tensile or close to zero (typically for \( p \leq 1 \text{ kPa} \)).

### 5.2.2 Calculation of residual load vector \( \{ \psi \} \)

The residual load vector \( \{ \psi \} \) for each iteration is calculated by integrating the constitutive relationships over the current estimate of the incremental displacements. These are given by Equation 3.22, the solution of which, as described in Section 3.4.1, requires the following terms to be calculated:

- a) Yield function derivatives \( \{ \partial F / \partial \sigma \} \)
- b) Plastic potential surface derivatives \( \{ \partial G / \partial \sigma \} \)
- c) Parameter \( A \)
- d) Elastic strains due to changes in suction \( \{ \Delta e_s^e \} \)
- e) Plastic strains due to changes in suction \( \{ \Delta e_s^p \} \)
- f) Yield function derivatives with respect to equivalent suction \( \{ \partial F / \partial s_{eq} \} \)

By employing the two constitutive models presented in Chapter 4, these terms are calculated as follows:

\[ \{ \partial F / \partial \sigma \} = \partial F \left( \frac{\partial p}{\partial \sigma} \right) + \partial F \left( \frac{\partial \eta}{\partial J_{2n}} \left( \frac{\partial \Delta \sigma_{2n}}{\partial \sigma} \right) \right) + \frac{\partial \eta}{\partial J_{2n}} \frac{\partial \Delta \sigma_{2n}}{\partial \sigma} \left( \frac{\partial \theta}{\partial \sigma} \right) \]  

(5.3)

133
The derivatives of the stress invariants are independent of the adopted constitutive model and are given by:

\[
\begin{align*}
\left\{ \frac{\partial p}{\partial \sigma} \right\}^T &= \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \end{bmatrix} \\
\left\{ \frac{\partial J}{\partial \sigma} \right\}^T &= \begin{bmatrix} \sigma_x - p & \sigma_y - p & \sigma_z - p & \tau_{xy} & \tau_{xz} & \tau_{yz} \end{bmatrix} \\
\left\{ \frac{\partial \theta}{\partial \sigma} \right\} &= \frac{\sqrt{3}}{2 \cos(3\theta)J^3} \left[ 3 \text{det } s \left\{ \frac{\partial J}{\partial \sigma} \right\} - \left\{ \frac{\partial \text{det } s}{\partial \sigma} \right\} \right]
\end{align*}
\]

where:

\[
\text{det } s = (\sigma_x - p)(\sigma_y - p)(\sigma_z - p) - (\sigma_x - p) \tau_{yz}^2 - \\
(\sigma_y - p) \tau_{xz}^2 - (\sigma_z - p) \tau_{xy}^2 + 2 \tau_{xz} \tau_{yz} \tau_{xy}
\]

\[
\frac{\partial \text{det } s}{\partial \sigma_x} = \frac{1}{3} \left[ (\sigma_x - p)(\sigma_z - \sigma_x) + (\sigma_z - p)(\sigma_x - \sigma_z) - 2 \tau_{xz}^2 + \tau_{xy}^2 + \tau_{yz}^2 \right]
\]

\[
\frac{\partial \text{det } s}{\partial \sigma_y} = \frac{1}{3} \left[ (\sigma_y - p)(\sigma_z - \sigma_y) + (\sigma_z - p)(\sigma_y - \sigma_z) - 2 \tau_{xz}^2 + \tau_{xy}^2 + \tau_{yz}^2 \right]
\]

\[
\frac{\partial \text{det } s}{\partial \sigma_z} = \frac{1}{3} \left[ (\sigma_z - p)(\sigma_y - \sigma_z) + (\sigma_y - p)(\sigma_z - \sigma_y) - 2 \tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2 \right]
\]

\[
\frac{\partial \text{det } s}{\partial \tau_{xy}} = -2(\sigma_z - p) \tau_{xy} + 2 \tau_{xz} \tau_{yz}
\]

\[
\frac{\partial \text{det } s}{\partial \tau_{xz}} = -2(\sigma_y - p) \tau_{xz} + 2 \tau_{xy} \tau_{yz}
\]

\[
\frac{\partial \text{det } s}{\partial \tau_{yz}} = -2(\sigma_x - p) \tau_{yz} + 2 \tau_{xy} \tau_{xz}
\]

The remainder of the terms in Equation 5.3 are given by:

\[
\frac{\partial F}{\partial p} = \frac{1}{p_o + f(s_{eq})}
\]

\[
\frac{\partial F}{\partial \eta} = -\frac{1}{\beta_f} \left( 1 + \frac{\eta}{K_2} \right) \left( \frac{1}{1 + \frac{\eta}{K_2}} \right)
\]

\[
\frac{1}{1 + \frac{\eta}{K_1}} \left( \frac{1}{1 + \frac{\eta}{K_1}} \right)
\]

\[
\frac{1}{1 + \frac{\eta}{K_1}} \left( \frac{1}{1 + \frac{\eta}{K_1}} \right)
\]

134
\[
\frac{\partial \eta}{\partial J_{2n}} = \frac{1}{2J_{2nf}} \eta 
\]  
(5.16)

\[
\frac{\partial \eta}{\partial J_{2nf}} = -\frac{\eta}{2J_{2nf}} 
\]  
(5.17)

\[
\frac{\partial J_{2nf}}{\partial \theta} = -\frac{6}{\sqrt{27}} C \cos(3\theta) \sqrt{J_{2nf}}^3 
\]  
(5.18)

where \(f(s_{eq})\) represents the expansion of the yield surface into the tensile stress region as defined in section 4.2.2, \(K_1, K_2\) and \(b_f\) are calculated from the model parameters \(\alpha_f\) and \(\mu_f\) through Equations 4.7 and 4.8, and \(C\) is a function of the parameter \(M_f\) given by Equation 4.12.

b) Calculation of plastic potential surface derivatives

\[
\left\{ \frac{\partial G}{\partial \sigma} \right\} = \frac{\partial G}{\partial p} \left[ \frac{\partial p}{\partial \sigma} \right] + \frac{\partial G}{\partial \eta} \left[ \frac{\partial \eta}{\partial J_{2n}} \right] \left[ \frac{\partial J_{2n}}{\partial \sigma} \right] + \frac{\partial G}{\partial J_{2nf}} \left[ \frac{\partial J_{2nf}}{\partial \sigma} \right] - \frac{\partial G}{\partial \theta} \left[ \frac{\partial \theta}{\partial \sigma} \right] 
\]  
(5.19)

The terms in the above equation are obtained from similar equations to those used to calculate the yield surface derivatives (Equations 5.4 –5.18). \(J_{2nf}\) is replaced by \(J_{2ng}\) and \(K_1, K_2, b_f\) and \(C\) are calculated from the plastic potential parameters \(\alpha_g, \mu_g\) and \(M_g\) instead of the yield function parameters \(\alpha_f, \mu_f\) and \(M_f\). In addition, as \(p_o\) is related to the yield function it is necessary, instead of using an equation similar to 5.14, to obtain the derivative of the plastic potential surface with respect to the mean total stress, \(p\), from the following equivalent expression:

\[
\frac{\partial G}{\partial p} = \frac{1}{p + f(s_{eq})} \left( 1 + \frac{\eta}{K_1} \right) \left( 1 + \frac{\eta}{K_2} \right) \left( 1 + \frac{\eta}{b_f} \right) 
\]  
(5.20)

c) Calculation of the parameter \(A\)

Equation 3.28 can be rewritten as follows:
\[
A = -\frac{1}{\Lambda} \frac{\partial F}{\partial p_o} \, d \, p_o^* \tag{5.21}
\]

where \( \Lambda \) is the plastic strain multiplier and \( p_o^* \) is the hardening parameter. From the hardening rule we have:

\[
d \, p_o^* = \frac{\nu}{\lambda(0) - \kappa} \, p_o^* \, d \, \varepsilon_p^p \tag{5.22}
\]

where the volumetric plastic strains are given by:

\[
d \, \varepsilon_p^p = \Lambda \frac{\partial G}{\partial p_o} \tag{5.23}
\]

Combining Equations 5.21, 5.22 and 5.23 gives:

\[
A = -\frac{\nu}{\lambda(0) - \kappa} \, p_o^* \, \frac{\partial F}{\partial p_o^*} \, \frac{\partial G}{\partial p_o} \tag{5.24}
\]

where

\[
\frac{\partial F}{\partial p_o^*} = \frac{\partial F}{\partial p_o} \, \frac{\partial p_o}{\partial p_o^*} \tag{5.25}
\]

\[
\frac{\partial F}{\partial p_o} = -\frac{p + f(s_{eq})}{(p_o + f(s_{eq}))^2} \tag{5.26}
\]

Depending on the adopted constitutive model, the derivative of the isotropic yield stress, \( p_o \), with respect to the equivalent fully saturated yield stress (hardening parameter), \( p_o^* \), is given by:

Model 1 (Option 1):

\[
\frac{\partial p_o}{\partial p_o^*} = \frac{\lambda(0) - \kappa}{\lambda(s_{eq}) - \kappa} \left( \frac{p_o^*}{p^*} \right)^{\frac{j(0) - j(s_{eq})}{j(s_{eq}) - \kappa}} \tag{5.27}
\]

Model 1 (Option 2):

\[
\frac{\partial p_o}{\partial p_o^*} = \alpha_c \left( \frac{j(0) - j(s_{eq})}{j(s_{eq}) - \kappa} \right) \tag{5.28}
\]
Model 2:  \[
\frac{\partial p_e}{\partial p_o} = \frac{p_o}{p_o^*} \left( \frac{p_o}{p_e^*} \right)^{-b} \left[ \lambda(0) - \kappa \right]^{-1} \left( b \ln \left( \frac{p_o}{p_e^*} \right) - 1 \right) + \lambda(0) - \kappa
\]  (5.29)

The parameter \( A \) can therefore be obtained from the following equations depending on the constitutive model:

Model 1 (Option 1): \[
A = \frac{v}{\lambda(s_{eq}) - \kappa} \left( p_o + f(s_{eq}) \right)^2 \left( 1 + \frac{\eta}{K_1} \right) \frac{K_o}{\kappa_{p_l}} \right)
\]  (5.30)

Model 1 (Option 2): \[
A = \frac{v}{\lambda(0) - \kappa} \left( p_o + f(s_{eq}) \right)^2 \left( 1 + \frac{\eta}{K_1} \right) \frac{K_o}{\kappa_{p_l}} \right)
\]  (5.31)

Model 2:

\[
A = \frac{v}{\lambda(s_{eq}) - \kappa} \left( p_o + f(s_{eq}) \right)^2 \left( 1 + \frac{\eta}{K_1} \right) \frac{K_o}{\kappa_{p_l}} \right)
\]  (5.32)

d) Calculation of the elastic strains due to changes in suction \( \{\Delta e_s^e\} \)

The elastic strain vector due to changes in equivalent suction are given by:

\[
\{\Delta e_s^e\} = \{1 \ 1 \ 1 \ 0 \ 0\}^T \frac{\kappa_s}{3v(s_{eq} + p_{aim})} \Delta s_{eq}
\]  (5.33)
e) Calculation of the plastic strains due to changes in suction \( \{ \Delta \varepsilon^p_s \} \)

If the secondary yield surface is active:

\[
\{ \Delta \varepsilon^p_s \} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix}^T \frac{\lambda_s - \kappa_s}{3v(\sigma_\text{eq} + P_{\text{am}})} \Delta \sigma_{\text{eq}}
\]

(5.34)

If the secondary yield surface is not active:

\[
\{ \Delta \varepsilon^p_s \} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T
\]

(5.35)

f) Calculation of the yield function derivatives with respect to equivalent suction \( \partial F/\partial s_{\text{eq}} \)

The vector of the yield function derivatives with respect to suction is calculated as follows:

\[
\left\{ \frac{\partial F}{\partial s_{\text{eq}}} \right\} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix}^T \frac{\partial F}{\partial s_{\text{eq}}}
\]

(5.36)

\[
\frac{\partial F}{\partial s_{\text{eq}}} = \frac{\partial f (s_{\text{eq}}) (p_o + f (s_{\text{eq}})) - (p + f (s_{\text{eq}})) \left( \frac{\partial p_o}{\partial s_{\text{eq}}} + \frac{\partial f (s_{\text{eq}})}{\partial s_{\text{eq}}} \right)}{(p_o + f (s_{\text{eq}}))^2}
\]

(5.37)

where, depending on the adopted variation of the apparent cohesion with equivalent suction:

For a linear increase:

\[
\frac{\partial f (s_{\text{eq}})}{\partial s_{\text{eq}}} = k
\]

(5.38)

For a nonlinear increase:

\[
\frac{\partial f (s_{\text{eq}})}{\partial s_{\text{eq}}} = S_r
\]

(5.39)

The derivative of the isotropic yield stress with respect to equivalent suction is given by:
Model 1 (Option 1):
\[
\frac{\partial p_o}{\partial s_{eq}} = \frac{\lambda(0) - \kappa}{\lambda(s_{eq}) - \kappa} \lambda(0)(1-r) \beta e^{-\beta s_{eq}} p_o \ln \frac{p_o^*}{p^*} \]  
\hspace{1cm} (5.40)

Model 1 (Option 2):
\[
\frac{\partial p_o}{\partial s_{eq}} = \frac{\lambda(0) - \kappa}{\lambda(s_{eq}) - \kappa} \lambda(0)(1-r) \beta e^{-\beta s_{eq}} p_o \ln \alpha_c \]  
\hspace{1cm} (5.41)

Model 2:
\[
\frac{\partial p_o}{\partial s_{eq}} = \frac{\lambda(0)(1-r) e^{-\beta s_{eq}} p_o \left( \frac{p_o}{p^*} \right)^b \left( \ln \frac{p_o^*}{p^*} \right)^2}{\ln \frac{p_o^*}{p^*} + b \ln \frac{p_o}{p_o^*} \ln \frac{p_o^*}{p^*}} \]  
\hspace{1cm} (5.42)

The flow chart of the ICFEP subroutine in which the above calculations are performed is given in Appendix I.

5.3 VALIDATION

This section presents single element analyses performed in order to validate the modifications made to the finite element program ICFEP and the implementation of the two constitutive models, demonstrate the models’ capabilities and limitations, and compare their predictions to experimental data. The analyses simulate the response of partially or fully saturated axisymmetric soil samples to a variety of stress paths.

5.3.1 Single finite element analyses of isotropic stress paths

The selected stress paths demonstrate the following features of the two models:

- Primary yield surface

- Secondary yield surface
- Coupling between the two yield surfaces
- Transition between fully and partially saturated states

The finite element results are compared to the theoretical solutions obtained from the model equations using simple spreadsheet calculations. As the problem geometry is not updated during conventional small strain finite element analyses, the predictions are not expected to match exactly the theoretical solutions. A few large displacement finite element analyses were therefore also performed to confirm that the expected small discrepancies are due to this fact only.

Stress increments of 25kPa and suction increments of 10kPa were used in most of the analyses presented in this section. Analyses were, however, also performed with different increment sizes, which indicated insignificant increment size dependency.

The simulated stress paths are presented in the $p$-$s_{eq}$ space and the predictions in the $v$-$\ln p$ space.

### 5.3.1.1 Primary yield surface

The fundamental difference between the two constitutive models presented in Chapter 4 is the assumed shape of the isotropic compression line and consequently of the primary yield surface. Model 1 approximates soil behaviour by adopting a linear (option 1) or bilinear (option 2) expression, while model 2 adopts a non-linear exponential expression. As discussed in the previous chapter this model feature controls the amount of collapse predicted along wetting paths.

- **Isotropic compression line**

Two isotropic loading paths at constant values of equivalent suction, of 0MPa and 0.2Mpa, are first considered. Figure 5-1 shows the initial stress state ($p = 0.1$MPa and $s_{eq} = 0.2$MPa) and the initial position of the primary yield surface (line $E_A$ in figure), which is selected to be a straight vertical line ($p_o^* = p_o = p^* = 0.1$Mpa). When the sample is loaded from A to B the primary yield surface
becomes curved and expands as it is dragged along the stress path. The final shape depends on the adopted constitutive model \( F_{B1a} \) for linear model 1, \( F_{B1b} \) for bi-linear model 1 and \( F_{B2} \) for model 2). The equivalent fully saturated compression line is obtained by first saturating the sample (path A to C), and then loading to point D. The final shapes of the yield surface at point D are also indicated on Figure 5-1 \( (F_{D1a} \) for linear model 1, \( F_{D1b} \) for bi-linear model 1 and \( F_{D2} \) for model 2). The model parameters are given in Table 5.1.

Figure 5-1: Validation stress paths – isotropic compression lines

| \( \alpha_f \) | 0.4 | \( \beta \) | 0.02 kPa\(^{-1} \) |
| \( \mu_f \) | 0.9 | \( b \) | 0.3 |
| \( M_f \) | 1 | \( \kappa_s \) | 0.0077 |
| \( \alpha_g \) | 0.4 | \( \nu_l \) | 3 |
| \( \mu_g \) | 0.9 | \( k \) | 0.8 |
| \( M_g \) | 1 | \( p_{atm} \) | 100 kPa |
| \( p^c \) | 100 | \( s_{air} \) | 0 kPa |
| \( \alpha_c \) | 1.667 | \( \mu \) | 0.2 |
| \( \lambda(0) \) | 0.066 | \( K_{min} \) | 100 kPa |
| \( \kappa \) | 0.0077 | \( S_o \) | 1000 kPa |
| \( r \) | 0.25 | \( A_s \) | 0.05 |

Table 5.1: Material properties
The predicted and calculated compression lines are shown in Figure 5-2. Curves P1a and P1b show the predicted and calculated response for the partially saturated case (path A to B) for model 1 with option 1 (linear compression line) and option 2 (bi-linear compression line), respectively, while curves P2 correspond to model 2. The equivalent fully saturated compression lines are indicated as S0 and are independent of the adopted constitutive model, as both models reduce to the same fully saturated constitutive model when the equivalent suction becomes equal to zero.

The finite element results are in excellent agreement with the theoretical calculations giving a maximum difference in the final specific volume of 0.15%. As noted above this difference is due to the fact that the geometry of the single element is not updated during the analyses. Large displacement analyses confirmed this, giving almost identical results to the theoretical solution (maximum difference of 0.0002%).

Figure 5-2: Validation analyses – isotropic compression lines
The Model 2 parameter $b$ controls the amount of the maximum predicted collapse, $\Delta v_{p_{\text{max}}}$, through Equation 4.44, and the value of the confining stress at which this takes place, $p_m$, through Equation 4.43. If $b$ is set to zero both $\Delta v_{p_{\text{max}}}$ and $p_m$ are infinite and the model becomes equivalent to the linear option of Model 1 (option 1). On the other hand if a high value is set for $b$ the amount of maximum collapse tends to zero and the predicted isotropic compression line lies below the equivalent fully saturated compression line and is parallel to it; the distance between the two lines being equal to the elastic swelling given by Equation 4.33 which corresponds to the current value of equivalent suction, $s_{eq}$. This feature of the model was investigated by repeating the above analyses with values of $b$ of 0.0001 and 1000. As expected the analyses that simulated stress path A-C-D gave identical results to those predicted for $b = 0.3$. For path A to B, the analysis with $b = 0.0001$ gave almost identical results to Model 1 – option 1 (maximum difference of 0.002%). The analysis with $b = 1000$ also gave the expected results; the predicted compression line is indicated by a dashed line in Figure 5-2 and is parallel to the fully saturated line.

- Wetting induced collapse

The capability of the models to predict collapse upon wetting and the influence of the confining stress on the predicted collapse are investigated through the set of stress paths shown in Figure 5-3. All stress paths start from the initial position $A_1$ ($p = 0.1\text{MPa}$ and $s_{eq} = 0.2\text{MPa}$) and their corresponding primary yield surface $F_{in}$ passes through point $A_2$ ($p_o = 0.2\text{MPa}$ and $s_{eq} = 0.2\text{MPa}$). The shape of the initial, $F_{in}$, and final, $F_f$, yield surfaces depends not only on the adopted model but also on the model parameters. The first stress path involved elastic wetting from $A_1$ to $B_1$ followed by loading to point $B_n$ ($p = 4\text{MPa}$ and $s_{eq} = 0\text{MPa}$). All the other stress paths involved an initial loading under constant suction to a given value of confining stress (points $A_i$), followed by elastoplastic wetting (collapse) at constant stress (to points $B_i$) and loading to the final stress state $B_n$. 

The model parameters are the same as those used in the previous set of analyses, given in Table 5.1, with the exception of the elastic compressibility coefficient due to changes in suction, $\kappa_s$, for which a low value of 0.001 was selected in order to reduce the elastic component of the wetting induced volume changes. For model 2, analyses were performed with two alternative values of the parameter $b$ of 0.1 and 0.6, in order to investigate the influence of this parameter.

Figure 5-4 shows the predicted volumetric deformations in terms of specific volume for stress path A$_1$-B$_1$-B$_n$. Curves S1a and S1b correspond to the linear and bilinear Model 1, respectively, and curves S2-0.1 and S2-0.6 to Model 2 for values of $b$ of 0.1 and 0.6. Also shown on the same figure are the results for path A$_1$-A$_n$-B$_n$ (curves P1a, P1b, P2-0.1 and P2-0.6).

The initial specific volume, $v_{in}$, (point A$_1$) is calculated from the input parameter $v_1$ (specific volume at unit pressure) and depends on the shape of the isotropic compression line and consequently on the shape of the initial primary yield surface. The initial specific volume for a given initial stress state and yield stress, $p_\sigma$, therefore varies with the choice of constitutive model and those model parameters that influence the primary yield surface. In the case analysed here, a value of 2.913 was predicted for model 1, while for model 2 depending on the
parameter $b$, values of 2.911 (for $b = 0.1$) and 2.902 (for $b = 0.6$) were obtained. These values matched the analytically calculated initial specific volumes.

![Figure 5-4: Validation analyses – wetting induced collapse](image)

In the first set of stress paths analysed in this section (Figure 5-1) the initial specific volume was the same for all cases since the initial yield stress, $p_0$, was set equal to the characteristic pressure, $p_c^*$, therefore defining a primary yield surface perpendicular to the $p$ axis in the $s_{eq} - p$ plane, irrespective of the adopted model and the model parameters.

The elastic swelling experienced along path $A_1$ to $B_1$, is independent of the adopted constitutive model and was predicted in all cases equal to 0.001, in agreement with the analytical calculation. Following the elastic swelling, during the initial stage of loading (path $B_1$ to $B_n$), the behaviour remained elastic until the equivalent fully saturated yield stress, $p_0^{*}$, (hardening parameter) was reached. Since in this set of analyses the partially saturated rather than the equivalent fully saturated yield stress was specified, the latter varied with the
constitutive model and the model parameters. The predicted and calculated values of $p_o^*$ were around 0.112MPa for model 1, and 0.117MPa and 0.137MPa for model 2 with $b$ equal to 0.1 and 0.6, respectively. Beyond the yield point all four analyses (curves S1a, S1b, S2-0.1 and S2-0.6) give the same fully saturated isotropic compression line, as seen in Figure 5-4. The associated analytically calculated curves were in very good agreement to the finite element predictions, giving a maximum difference of 0.3% at a confining stress of 4MPa, and have been omitted from the figure for clarity. As mentioned previously, this difference is eliminated if large displacement analyses are performed.

Curves P1a, P1b, P2-0.1 and P2-0.6 (stress path A1-Aa-Bn) in Figure 5-4 are linear elastic in the $v$-ln$p$ plane, up to the isotropic yield stress, $p_o$, of 0.2MPa. Beyond this value they follow the isotropic compression lines, as defined by the model used in each case and the model parameters. As expected, curve P2-0.6 ($b = 0.6$) is closer to the fully saturated isotropic compression line than curve P2-0.1 ($b = 0.1$). Both the amount of maximum collapse and the confining stress at which this takes place increase with a reducing value of $b$. At the end of the loading stage at a confining stress of 4000kPa (point A_n) the maximum difference between the predicted and calculated specific volumes is 0.29%.

After loading to point A_n, saturation of the soil samples was simulated (to point B_n). As seen in Figure 5-4, collapse was predicted in all cases and the final state lied on the equivalent fully saturated isotropic compression line. The overall volume change from the initial point A_1 to point B_n was found to be independent of the path followed, which was in agreement with the theoretical predictions.

The volumetric deformations are controlled by the change in the hardening parameter, $p_o^*$ (equivalent fully saturated yield stress), which was the same for the analyses performed using the same model and model parameters.

Similar tests to the ones discussed above were performed involving isotropic loading from point A_1 to different values of confining stress (points A_i), followed by saturation (points B_i) and subsequent loading to point B_n, as seen in Figure 5-3. Stress path independency for this particular case was confirmed as all analyses, with the same constitutive model and model parameters predicted the
same final specific volume at point $B_n$ equal to that predicted by stress paths $A_1-B_1-B_n$ and $A_1-A_n-B_n$.

Analyses with linear model 1 predicted a linear increase of the amount of wetting induced collapse (expressed in terms of change of specific volume) with confining stress in the $\nu$-$\ln p$ plane, while analyses with the bilinear model 1 predicted a constant amount of collapse, which was in accordance with the theoretical calculations. The predicted and calculated amount of wetting induced collapse, using model 2 for the two different values of the parameter $b$ considered above ($b = 0.1$ or $0.6$) is shown in Figure 5-5.

![Figure 5-5: Validation analyses – wetting induced collapse at various constant values of mean net stress](image)

Relatively large discrepancies are observed between the changes of specific volume due to wetting, predicted by the finite element analyses and those calculated theoretically (maximum difference of 7.5%). These however are mostly due to the fact that the analyses were performed using the conventional
small strain finite element formulation, while the analytical solution implies large strains. Large displacement analyses were performed to confirm this observation and their predictions are also plotted on Figure 5-5. The maximum difference between the predictions of the large displacement analyses and the theoretical calculations is 0.3%.

For a value of 0.6 of the parameter $b$, the predicted and calculated amount of collapse initially increases nonlinearly with confining stress, reaches a maximum value and then decreases tending towards zero at very high confining stresses. The confining stress corresponding to the maximum collapse, $p_m$, can be calculated from Equation 4.43 and is equal to 529.5kPa. The calculated maximum collapse in terms of change of specific volume in this case is equal to 0.029. A value of $b$ equal to 0.1 gives a very large confining stress $p_m$ of approximately 2.2GPa and a much larger value of the maximum potential collapse of 0.178.

- **Stress path dependency**

For the set of stress paths considered above (Figure 5-3) the final specific volume (point $B_n$) was found to be independent of the stress path followed. This however is a special case since all the stress paths considered resulted in the same change of the hardening parameter, $p_o$ (equivalent fully saturated isotropic yield stress). In general, the final volumetric deformations are stress path dependent. In order to demonstrate this, two stress paths are considered, as shown in Figure 5-6.

Path A-B-D involves an initial elastic compression due to drying up to an equivalent suction of 0.4MPa, during which the yield surface doesn’t move, followed by isotropic loading to a final confining stress of 1MPa. At the end of the stress path the primary yield surface has expanded and passes through the final point D, as seen in Figure 5-6. Along path A-C-D, isotropic loading first takes place, during which the primary yield surface is dragged to point C, as shown in Figure 5-6, and remains unchanged during the subsequent drying to the final point D. The final position of the yield surface is therefore different for the two stress paths and consequently the change of the hardening parameter is also
different. Since a larger change of the hardening parameter is caused by stress path A-C-D it is expected that the change of specific volume is also larger. This can be seen in Figure 5-7, which presents the finite element predictions for the change in specific volume along the considered stress paths. Curves M1a and M1b correspond to linear and bilinear constitutive model 1, respectively, and curves M2 to constitutive model 2. The parameters given in Table 5.1 were used in all analyses.

Figure 5-6: Validation stress paths – stress path dependency

As expected, the final predicted and calculated specific volumes for stress path A-C-D are always lower than those calculated for path A-B-D. It is interesting to note that the isotropic compression line for the bilinear option of model 1 is independent of the value of the equivalent suction, $s_{eq}$, as it is controlled entirely from the value of the characteristic ratio, $a_c$. The finite element predictions were in good agreement with the theoretical calculations giving a maximum difference of 0.1%.
5.3.1.2 Secondary yield surface

The performance of the secondary yield surface is investigated through the simulation of a simple drying-wetting cycle at constant applied confining stress (Figure 5-8). The model parameters are given in Table 5.1. A value of 0.2MPa is set however for the yield suction, \( s_o \), instead of the high value used in the previous analyses.

Figure 5-9 shows the predicted and calculated volumetric response (in terms of specific volume) to the above stress path. The shape of the primary yield surface does not affect the models’ predictions and therefore all models give identical results. The finite element predictions are in excellent agreement with the theoretical calculations (maximum difference 0.007%).

The response during the initial stage of drying (up to 0.2MPa, path AC) is elastic giving a relatively small reduction of the specific volume. The drying line (AC) in Figure 5-9 is not linear in the \( \nu \)-ln\( s_{eq} \) plane since the deformations are controlled by the sum of the equivalent suction and the atmospheric pressure.
Beyond the yield suction the behaviour is elastoplastic leading to larger deformations (path CB). The behaviour during wetting (path BD) is purely elastic and the value of yield suction remains unchanged at 0.4MPa.

Figure 5-8: Validation stress paths – secondary yield surface

Figure 5-9: Validation analyses – secondary yield surface
5.3.1.3 Coupling between the two yield surfaces

The coupling between the two yield surfaces is investigated through the example of the stress paths shown in Figure 5-10. The movements of the primary and secondary yield surfaces are related to each other through Equation 4.57. Consequently the volumetric strains associated with the activation of either of the two yield surfaces can be calculated from the change in the primary hardening parameter (equivalent fully saturated isotropic yield stress), \( p_o^* \), only, through Equation 4.55. This method was used to analytically calculate the volumetric response to the two stress paths under consideration. The two constitutive models give identical results since the shape of the primary yield surface doesn’t affect the calculations. The soil properties used in the finite element analyses and the analytical calculations are listed in Table 5.1.

![Figure 5-10: Validation stress paths – coupling between the two yield surfaces](image)

The predicted and calculated changes in specific volume along stress paths A-B-E-C and A-C are shown in Figure 5-11. The initial stress state (point A) is on the fully saturated isotropic compression line. As the soil is loaded (path A-C) the stress state moves along this line to point C; the difference between the predicted and calculated specific volume at the end of the path is 0.14\%. Along path A-B
the stress state initially moves away from the yield surface, entering the elastic zone, and small elastic compression takes place up to point D ($s_{eq} = 0.05\text{MPa}$) in Figure 5-11. Thereafter the secondary yield surface is activated and large elastoplastic deformations are incurred (point B). The movement of the secondary yield surface ($\Delta s_{eq} = 0.45\text{MPa}$) causes a movement of the primary yield surface ($\Delta p_0^* \approx 0.347\text{MPa}$) and after subsequent saturation (point E) the soil state is now well inside the elastic region. When the soil is now loaded it compresses elastically to point F as shown in Figure 5-11 before meeting the fully saturated compression line. The final specific volume is the same for stress paths A-C and A-B-E-C since the primary hardening parameter at the end of the stress paths is also the same.

Figure 5-11: Validation analyses – coupling between the two yield surfaces
5.3.1.4 Transition between partially and fully saturated states

In most engineering problems involving partial soil saturation only regions of the soil are partially saturated (e.g. capillary zone). Moreover the boundaries between partially and fully saturated soil regions are likely to change with time; in the most common case due to fluctuations of the ground water table. An appropriate soil model must therefore be applicable to both soil states. The two constitutive models presented in this thesis are formulated in total stress terms for partially saturated conditions \( s_{eq} > 0 \text{MPa} \) and switch to effective stress terms at the transition to fully saturated conditions.

All the analyses presented above were performed assuming a zero air entry suction value \( s_{air} = 0 \text{kPa} \) and therefore in all cases \( s_{eq} = s \). In addition, only partially saturated \( (s_{eq} > 0 \text{MPa}) \) and fully saturated with zero equivalent suction \( (s_{eq} = 0 \text{kPa}) \) conditions were considered. Consequently all analyses were essentially in total stress terms. This section investigates the performance of the models during the transition from partially to fully saturated states and vice versa. The more general case of a non-zero value of the air entry value, \( s_{air} \), is examined.

Figure 5-12: Validation stress paths – transition between partially and fully saturated states
The stress path shown in Figure 5-12 is considered. The soil during initial wetting (path A to B) is partially saturated. Once the air entry suction ($s_{air} = 0.025\text{MPa}, s_{eq} = 0\text{MPa}$) is reached (point B), the soil becomes fully saturated and the behaviour is controlled by the effective stress principle. Upon loading at a constant compressive pore water pressure of 50kPa (path C to D) the soil reaches first yield at a total stress of 0.275MPa ($\sigma’ = 0.225\text{MPa}$). During subsequent drying at constant total stress the soil continues to deform elastoplastically until the transition to partially saturated conditions (point E). Thereafter only elastic compression takes place.

![Figure 5-13: Validation analyses – transition between partially and fully saturated states](image)

The changes of the specific volume along the above path predicted by the finite element analysis and the analytical calculation are presented in Figure 5-13. The analysis and calculations were performed using the soil properties listed in Table 5.1, except for the air entry value which was set to 0.025MPa. The predictions can be seen to be in good agreement with the calculations, giving a maximum difference of 0.02%. As with the previous set of analyses the shape of the Primary Yield surface does not affect the predictions and therefore both models give the same results.
5.3.2 Single finite element analyses of deviatoric stress paths

The selected stress paths demonstrate the following features of the two models:

- Primary yield surface
- Plastic potential surface
- Critical state

The stress paths considered are presented in the \( p-s \) stress space and the finite element results are given in the form of \( J-p, J-E_d \) and \( \varepsilon_v-E_d \) plots. The deviatoric strain, \( E_d \), and the volumetric strain, \( \varepsilon_v \), are given as follows:

\[
E_d = \frac{2}{\sqrt{6}} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2} \tag{5.43}
\]

\[
\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{5.44}
\]

The analytically calculated positions of the yield and plastic potential surfaces at the beginning and end of each analysis are also plotted in the \( J-p \) plots.

As discussed previously, the main difference between the two models introduced in this thesis regards the shape of the isotropic compression line and consequently the shape of the primary yield surface in the \( p-s_{eq} \) plane. This feature affects the predicted strains along deviatoric stress paths, as it controls the size of the yield surface in the \( J-p \) plane. However, since the differences between the two models were illustrated in length in the previous section, only the results of the more general model 2 are discussed here.

The analyses presented in this section were increment size independent.

5.3.2.1 Primary yield surface and Critical State

Five constant mean net stress, \( p \), shearing paths at different values of equivalent suction are first considered. The initial stress state is isotropic in all cases and is shown in the \( p-s \) plane in Figure 5-14 (points A, B, C, D and E). Also shown in
the same figure is the initial position of the primary yield surface. The increase of the apparent cohesion with suction is assumed to be linear as indicated by the straight line representing the extension of the elastic area into the tensile mean net stress region.

![Graph showing initial stress states and position of the Primary Yield surface](image)

Figure 5-14: Initial stress states and position of the Primary Yield surface

The same soil properties as those used in the isotropic analyses were adopted (Table 5.1), with exception of the air entry value, which was set to 25kPa. An associated flow rule is assumed, as equal values are selected for the yield surface parameters $\alpha_f$, $\mu_f$ and $M_f$ and the respective plastic potential parameters $\alpha_g$, $\mu_g$ and $M_g$.

In all five tests the deviatoric stress, $J$, is increased while maintaining constant mean net stress and suction, until critical state is reached. Figure 5-15 shows the final stress states (critical state) predicted by the finite element analyses and the calculated initial and final positions of the primary yield surface.

As expected the analyses at higher suction values reach critical state at higher values of the deviatoric stress. The fully saturated analysis ($s = 0.025\text{MPa}, s_{eq} = 0\text{MPa}$) reaches a maximum deviatoric stress of 129kPa, while for the analysis at 0.425MPa suction ($s_{eq} = 0.400\text{MPa}$) the ultimate deviatoric stress is 0.313MPa (143% increase). This large difference is due to the increase of apparent cohesion.
with suction, which controls both the initial and final positions of the yield surface.

Figure 5-15: Initial and final positions of the Primary Yield surface

The stress-strain behaviour predicted by the analyses can be seen in Figure 5-16. While the fully saturated analysis is entirely elastoplastic, the partially saturated analyses are initially elastic and show a sharp change of behaviour when the yield surface is first reached. The partially saturated stress-strain curves are identical in the elastic region, since the shear modulus, $G$, is independent of suction.

The deviatoric stress at which the behaviour changes from purely elastic to elastoplastic depends not only on the assumed increase of the apparent cohesion but also on the increase of the isotropic yield stress with suction. It is therefore constitutive model dependent. Clearly use of linear model 1 would predict higher isotropic yield stresses and therefore more elastic behaviour. However, the final deviatoric stress would not be affected.
Figure 5-16: Stress-strain curves

Figure 5-17: Relationship between the volumetric and deviatoric strains
Figure 5-17 illustrates the relationship between the predicted volumetric and deviatoric strains. In the elastic region zero volumetric strains are predicted since the mean net stress doesn’t change. In the elastoplastic region the ratio $d\varepsilon_v/dE_d^p$ is related to the derivative $dG/dp$ at the current stress state. At critical state $dG/dp = 0$ and $d\varepsilon_v/dE_d^p = 0$, as seen in all five curves, while for isotropic conditions $dG/dp = \infty$ and $d\varepsilon_v/dE_d^p = \infty$, which is consistent with the predictions for the fully saturated case.

5.3.2.2 Non-linear increase of apparent cohesion

The test starting at point E ($p = 0.2$MPa, $s = 0.425$MPa) presented above was repeated assuming a non-linear relationship for the increase of apparent cohesion with suction. As discussed in Chapter 4, this can be done by relating the parameter $\chi$ to the degree of saturation, $S_r$. The variation of the degree of saturation with suction was modelled with the Van Genuchten expression given in Chapter 2. The following additional parameters were selected: $\psi = 0.014$, $n = 5$, and $m = 0.4$. The resulting water retention curve is shown in Figure 5-18, and the increase of apparent cohesion with suction in Figure 5-19.

The predicted final deviatoric stress and the associated yield surface for a suction of 0.425MPa is shown in Figure 5-20. These are compared to the results of the equivalent analysis with a linear increase of the apparent cohesion with suction. In the non-linear case the soil is on the dry side and critical state could not be reached, as the analysis was stress controlled; the final stress state however was close enough to assume critical state for comparison purposes. This example, although maybe slightly exaggerated with regard to the adopted model parameters, is indicative of the potential overestimation of the shear strength the linear approach can lead to (approximately 80% overestimation).
Figure 5-18: Water retention curve

Figure 5-19: Relationship between apparent cohesion and suction
5.3.2.3 Plastic potential surface – non-associated flow rule

The plastic potential surface is defined independently of the yield surface. An associated flow rule is attained only in the special case when the relative yield and plastic potential parameters are equal. The more general case of non-associated flow rule is examined here for the stress path starting at $s = 225\text{kPa}$ described in section 5.3.2.1. The plastic potential parameters are taken as: $\alpha_g = 0.1, \mu_g = 0.9$ and $M_g = 1$, and the resulting plastic potential surface is illustrated in Figure 5-21. Since the parameter $M_g$ is equal to the yield surface parameter $M_f$, the ultimate deviatoric stress is the same as in the analysis with associated flow rule, as are the volumetric strains. The predicted deviatoric strains are however as expected higher, as seen in Figures 5-22 and 5-23.
Figure 5-21: Initial Primary Yield and Plastic Potential surfaces

Figure 5-22: Stress-strain curves for associated and non-associated flow rule
5.3.3 Comparison with experimental results

Two sets of experimental data are considered in this section:

a) Set 1

The first set involves tests on partially saturated compacted kaolin reported by Josa (1988). Alonso et al. (1990) compared this set of data to the predictions of the Barcelona Basic model, and Thomas & He (1998) used the set to validate their finite element implementation of the same constitutive model. The model parameters used by the above authors are presented in Table 5.2. The notation used in Table 5.2 is that of model 1 – option 1 (linear isotropic compression line), which for the particular set of parameters is equivalent to the Barcelona Basic model.
The two stress paths (A-K-L-I and A-K-C-I) shown in Figure 5-24 are first considered. The initial conditions (point A) are the following: $J = 0$ kPa, $p = 45$ kPa, $s = s_{eq} = 10$ kPa and $\nu = 1.915$. The initial equivalent fully saturated isotropic yield stress (primary hardening/softening parameter) is $p_o^* = 55$ kPa.

<table>
<thead>
<tr>
<th>$\alpha_f$</th>
<th>0.4</th>
<th>$\beta$</th>
<th>0.0164 kPa$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_f$</td>
<td>0.9</td>
<td>$\kappa_s$</td>
<td>0.01</td>
</tr>
<tr>
<td>$M_f$</td>
<td>0.82</td>
<td>$\nu_f$</td>
<td>2.15</td>
</tr>
<tr>
<td>$\alpha_s$</td>
<td>0.4</td>
<td>$k$</td>
<td>1.24</td>
</tr>
<tr>
<td>$\mu_g$</td>
<td>0.9</td>
<td>$p_{atm}$</td>
<td>100 kPa</td>
</tr>
<tr>
<td>$M_g$</td>
<td>0.82</td>
<td>$s_{air}$</td>
<td>0 kPa</td>
</tr>
<tr>
<td>$p^*$</td>
<td>43</td>
<td>$G$</td>
<td>3300 kPa</td>
</tr>
<tr>
<td>$\lambda(0)$</td>
<td>0.14</td>
<td>$K_{min}$</td>
<td>100 kPa</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.015</td>
<td>$s_o$</td>
<td>30 kPa</td>
</tr>
<tr>
<td>$r$</td>
<td>0.26</td>
<td>$\lambda_s$</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 5.2: Material properties for set 1 (after Alonso et al. (1990))
The measured variation of the specific volume along the two paths is shown in Figure 5-25. Also shown on the same figure are the finite element predictions obtained with model 1 – option 1 using the material properties listed in Table 5.2. As seen the predicted and measured response is in reasonably good agreement. The largest difference between the finite element predictions and the measurements is observed during the initial drying section of path A-K-L-I. It should be noted though that the reported response along this section may not be very accurate, since approximately the same specific volume change was measured for suction changes of 50kPa (path AK) and 80kPa (path AC).

![Figure 5-25: Change of specific volume along stress paths A-C-I and A-K-L-I](image)

The values of the isotropic yield stress and the shapes of the isotropic compression lines cannot be clearly identified from the experimental curves of specific volume versus the natural logarithm of the mean net stress. It seems however that particularly for the path involving loading at the higher applied suction (A-C-I) of 90kPa, the finite element predictions overestimate the yield stress, and in addition the isotropic compression is slightly non-linear. A better
agreement with the experimental results is obtained when the non-linear model (model 2) is used with $b = 0.03$, and the following modified parameters: $p^C = 55\text{kPa}$, $r = 0.2$ and $\beta = 0.03$. The predicted specific volume variation in this case is shown in Figure 5-26. Also shown in the same figure are the finite element predictions with model 1 – option 1 and the modified parameters.

![Figure 5-26: Change of specific volume along stress path A-C-I – modified parameteres](image)

Numerical simulation of a constant mean net stress shearing test reported by Josa (1988) was also performed. The stress state at the beginning of shearing was the same as that at point I in the above test. Figure 5-27 shows the measured relationship between the deviatoric stress, $J$, and the deviatoric strain, $E_d$. Also shown on the same figure are the numerical predictions for model 1 – option 1 with the material properties shown in Table 5.2, and model 2 with the modified material properties. It can be seen that the numerical predictions are in very good agreement with the experimental results. In addition, in this case the shape of the isotropic compression line does not affect the predictions, as both models give almost identical stress-strain curves.
b) Set 2

The second set of laboratory tests simulated using ICFEP were two tests on partially saturated compacted kaolin reported by Karube (1986) and considered by Alonso et al. (1990). The model parameters used by Alonso et al. (1990) were also adopted here and are listed in Table 5.3. The parameter values not shown in Table 5.3 were not relevant for the two stress paths under consideration as both were isotropic.

The stress paths analysed are shown in Figure 5-28. Both start and end at the same position (point A): $J = 0$ kPa, $p = 20$ kPa, $s = s_{eq} = 50$ kPa. The initial specific volume is equal to 2.098. The initial equivalent fully saturated isotropic yield stress (primary hardening/softening parameter) is $p_o^* = 40$ kPa.
Table 5.3: Material properties for set 2 (after Alonso et al. (1990))

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>-</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.02 kPa$^{-1}$</td>
</tr>
<tr>
<td>$\mu_f$</td>
<td>-</td>
</tr>
<tr>
<td>$\kappa_s$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\sigma_f$</td>
<td>-</td>
</tr>
<tr>
<td>$\mu_g$</td>
<td>-</td>
</tr>
<tr>
<td>$p_{atm}$</td>
<td>100 kPa</td>
</tr>
<tr>
<td>$M_g$</td>
<td>-</td>
</tr>
<tr>
<td>$s_{air}$</td>
<td>0 kPa</td>
</tr>
<tr>
<td>$p^c$</td>
<td>10</td>
</tr>
<tr>
<td>$\lambda(0)$</td>
<td>0.065</td>
</tr>
<tr>
<td>$K_{min}$</td>
<td>100 kPa</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.011</td>
</tr>
<tr>
<td>$s_o$</td>
<td>70 kPa</td>
</tr>
<tr>
<td>$r$</td>
<td>0.75</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Figure 5-28: Stress paths for set 2

Figure 5-29 shows the measured and predicted (with model 1 – option 1) variation of the specific volume along paths A-B-C-D-A and A-D-C-B-A. As seen the finite element predictions are in good agreement with the measurements.
5.4 CONCLUSIONS

The two generalised constitutive models for partially and fully saturated soils, which were presented in Chapter 4, were implemented into the Imperial College Finite Element Program (ICFEP) and validated through a series of single finite element analyses.

a) Implementation

The choice of constitutive model influences both the formulation and the solution of the global equations. More specifically the constitutive model is used to determine the incremental elastic matrix, \([D]\), from which the incremental global stiffness matrix \([K_g]\) is calculated to form the global set of equations. The constitutive model is then used during the solution of the global equations in
order to determine the residual load vector \( \{ \psi' \} \) for each solution iteration. The terms required for the determination of the elastic matrix, \([D]\), and the stress-strain relationship were presented in this chapter.

b) Validation

A series of single finite element analyses using both constitutive models were presented in this chapter. The results of these analyses highlighted the models’ features for both isotropic and anisotropic conditions, and were compared to analytical (spreadsheet) calculations, validating in this way the models’ implementation.

The two models were also used to simulate two sets of experimental tests. The comparison of the finite element results with the measurements was satisfactory.
Chapter 6

Influence of partial soil saturation on the behaviour of footings

6.1 INTRODUCTION

A common engineering problem which often involves partially saturated soils is that of a shallow foundation that rests above the groundwater table. In many cases a capillary zone exists above the groundwater table, where the soil is partially saturated, and which can be very large depending on the soil type. Typical footing analyses ignore this zone and assume that the soil above the groundwater table is dry.

Nesnas (1995) presented partially saturated finite element analyses of a strip footing using the Barcelona Basic model and assuming weightless soil and a constant value of suction throughout the soil domain. These analyses indicated an increase of the bearing capacity and a decrease of the settlements with increasing suction, but the predicted load-settlement curves were increment size dependent and did not tend to an ultimate load even at settlements of several meters. An additional analysis simulating the rise of the groundwater table at constant load, in which the soil weight was taken into account indicated the possibility that further settlement may take place due to wetting.

This chapter presents a series of partially saturated and conventional (fully saturated soil below and dry soil above the groundwater table) finite element analyses of a strip footing performed with ICFEP using the two constitutive models introduced in this thesis. The influence of partial soil saturation and of fluctuations of the groundwater table on the behaviour of footings is investigated. Comparison is made between the predictions of partially saturated
and conventional analysis and also between the finite element predictions and the analytically calculated bearing capacity.

6.2 DETAILS OF ANALYSES

All analyses involved a 2m wide rough rigid strip footing bearing on a uniform soil. Four different depths of the groundwater table were considered: 0m (fully saturated soil), -1m, -2m and -4m and two different pore pressure profiles. The pore pressure profiles are shown in Figure 6-1. Profile 1 is the typical profile which is used in conventional analyses assuming that the soil is completely dry above the ground water table, while profile 2 assumes a hydrostatic pore pressure profile up to the ground surface. An air entry suction value of zero was used in all the profile 2 analyses and therefore the soil in these analyses was treated as partially saturated from the water table to the ground surface. All the analyses were performed drained.

Three sets of analyses were performed:

Set 1: Consists of seven analyses. The footing was loaded to failure for each of the three groundwater table depths (-1m, -2m and -4m), and for both pore pressure profiles. It also contains an analysis for the fully saturated case (groundwater table at ground level).

Set 2: Consists of six analyses. The footing was loaded to a certain load with the water table at -2m and subsequently the groundwater table was raised to the ground level at constant applied load. Three different loads were considered: 100kN, 175kN and 350kN, and both pore pressure profiles.

Set 3: Consists of three analyses, which involved loading the footing to the same three different loads as before, with the water table at -2m, raising the water table to -1m and then loading the footing to failure. Only pore pressure profile 2 was considered in this set of analyses.

The finite element mesh used in the analyses is shown in Figure 6-2. Eight noded plane strain elements with reduced integration were used.
The finite element solutions presented in this chapter were not dependent on the increment size of the analyses.

Figure 6-1: Assumed pore pressure profiles

Figure 6-2: Finite element mesh
6.3 CONSTITUTIVE MODEL AND PARAMETERS

All sets of analyses outlined above were performed with constitutive model 1 – option 1 (linear isotropic compression line). The use of this model is justified by the fact that the stress and suction levels are relatively small in the partially saturated soil region as this is limited to a small depth from the ground surface in all cases analysed. However, several analyses were also performed with constitutive model 2 for different values of the parameter $b$ in order to investigate the influence of the shape of the isotropic compression line on the footing behaviour. These analyses are discussed in Section 6.6.

The soil parameters used in the analyses are shown in Table 6.1. The yield function and plastic potential parameters $\alpha_f$, $\alpha_g$, $\mu_f$ and $\mu_g$ were selected such that the model in the fully saturated case becomes equivalent to the modified Cam-Clay model. The soil parameters are for Lower Cromer till and are taken from the studies of Gens (1982), Gens & Potts (1982) and Maswoswe (1985). A constant Poisson’s ratio, $\mu$, was used instead of a shear modulus $G$, in order to have an increasing value of $G$ with depth.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_f$</td>
<td>0.4</td>
</tr>
<tr>
<td>$\mu_f$</td>
<td>0.9</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.0164kPa$^{-1}$</td>
</tr>
<tr>
<td>$M_f$</td>
<td>1.2</td>
</tr>
<tr>
<td>$\alpha_g$</td>
<td>0.4</td>
</tr>
<tr>
<td>$\mu_g$</td>
<td>0.9</td>
</tr>
<tr>
<td>$M_g$</td>
<td>1.2</td>
</tr>
<tr>
<td>$p^c$</td>
<td>12.0kPa</td>
</tr>
<tr>
<td>$S_{air}$</td>
<td>0.0kPa</td>
</tr>
<tr>
<td>$\lambda(0)$</td>
<td>0.066</td>
</tr>
<tr>
<td>$K_o$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.0077</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>17.0kN/m$^3$</td>
</tr>
</tbody>
</table>

Table 6.1: Soil parameters for constitutive model 1 – option 1
A constant value with depth of 1.5 was assumed for the OCR throughout the soil. In the partially saturated soil region OCR defines the ratio of the equivalent fully saturated isotropic yield stress, $p_o^*$, over the mean net stress, $p$, (option 1 in section 4.3.3). The yield stress ratio (YSR) is defined as the ratio of the current partially saturated yield stress, $p_o$, over the mean net stress, $p$, and therefore varies with depth. The yield stress increases with suction, giving high YSR values close to the ground surface.

For simplicity and since the problem analysed does not involve high values of suction ($s \leq 39.24 \text{kPa}$) a linear increase of the apparent cohesion with suction was assumed ($k = \text{const.}$).

A high value of the yield suction $s_o$ was selected, as no depression of the groundwater table (increase of suction) was modelled and therefore this parameter was not relevant.

Finally, the same unit weight of 17kN/m$^3$ was assigned to the soil above and below the groundwater table for all analyses. This simplification was made to aid comparison between the results of different analyses and is justified since for the suction levels involved in the problem analysed (maximum suction of 39.24kPa) the degree of saturation, $S_r$, is expected to be relatively high.

6.4 ANALYTICAL CALCULATION OF BEARING CAPACITY FOR CONVENTIONAL ANALYSIS

Consider the idealised failure mechanism illustrated in Figure 6-3. The bearing capacity equation for this case can be calculated from:

$$q_{ult} = 0.5\gamma_e BN_f$$

(6.1)

where $\gamma_e$ is the average effective unit weight of the soil wedge beneath the footing, $B$ is the width of the footing, and $N_f$ is the bearing capacity factor.

The average effective unit weight, $\gamma_e$, can be calculated as follows (after Bowles 1997):
\[
\gamma_c = (2H - D) \frac{D}{H^2} \gamma + \frac{\gamma'}{H^2} (H - D)^2
\]  
(6.2)

where \(\gamma\) and \(\gamma'\) are the wet and submerged unit weights, respectively, and \(H\) is the depth of the wedge below the footing:

\[
H = 0.5B \tan(45^\circ + \phi/2)
\]  
(6.3)

For the adopted value of the parameter \(M_k = 1.2\), the angle of shearing resistance in triaxial compression is equal to 30°. However since the problem analysed is in plane strain conditions the associated angle of shearing resistance needs to be used. The angle of shearing resistance in plane strain is equal to 32.9° (the analytical calculation is given in Appendix II). The depth of the wedge below the footing, \(H\), for this value of \(\phi\) is equal to 1.84m.

The bearing capacity factor, \(N_p\), was determined with the Meyerhof, Hansen, Vesic and Terzaghi methods, which gave values of 26.2, 24.8, 35.2 and 31.9, respectively.

Figure 6-3: Schematic failure mechanism
6.5  RESULTS

6.5.1 Set 1 – Loading to failure

The load-settlement curves for the first set of analyses, which involved drained loading of the footing, are shown in Figures 6-4 and 6-5. Curves P1, P2 and P4 resulted from the analyses with the G.W.T. at -1m, -2m and -4m, respectively and pore pressure profile 2. Curves D1, D2 and D4 are for the G.W.T at -1m, -2m and -4m, respectively and pore pressure profile 1. Finally curve S0 resulted from the fully saturated analysis (G.W.T. at 0m).

Plotted on the same figures are also the ultimate loads for the fully saturated case, calculated from the theoretical bearing capacity expression (Equation 6.1) using the factors proposed by Meyerhof, Hansen, Vesic and Terzaghi.

All analyses reached an ultimate load, however in most cases this was achieved at settlement values larger than 2m. Since such settlements are not likely in practice, the load corresponding to a settlement of 2m (equal to the footing width) will be regarded as the ultimate load for the remainder of this chapter.

Figure 6-4: Load-settlement curves for partially saturated drained loading analyses
Figures 6-4 and 6-5 demonstrate that the bearing capacity of the footing increases with the increase of the depth of the groundwater table and that this increase is much larger for the partially saturated analyses (profile 2). For example, an increase of the groundwater table depth from 1m to 2m resulted in a 40% increase of the ultimate load in the partially saturated analysis (pore pressure profile 2), while the increase in the conventional analysis (pore pressure profile 1) was only 15%.

It is also illustrated in Figures 6-4 and 6-5 that conventional analyses (pore pressure profile 1) underestimate significantly the predicted ultimate loads. For example, the partially saturated analysis for -1m G.W.T.(curve P1) predicts an ultimate load approximately 15% higher than that predicted by the equivalent dry analysis (curve D1). This difference increases with the increase of the depth of the groundwater table. For –2m G.W.T. (curves P2 and D2) the increase is 40%, while for –4m GWT (curves P4 and D4) the increase is 90%.

Figure 6-5: Load-settlement curves for conventional drained loading analyses
The finite element analysis result of the fully saturated case (curve S0) is in good agreement with theoretical predictions of the bearing capacity made with Equation 6.1 when the bearing capacity factors of Hansen and Meyerhof are used, but is generally more conservative.

The ultimate loads predicted by the partially saturated and conventional finite element analyses and those calculated analytically through Equation 6.1, are plotted against the normalized depth of the groundwater table (\(=D/B\)) on Figure 6-6. The analytically calculated ultimate load increases non-linearly with the depth of the G.W.T. and reaches a maximum value when \(D = H\). Beyond this depth the calculated average effective unit weight, \(\gamma_e\), is equal to the wet unit weight, \(\gamma\), and therefore the calculated ultimate load remains constant. A similar variation can be observed for the results of the conventional finite element analyses, which are generally more conservative, but in good agreement with the
Hansen and Meyerhof solutions. In contrast, the partially saturated analyses predict that the ultimate load continuously increases with the depth of the groundwater table. As indicated by the dashed line on the same figure, the influence of partial saturation on the predicted ultimate load increases constantly with the depth of the G.W.T., a behaviour that can mainly be attributed to the increase of the apparent cohesion with suction. Ideally this could be dealt with analytically by including a cohesion term in the bearing capacity equation. However, such an approach is not straightforward since suction varies with depth and so does the value of the apparent cohesion.

6.5.2 Set 2 – Rise of groundwater table

Figure 6-7 shows the load-settlement relationships predicted by three analyses in which the footing was loaded to three different loads 100kN (PW1), 175kN (PW2) and 350kN (PW3) and subsequently the water table was raised from -2m to the ground level. All three analyses were partially saturated (pore pressure profile 2). On the same figure are also plotted for comparison the load-settlement curves S0 and P2 presented in the previous section.

Figure 6-7: Load-settlement curves for partially saturated wetting analyses
It can be seen in this figure that the higher the load at which wetting took place the larger the induced settlement. Partially saturated soils may swell or collapse upon wetting, depending on how close their stress state is to the yield surface (Primary yield surface). Obviously, for higher footing loads, the soil zone that is close to collapse is larger and consequently the wetting induced settlements are also larger. In fact, in analysis PW3 the footing failed when the water table rose to -0.8m. The vectors of incremental displacements of this analysis just before failure are presented in Figure 6-8.

Figure 6-9 shows the progression of the vertical movement of the footing as the water table rose, for the three analyses mentioned above. It also shows the displacement vs. rise of the groundwater table relationship obtained from a conventional dry analysis (DW2) with pore pressure profile 1, in which the loading and wetting sequence of analysis PW2 was followed. It is clear that the conventional analysis is not able to produce realistic behaviour and predicts heave of the footing. Another interesting aspect shown in Figure 6-9 is that for the analyses PW1 and PW2 the vertical displacement initially increases almost linearly as the water table rises, but eventually comes to a constant value when
the water table comes close to the ground surface and even slightly reduces for analysis PW1. The reason for this is that as the water level rises, the partially saturated zone, which is where most of the compression occurs, reduces. On the other hand for analysis PW3 (wetting at high load) the vertical displacement initially increases linearly with the rise of the ground water table, but then, close to failure, tends to infinity.

![Figure 6-9: Progression of vertical movement with rise of groundwater table](image)

6.5.3 Set 3 – Subsequent loading

Figure 6-10 presents the load-settlement curves for the third set of analyses. Curves PR1, PR2 and PR3 were predicted from analyses which followed initial loadings to 100kN, 175kN and 350kN, respectively, with the groundwater table at –2m and pore pressure profile 2. In all cases the G.W.T. was raised to –1m and the footing was loaded to failure. It is clearly illustrated in the figure that the analyses PR1, PR2 and PR3 give very similar ultimate loads, which are larger than the ultimate load of analysis P1, which corresponds to loading to failure.
from an initial groundwater table at –1m. This behaviour is not surprising as in the subsequent loading analyses, although the final G.W.T. in all four cases was at –1m, the footing was able to sustain the same initial load with significantly less settlement. After wetting the overall settlement was still significantly smaller than that experienced in analysis P1 for the same load, indicating a different stress distribution and extent of the elastoplastic zone in the soil.

The difference between the soil response to wetting and that to subsequent loading is shown in Figure 6-11 for one of the analysed cases. Figure 6-11a shows the vectors of incremental displacements at the end of wetting from –2m to –1m (analyses PW2 and PR2) and Figure 6-11b shows the vectors at the beginning of subsequent loading (analysis PR2). The patterns of movements are very different in the two cases. Figure 6-11a indicates that during the rise in the G.W.T. the soil is beginning to form a failure mechanism. However, Figure 6-11b indicates that on subsequent loading the soil is behaving as if the footing was not near to failure.
6.6 INFLUENCE OF THE SHAPE OF THE ISOTROPIC COMPRESSION LINE

All the analyses presented above were performed assuming a linear partially saturated isotropic compression line. To investigate the influence of the shape of the compression line on the predictions, loading (set 1) and wetting (set 2) finite element analyses were performed with model 2 using different values of the parameter $b$. Three values of $b$ were considered: 0.1, 0.226 and 0.472, which correspond to maximum potential collapse at a confining stresses, $p_m$, of approximately 265MPa, 1000kPa and 100kPa, respectively. Analyses were also performed, as a form of validation, using a very small value of $b$ of 0.001 for which model 2 theoretically becomes equivalent to model 1 (option 1). The results of these latter analyses were indeed identical to the model 1 results and therefore will not be presented. The isotropic compression lines predicted with these three values of the parameter $b$ for a value of suction equal to 19.62kPa (at ground surface) are plotted in Figure 6-12.
Figure 6-12: Isotropic compression lines for different values of the parameter $b$

### 6.6.1 Load-settlement curve

Three loading analyses similar to analysis P2 (G.W.T. at –2m and pore pressure profile 2) were performed with the three different $b$ values. The predicted load-settlement curves are plotted in Figure 6-13. It can be seen that the parameter $b$ does not have any significant influence on the predicted curves. Consequently neither do the shape of the isotropic compression line nor the shape of primary yield surface in the isotropic plane (i.e. the Loading-Collapse curve). It is evident from this that for the low suction levels of this particular problem it is the increase of apparent cohesion that controls the soil strength and not the value of the isotropic yield stress, $p_0$. At these low values of suction the value of the isotropic yield stress does not vary sufficiently enough to significantly affect the size of the primary yield surface.
6.6.2 Settlement due to the rise of the groundwater table

Analyses PW2 and PW3 (rise of the groundwater table at a constant load of 175kN and 350kN respectively) were repeated using model 2 with the three different values of parameter $b$. The progression of the settlement with the rise of the G.W.T. predicted by these analyses is shown in Figure 6-14. The results are directly comparable as the initial stress-strain conditions at the beginning of wetting are very similar for the given loads.

The settlements predicted with constitutive model 2 follow the same pattern as that observed in the model 1 analyses. For the lower load of 175kN only small settlements take place, which initially increase linearly with the rise of the groundwater table but level off as the G.W.T. approaches the ground surface. For the higher load of 350kN much larger settlements are predicted indicating failure.
Unlike the predictions for the load-settlement curve, the shape of the isotropic compression line can be seen to greatly affect the behaviour of the footing due to wetting. The settlements reduce significantly with increasing $b$. For the lower load of 175kN an increase of the parameter $b$ from 0.001 (model 1) to 0.472 leads to a decrease of the final predicted settlement of approximately 73%. For the larger load of 350kN the effect of the parameter $b$ is even greater. At a rise of the G.W.T. from –2m to –1m the settlement predicted for $b = 0.001$ is 150% larger than that predicted for $b = 0.472$. For isotropic stress states it is only the relationship between the yield stress, $p_0$, and the equivalent fully saturated yield stress, $p_0^*$, that controls the amount of wetting induced collapse. In any other case the change in apparent cohesion also affects the predicted amount of collapse, but generally to a much lesser extent.
The influence of partial soil saturation on the behaviour of footings was investigated through a series of finite element analyses. Both partially saturated and conventional (dry soil above the ground water table) analyses were performed, and the results compared to analytically calculated values of the bearing capacity using the basic bearing capacity equation for dry soil above fully saturated soil. The following conclusions can be drawn from this study:

- Both partially saturated and conventional analyses predict higher values of the bearing capacity for larger depths of the groundwater table. The analyses also show a reduction of the settlements (at the same footing load) with the increase of the G.W.T. depth. The predictions of the conventional finite element analyses are in good agreement with the analytically calculated ultimate loads when these are obtained using the Hansen and Meyerhof bearing capacity factors.

- Conventional analyses significantly underestimate the bearing capacity. The difference between the ultimate loads predicted by the conventional and partially saturated analyses increases with increase of the depth of the groundwater table.

- Raising the G.W.T induces settlements, the value of which was found to depend on the value of the load at which the wetting takes place (the higher the load the higher the settlement). If the load is sufficiently high, a rise of the ground water table may cause the footing to collapse. The limitations of the conventional type of analysis are more evident in this case, as this type of analysis predicts heave even at high applied loads and never failure of the footing.

- The bearing capacity of a footing after a rise of the water table seems to be independent of the load at which this rise took place. This load is larger than that corresponding to loading at a constant G.W.T., indicating a stress history dependency of the final bearing capacity.

- The shape of the isotropic compression line does not affect the predicted load-settlement curve for the G.W.T. depth considered ($D = 2m$). This
indicates that for the suction and stress range involved in the problem examined, it is the variation of apparent cohesion with suction that controls the shear strength of the soil. It is however likely that for large depths of the G.W.T. (possibly several or more meters) the shape of the isotropic compression line becomes important.

- The footing response to rising groundwater table, predicted with different shapes of the isotropic compression line (different values of the parameter $b$), is similar to that predicted with a linear isotropic compression line (model 1 – option 1), producing moderate additional settlement at low applied load and collapse at higher applied load. However, this feature of model 2 has a significant influence on the predicted settlement values, giving much lower settlements as the confining stress, $p_m$, associated with maximum potential collapse decreases (increase of parameter $b$). This behaviour demonstrates that in the case of rising G.W.T. the shape of the isotropic compression line is more dominant than the increase of apparent cohesion due to suction.
Chapter 7

Influence of partial soil saturation on pile behaviour

7.1 INTRODUCTION

This chapter investigates the influence of partial soil saturation on the behaviour of piles. The partially saturated constitutive models presented in Chapter 4 were used to perform a series of single pile analyses. Two studies are presented in this chapter. The first involves a single pile in a uniform soil subjected to vertical loading and fluctuations of the groundwater table. The second study is based on a case history concerning the foundation piles of a high rise building in Canary Wharf, London.

7.2 SINGLE PILE ANALYSES

As mentioned in the previous chapter, in many cases a capillary zone exists above the groundwater table, where the soil is partially saturated. The presence of such a zone has been found to be of significant importance in the case of shallow foundations, giving high values of bearing capacity, but also the danger of collapse upon wetting. A similar effect of the capillary zone is expected in the case of pile foundations. Moreover, when bored piles are constructed, it is often required that the groundwater level is reduced prior to the excavation and is allowed to gradually return to its original level at some point after the end of the works. As this practice is usually followed when the foundation soil comprises sands or silts (soils with a relatively low air entry suction value), it often leads to desaturation and subsequent resaturation of the soil. This section investigates the influence of the presence of a partially saturated soil zone above the ground
water table and of fluctuations of the groundwater table on the behaviour of a single pile in a uniform soil.

### 7.2.1 Details of analyses

All analyses involved a wished in place single pile in uniform soil. The pile was of 20m length and 1m diameter. Thin solid elements were used in the soil adjacent to the pile shaft rather than interface elements, therefore the concrete-soil angle of interface friction was taken to be equal to the angle of shearing resistance of the soil, \( \phi' \). Different depths of the groundwater table were considered, 0m (fully saturated soil), -2m, -4m, -6m, -8m, -10m and -25m. Two sets of analyses were performed:

Set 1: involved loading the pile to failure for each one of the groundwater table depths, considering two different pore pressure profiles for each depth (Figure 7-1). Profile 1 is the typical profile, which is used in effective stress conventional analyses assuming that the soil is dry above the ground water table, while profile 2 assumes a hydrostatic pore pressure profile up to the ground surface. An air entry suction value of zero was used in all the profile 2 analyses and therefore the soil in these analyses was considered to be partially saturated from the water table to the ground surface.

Set 2: involved loading of the pile to different load levels for both pore pressure profiles, and subsequently the water table was raised to ground level. The analyses were performed for several initial depths of the groundwater table.

The Finite Element mesh used in the analyses is shown in Figure 7-2. Eight noded axisymmetric isoparametric quadrilateral elements with reduced integration were used.
Figure 7-1: Single pile analysis: pore pressure profiles

Figure 7-2: Single pile analyses: Finite element mesh
7.2.2 Constitutive model and material properties

For the partially saturated analyses at which small depths of the groundwater table were considered ($D = 0m \sim 10m$) the stress and suction levels in the partially saturated soil zone are relatively low and therefore constitutive model 1 with a linear isotropic compression line (option 1) was used. However use of this constitutive model in the analyses with $D = 25m$, where the partially saturated zone extended to high stress regions, resulted in an unrealistic $YSR$ (Yield Stress Ratio $= \frac{p_o}{\bar{p}}$) profile with depth, as shown in Figure 7-3. The variation of $YSR$ with depth depends on the adopted relationship between the current yield stress $p_o$ and the equivalent fully saturated yield stress $p_o^*$. As discussed in Chapter 4, the particular relationship adopted by Model 1 – option 1 (Equation 4.26) leads to the calculation of unrealistically high $p_o$ values at high applied stresses, since it is based on the assumption that the partially saturated isotropic compression line continuously diverges from the fully saturated line. A modification of the model parameters could lead to more realistic predictions of the isotropic yield stress. Such an approach, however, is arbitrary, but most importantly would only give realistic predictions for the narrow stress range for which the parameters were adjusted. Model 1 - option 2, is more appropriate as it assumes a constant $\frac{p_o}{p_o^*}$ ratio at high applied stresses (Equation 4.27), and was therefore used instead in these analyses. The $YSR$ profile obtained in this way is shown in Figure 7-4.

The soil parameters used in the analyses are shown in Table 7.1 (see also Chapter 6). The yield function and plastic potential parameters $\alpha_f$, $\alpha_g$, $\mu_f$ and $\mu_g$ were selected such that the model in the fully saturated case becomes equivalent to the modified Cam-Clay model. The soil properties are for Lower Cromer till and are taken from the studies of Gens (1982), Gens & Potts (1982) and Maswoswe (1985). A Poisson’s ratio was used instead of a value for the shear modulus $G$, in order to have a varying value of $G$ with depth. For simplicity a linear increase of the apparent cohesion with suction was assumed ($f(s_{eq}) = k s_{eq}$). A value of 12kPa was taken for the characteristic pressure, $p^c$, for the constitutive model 1 – option 1 analyses, while for the analyses using option 2 a characteristic stress ratio $\alpha_c$ equal to 1.667 was selected. A constant unit weight, $\gamma$, of 17kN/m$^3$ was assumed throughout the soil layer.
Table 7.1: Soil properties – single pile in uniform soil analyses

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_f$</td>
<td>0.4</td>
<td>$r$</td>
<td>0.35</td>
</tr>
<tr>
<td>$\mu_f$</td>
<td>0.9</td>
<td>$\beta$</td>
<td>0.0164kPa$^{-1}$</td>
</tr>
<tr>
<td>$M_f$</td>
<td>1.2</td>
<td>$\kappa_s$</td>
<td>0.001</td>
</tr>
<tr>
<td>$\alpha_g$</td>
<td>0.4</td>
<td>$\nu_l$</td>
<td>2.0</td>
</tr>
<tr>
<td>$\mu_g$</td>
<td>0.9</td>
<td>$k$</td>
<td>0.8</td>
</tr>
<tr>
<td>$M_g$</td>
<td>1.2</td>
<td>$\mu$</td>
<td>0.2</td>
</tr>
<tr>
<td>$p^e$</td>
<td>12.0kPa</td>
<td>$S_{air}$</td>
<td>0.0kPa</td>
</tr>
<tr>
<td>$\lambda(0)$</td>
<td>0.066</td>
<td>$K_o$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.0077</td>
<td>$\gamma$</td>
<td>17.0kN/m$^3$</td>
</tr>
</tbody>
</table>

Figure 7-3: Predicted YSR profile for $D = 25$m from model 1 - option 1
None of the analyses involved depression of the water table (drying) and consequently the secondary yield surface was not relevant. A high value of the yield suction $s_o$ was therefore selected.

An equivalent fully saturated overconsolidation ratio $OCR$ of 1.5 was set instead of $YSR$, as it is independent of suction (option 1 in section 4.3.3) therefore specifying the equivalent fully saturated isotropic yield stress, $p_{o*}$, (hardening/softening parameter) profile. The partially saturated isotropic yield stress and effectively the $YSR$ profile are obtained from Equations 4.26 and 4.27 for constitutive model 1 options 1 and 2, respectively. The selected $OCR$ value was also applied to the fully saturated soil zone.

The concrete pile behaviour was modelled as linear elastic. A Young’s Modulus of 20GPa and a Poisson’s Ratio of 0.15 were used in the analyses.
7.2.2 Results

7.2.2.1 Pile loading

Figure 7-5 shows the load-settlement curves for some cases in the first set of analyses, which involved drained loading of the pile. Curves P2, P6 and P10 are for the analyses with the G.W.T. at -2m, -6m and -10m, respectively and pore pressure profile 2, curves D2, D6 and D10 are for the G.W.T at -2m, -6m and -10m, respectively and pore pressure profile 1, while curve S0 is for the fully saturated analysis (G.W.T. at 0m).

As can be seen in this figure, the bearing capacity of the pile increases with the increase of the depth of the ground water table. This increase is mainly due to the increase of the shaft resistance, as seen on Figure 7-6. Once the ultimate shaft resistance is reached, at approximately 10mm~20mm displacement, the load-displacement curves for all analyses are close to parallel. Although the shaft resistance for all analyses is reached at small displacements (less than 1%~2% of the pile diameter) the base resistance continues to increase even at high values of vertical displacement. As stated in Potts & Zdravkovic (2001), it can be assumed
that the ultimate load for practical purposes is reached at vertical displacements of 10% of the pile diameter. This definition of ultimate load will be adopted for this study.

As seen in Figure 7-5 the influence of partial soil saturation is insignificant for small depths of the G.W.T. (curves P2 and D2 are almost identical) and becomes more important at larger depths of the G.W.T.. For the ground water table at 10m below ground surface (curves P10 and D10) the conventional analysis underestimates the load carrying capacity by approximately 17%. However, for the factors of safety used in practice (usually FOS > 2) the predictions of the partially saturated and conventional analyses are very similar (at the same displacement). The results of the analyses with the ground water table at -4m and -8m have been omitted from this figure for clarity, but gave similar results.

Figure 7-6: Shaft and base resistances for the fully saturated and conventional (D = 10m) analyses
The ultimate loads for different depths of the G.W.T., predicted by the conventional (profile 1) and partially saturated (profile 2) analyses are shown in Figure 7-7. The increase of the ultimate load with the depth of the groundwater table and, more importantly, the increase of the effect of partial soil saturation with the increase of the depth of the G.W.T. are noted. The ultimate loads predicted by the partially saturated analyses depend on the choice of the Loading-Collapse yield surface and the variation of the cohesion increase parameter, \( k \), with suction. In the extreme case where the Loading-Collapse yield surface is a straight vertical line in the \( p-s_{eq} \) stress space \( (p_o = p_o^*) \) and \( k = 0 \), the partially saturated and conventional predictions are identical.

Figure 7-8 shows the load-displacement curves predicted by the conventional (curve D25) and partially saturated (curve P25) analyses for the ground water table at 25m below ground level. The partially saturated analysis predicts approximately 75% higher ultimate load than the conventional analysis. Although these analyses appear to represent an extreme case, such a situation may occur when dewatering or underdrainage takes place, causing suppression of the ground water level to great depths. In these cases understanding the likely
overprediction of capacity that a pile load test may show, when compared to the capacity when the G.W.T. returns to a smaller final depth, may be important. In this particular case an overprediction of approximately 320% would be made if the G.W.T. was to return to ground level. Depending on the pile load at which the rise of the G.W.T. takes place, large settlements may also occur.

![Figure 7-8: Load-displacement curves for $D = 25$ m](image)

It should be noted that all the conventional analyses presented above were performed with pore pressure profile 1 and therefore did not allow for suctions above the groundwater table. Conventional fully saturated analyses accounting for suctions by using pore pressure profile 2 predict higher ultimate loads than partially saturated analyses (with the same pore pressure profile). This is clearly not conservative and is due to the fact that, while for fully saturated soils strength increases linearly with suction, this increase is nonlinear for partially saturated soils.
7.2.2.2 Rise of groundwater table

The second set of analyses was performed in order to investigate the potential for pile movements associated with changes in the G.W.T. depth. These analyses involved loading the pile to different loads for a range of G.W.T. depths and then raising the water table to ground level.

None of the analyses involving small depths of the G.W.T. ($D = 0\text{m} \sim 10\text{m}$) predicted additional settlement of the pile due to wetting. In fact, in all cases upward movement of the pile was observed, following the general soil heave. The response of the soil to wetting was mostly elastic. The reduction of the effective stresses in the fully saturated region caused only elastic swelling, while in the partially saturated region the wetting paths at constant total stress intersected the primary yield surface only in a small region close to the soil-pile interface. A typical pile response to wetting is shown in Figure 7-9 for a partially saturated analysis with an initial $D = 10\text{m}$ and a constant load of 4MN (approximately 50% of ultimate load).

![Figure 7-9: Progression of vertical movement of pile during rise of groundwater table ($D = 10\text{m}$, $L = 4\text{MN}$)](image)

Figure 7-9: Progression of vertical movement of pile during rise of groundwater table ($D = 10\text{m}$, $L = 4\text{MN}$)
A different response was predicted however for the case where the ground water table was initially at -25m. Raising the water table at a constant load of 10MN (approximately two thirds of the ultimate load) caused the pile to collapse, as shown in Figure 7-10. Also shown in this figure is the response predicted by a conventional (pore pressure profile 1) analysis in which the pile had the same initial settlement due to loading as that produced by the partially saturated analysis (though at a much lower load). This analysis showed no additional settlement due to rise of the groundwater table. It is clear that the conventional analysis is not able to reproduce a realistic behaviour of the pile due to the rise of the groundwater table, as it predicts small upward movement of the pile. The vectors of incremental displacement close to failure (at 8.5m) for the partially saturated analysis are plotted in Figure 7-11.

Figure 7-10: Progression of vertical movement of pile during rise of groundwater table for D = 25m.
Figure 7-11: Vectors of incremental displacement at failure
7.3 ANALYSIS OF CANARY WHARF PILES

The behaviour of bored piles endbearing in partially saturated soil is investigated in this section. This is common in the London City area where clay layers are underlain by sandy to silty soils. The groundwater level is often reduced prior to pile construction, the works are performed in partially saturated soil and finally the groundwater table gradually returns to the pre-dewatering level. In most cases pile load tests are performed prior to the return of the groundwater to the initial level and therefore overestimate the true pile capacity and underestimate the settlements. Moreover, the effect of any subsequent rise of the water table (and consequent saturation of the foundation soil), particularly when this takes place after pile loading, is not well understood, as it cannot be modelled by conventional fully saturated analyses. This uncertainty is usually dealt with in practice through conservative design.

Finite element analyses of the foundation piles of a high-rise building in Canary Wharf in London are presented in the following sections. The effect of dewatering and consequent desaturation of the foundation soil on the load-settlement curve of the piles is examined through both conventional and partially saturated analyses and comparison is made with the measured response of a pile during a pile load test. The expected rise of the groundwater table is then modelled in order to investigate its influence on the pile movement. Based on this example a more general study is then presented where different soil properties are used and the predictions of the two constitutive models introduced in this thesis are compared.

7.3.1 Ground profile and pore pressure conditions

The ground profile used in the finite element analyses comprised (from top to bottom) 10m of fill, 3.8m of Terrace Gravel, 3.9m of Lambeth Group Clay, 6.5m of Lambeth Group Sands, 12.8m of Thanet Sands underlain by Chalk (Figure 7-12).

The pore water pressure conditions in the vicinity of the site are complex. Two aquifers exist, the upper aquifer in the Terrace Gravel and above and the lower
aquifer in the Lambeth Sands, Thanet Sands and Chalk. The piezometric levels in the lower aquifer have changed significantly during the past. Large scale water abstraction during the 19th century reduced water levels by as much as 70m in places. The abstraction rate reduced during the 1960’s and the piezometric level of the lower aquifer rose to about -6mOD to -10mOD. During the construction of Canary Wharf Station for the Jubilee line Extension between 1994 and 1997, the piezometric level in the lower aquifer was again depressed to -22mOD to -25mOD, but returned to the previous state of approximately -6mOD to -10mOD after the end of construction. Prior to the construction of the piles at the site considered in this study, local dewatering was carried out in both aquifers. The water level was reduced to -5.5mOD in the upper aquifer and to -28mOD in the lower aquifer. After cessation of local dewatering the piezometric level in the upper aquifer remained unchanged, whereas in the lower aquifer the level was expected to rise to –5.5mOD. The two pore water pressure profiles considered in the analyses are shown in Figure 7-13.

Figure 7-12: Canary Wharf pile analyses – Ground profile
7.3.2 Pile analysis – Load test and rise of groundwater table

7.3.2.1 Material properties

The Thanet Sand and Lambeth Sand layers were modelled with the generalised constitutive model 1 presented in Chapter 4. Due to the large depth the stresses in these layers are relatively high and consequently option 2 (bilinear model) was used and only the second segment of the isotropic compression line (Equation 4.27 - constant amount of potential collapse) was relevant. The Thanet sand and Lambeth sand model parameters used in the numerical analyses were:

a) Yield function and plastic potential parameters
The best estimate of the shapes of the Yield and Plastic Potential surfaces for fully saturated conditions was derived based on triaxial tests on Thanet Sand (Coop (2002)) and is presented in Figure 7-14. Also shown in the same figure are the implied plastic strain increment vectors for this particular choice of shapes. This yield and elastoplastic behaviour is also in accordance with experimental results on other sands as reported by Coop (1999). The model parameters corresponding to the selected Y.F. and P.P. surfaces are given in Tables 7.2 and 7.3. The values of the slope of the critical state line, $M_g$, were calculated from the angles of shearing resistance for triaxial compression, obtained from the geotechnical site investigation, as 1.46 for the Thanet Sands and 1.32 for the Lambeth Sands. The corresponding $M_f$ values are 1.0 for the Thanet Sands and 0.9 for the Lambeth Sands.

Figure 7-14: Yield and plastic potential surfaces
Table 7.2: Soil Properties for Lambeth sand

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_f$</td>
<td>0.08</td>
<td>$\beta$</td>
<td>0.02 kPa$^{-1}$</td>
</tr>
<tr>
<td>$\mu_f$</td>
<td>2.0</td>
<td>$\kappa_s$</td>
<td>0.001</td>
</tr>
<tr>
<td>$M_f$</td>
<td>0.9</td>
<td>$\nu_l$</td>
<td>1.826</td>
</tr>
<tr>
<td>$\alpha_f$</td>
<td>0.01</td>
<td>$\nu$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\mu_g$</td>
<td>0.57</td>
<td>$s_{air}$</td>
<td>15.0 kPa</td>
</tr>
<tr>
<td>$M_g$</td>
<td>1.32</td>
<td>$\psi$</td>
<td>0.03 kPa$^{-1}$</td>
</tr>
<tr>
<td>$\alpha_c$</td>
<td>1.667</td>
<td>$m$</td>
<td>0.35</td>
</tr>
<tr>
<td>$\lambda(0)$</td>
<td>0.06</td>
<td>$n$</td>
<td>4.5</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.005</td>
<td>$S_{ro}$</td>
<td>0.15</td>
</tr>
<tr>
<td>$r$</td>
<td>0.25</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7.3: Soil Properties for Thanet sand

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_f$</td>
<td>0.08</td>
<td>$\beta$</td>
<td>0.02 kPa$^{-1}$</td>
</tr>
<tr>
<td>$\mu_f$</td>
<td>2.0</td>
<td>$\kappa_s$</td>
<td>0.001</td>
</tr>
<tr>
<td>$M_f$</td>
<td>1.0</td>
<td>$\nu_l$</td>
<td>1.872</td>
</tr>
<tr>
<td>$\alpha_f$</td>
<td>0.01</td>
<td>$\nu$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\mu_g$</td>
<td>0.57</td>
<td>$s_{air}$</td>
<td>13.0 kPa</td>
</tr>
<tr>
<td>$M_g$</td>
<td>1.46</td>
<td>$\psi$</td>
<td>0.014 kPa$^{-1}$</td>
</tr>
<tr>
<td>$\alpha_c$</td>
<td>1.667</td>
<td>$m$</td>
<td>0.4</td>
</tr>
<tr>
<td>$\lambda(0)$</td>
<td>0.06</td>
<td>$n$</td>
<td>5.0</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.005</td>
<td>$S_{ro}$</td>
<td>0.13</td>
</tr>
<tr>
<td>$r$</td>
<td>0.25</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
b) Hardening and softening parameters

Due to the lack of partially saturated experimental data on Lambeth Sand and Thanet Sand, the partially saturated parameters $\alpha_c$, $\kappa_s$, $r$ and $\beta$ were taken from the study of Maswoswe (1985) on Lower Cromer Till. The use of parameters for till in place of sands was decided after comparison of the grading curves of these soils with those for which partially saturated experimental data was available. As this study is for illustrative purposes it was thought reasonable to proceed with these values. The compressibility coefficient due to changes in suction, $\lambda_s$, is not relevant for this set of analyses as further lowering of the groundwater table (drying) is not modelled. Therefore a high value is given to this parameter. Moreover, since the ground at the site under investigation has undergone many cycles of lowering and raising of the groundwater table in the past, it is expected that even if dewatering was modelled, this would induce only elastic strains. The compressibility coefficient along elastic paths, $\kappa$, was calculated from the Poisson’s Ratio, $\mu$, and Young’s Modulus, $E$, as 0.005. The fully saturated compressibility coefficient, $\lambda(0)$, was taken 12 times larger than $\kappa$, equal to 0.06.

c) Initial hardening parameters

The specific volume at unit confining stress, $\nu_f$, was 1.82 for Lambeth Sand and 1.87 for Thanet Sand. An estimate of the OCR value was obtained from the relationships presented by Brooker & Ireland (1965). For a very low plasticity soil with a $K_o$ value of 1.15 a value of 6 was selected for the OCR. This value refers to the original fully saturated state.

d) Other parameters

Because of the large suctions involved in this problem the cohesion increase parameter, $k$, was set equal to the degree of saturation, $S_r$. The variation of the degree of saturation with suction was obtained from the particle size distribution curves of the materials using the Arya & Paris (1981) method. These were in
turn fitted into the Van Genuchten (1980) expression for the soil water characteristic curve, given in Chapter 2 in terms of volumetric water content, $\theta$. The expression can be written in terms of degree of saturation, $S_r$, as follows:

$$S_r = \left[ \frac{1}{1 + \left( \frac{S_{eq}}{\psi} \right)^m} \right]^n \left( 1 - S_{ro} \right) + S_{ro} \quad (7.1)$$

where, $\psi$, $m$ and $n$ are fitting parameters, and $S_{ro}$ is the residual degree of saturation at very high values of suction. The soil water characteristic curves obtained using Equation 7.1 are shown in Figure 7-15. The full list of parameters for the Thanet Sands and Lambeth Sands is given in Tables 7.2 and 7.3.

Figure 7-15: Soil-water characteristic curves for Thanet sands and Lambeth sands

The Terrace Gravel, Lambeth Clay and Chalk layers were modelled with the generalised Mohr-Coulomb model. The soil properties adopted for the analyses are shown in Table 7.4. A value of zero was set for the angle of dilation for these layers. The Lambeth Clay was assumed to behave undrained.
The $K_o$ values assigned to each soil layer were 0.5 for the Terrace Gravel, 1.15 for the Lambeth Clay, Lambeth Sands and Thanet Sands, and 1.0 for the Chalk.

The concrete pile behaviour was modelled as linear elastic. A Young’s Modulus of 20GPa and a Poisson’s Ratio of 0.15 were used in the analyses.

<table>
<thead>
<tr>
<th></th>
<th>Terrace Gravel</th>
<th>Lambeth Clay</th>
<th>Chalk</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi'$</td>
<td>33°</td>
<td>29°</td>
<td>34°</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>$E$</td>
<td>30 MPa</td>
<td>30 MPa</td>
<td>1000 MPa</td>
</tr>
</tbody>
</table>

Table 7.4: Soil Properties for Terrace Gravel, Lambeth Clay and Chalk

### 7.3.2.2 Details of analyses

The ground level at the beginning of the analyses was taken at +5.0mOD and the piezometric level for both aquifers at –5.5mOD. Before the construction of the pile the piezometric level in the lower aquifer was lowered to –28.0mOD and the top 10m of fill were excavated. The fill layer was simulated as a uniform surcharge acting throughout the top of the Finite Element mesh and was removed at the first stage of each analysis to simulate the excavation of the cofferdam.

The pile analysed was of 1.5m diameter and 20.5m length and was wished in place. The finite element mesh used in this study is illustrated in Figure 7-16. Two types of analyses were performed. The first type involved vertical loading of the pile to failure and the second type loading of the pile to a certain load and then raising the water table in the lower aquifer. Both conventional and partially saturated analyses were carried out, using the best estimate for the model parameters from the available data.
7.3.2.3 Results

Figure 7-17 shows the load-displacement curve obtained from the partially saturated analysis. Also shown on this figure is the loading-unloading path measured during the load test of the pile under consideration. It can be seen that the predicted and measured loading paths are in reasonable agreement. Figure 7-17 also shows the load-displacement curve predicted from a conventional analysis assuming that the Thanet Sands and Lambeth Sands switch from fully saturated to completely dry behaviour at values of suction equal to their respective air entry values of suction. The conventional analysis predicts an approximately 20% lower ultimate load than the partially saturated analysis. Although the test pile did not reach an ultimate load, the comparison in Figure
7-17 indicates that, if it had been loaded further, it would probably have agreed better with the partially saturated analysis.

Using the same set of parameters the pile was loaded to three different loads, 19MN (approx. 50% of the ultimate load predicted by the partially saturated analysis), 26MN (approx. 65% of the ultimate load) and 32MN (approx. 80% of the ultimate load) and then the water level of the lower aquifer was raised, simulating the rise of the water table after the end of construction in the real case. Figure 7-18 shows the progression of vertical movements with the rise of the G.W.T. predicted by the partially saturated analyses. It can be seen that the pile behaviour is very much dependent on the load at which wetting takes place. For the analysis with the lower load, upward movement of the pile is predicted which increases linearly with the rise of the water table. The analysis with $N = 26$MN also predicted heave initially, but seems to have levelled off after the G.W.T. had risen by approximately 8m. The analysis however at which wetting took place at the high load of $N = 32$MN predicted large settlements, larger than those induced by the pile loading.
Also shown in Figure 7-18 is the progression of vertical movements predicted by a conventional analysis. The initial pile settlement was 30mm, which is 50% larger than the settlement predicted by the partially saturated analysis for the same load and equal to the settlement predicted by the partially saturated analysis for $N = 32$ MN. Subsequently, when the G.W.T. was raised, the conventional analysis predicted a response to wetting very different from that predicted by the equivalent partially saturated analysis, producing heave and linear reduction of the initial 30mm settlement.

It should be noted that the results of the analyses described above do not indicate any danger to the structure under consideration due to the rise in the ground water table for the given working load of the pile (13.35MN) and the set of material properties adopted in the analyses.
7.3.3 Pile analysis – Influence of model parameters

The results obtained by changing some of the material properties will now be discussed. In particular the influence of OCR, r and β which control the position and shape of the primary yield surface is investigated. The set of parameters given in Tables 7.2, 7.3 and 7.4 will be referred to as the reference set for the remainder of this chapter.

7.3.3.1 Influence of OCR

The influence of OCR is first examined assuming pore pressure profile 2 (partially saturated analyses only). It should be noted here that OCR is a measure of the distance of the soil stress state from the yield surface, refers to the original fully saturated state and is in effective stress terms. The distance from the yield surface at the beginning of the pile loading (the soil is now partially saturated) is given by the yield stress ratio YSR (= po/p), which is in total stress terms and depends on the initial OCR value, but also on the suction values and the shape of the Loading-Collapse yield surface. YSR values are higher for higher OCR values and therefore the soil is less likely to collapse upon wetting.

Figure 7-19 shows the load-displacement curves obtained for values of the OCR in the Thanet sand and Lambeth sand layers of 6 (reference set of parameters), 2 and 1.5. It can be seen that by reducing the value of OCR, much lower ultimate loads are predicted.

Figure 7-20 shows the pile response to the rise of the ground water table at a constant load of approximately 19MN for the OCR = 1.5 and OCR = 6 cases. While for the OCR = 6 case the pile moved upwards, as mentioned before, very large settlements were predicted for OCR = 1.5, more than 10 times larger than those induced by the pile loading. Also shown in this figure are the responses of the pile to wetting for OCR = 1.5 and for loads of 13.5MN and 20.2MN. It is clearly illustrated that the pile response to wetting depends on the load at which it takes place. A small increase of the pile load from 19MN to 20.2MN increases settlements due to wetting by about 90%. Similar results were obtained from the
$OCR = 2$ analyses. The pile experiences large additional settlements due to wetting only at high loads, close to or after the shaft resistance has been exceeded.

Figure 7-19: Load-settlement curves for different $OCR$ values

Figure 7-20: Progression of vertical movement for different $OCR$ values
7.3.3.2 Influence of parameters $r$ and $\beta$

Loading and wetting partially saturated analyses were also performed using different values for the parameters $r$ and $\beta$ (with $OCR = 6$) which control the compressibility coefficient $\lambda(s_{eq})$ and consequently the isotropic yield stress $p_o$ through Equation 4.27.

Figure 7-21 shows the load-displacement curve predicted for $r = 0.35$ and $\beta = 0.01$ and is compared to the curve predicted with the reference parameter set. This choice of $r$ and $\beta$ gives a smaller increase of yield stress with suction than the reference values (Figure 7-22) and as expected predicts a lower ultimate load of the pile (the reference set of parameters gives approximately 16% higher ultimate load).

![Figure 7-21: Load-settlement curves for different partially saturated parameters](image)

The pile response to the rise of the water table at different constant values of the vertical load for the modified set of parameters is plotted in Figure 7-23. A similar response to that observed for the reference set of parameters can be seen, with the two analyses at relatively low loads (18MN and 23MN) giving upward
movement of the pile and the analysis at the higher load of 28MN predicting settlement.

Figure 7-22: Loading-collapse surface for different partially saturated parameters

Figure 7-23: Progression of vertical movement for different p. sat. parameters
7.3.4 Pile analysis – Comparison of the two constitutive models

All the analyses presented above were performed using constitutive model 1 – option 2 for the Thanet sand and Lambeth sand soil layers. A set of analyses using model 2 is presented in this section and a comparison is made between the predictions of the two models. The fundamental difference between the two models lies in the assumption regarding the shape of the isotropic compression line and consequently the shape of the primary yield surface. While model 1 assumes either a linear or bilinear expression, model 2 being non-linear is more flexible and, depending on the value of the parameter $b$, can reproduce a wide variety of shapes, ranging from that adopted in linear model 1 to one very close to the equivalent fully saturated compression line. The influence of the shape of the isotropic compression line on the finite element predictions is investigated here.

7.3.4.1 Model parameters

Three different values of the parameter $b$ are investigated; $b = 0.472$ which corresponds to maximum potential collapse at $p_m = 100kPa$, $b = 0.226$ which corresponds to $p_m = 1000kPa$, and $b = 0.1$ for which maximum potential collapse takes place at a very high confining stress $p_m \approx 26000kPa$. Typical isotropic compression lines for the Thanet sand and Lambeth sand layers predicted with these three values of the parameter $b$ are plotted in Figure 7-24. Also shown are the isotropic compression lines for model 1 (both options) and the equivalent fully saturated case. For a value of $b$ of 0.226 ($p_m = 1000kPa$) the isotropic compression line of model 2 is very close to that given by model 1 – option 2. It is therefore expected that the finite element results for this value will be close to those predicted with the reference set of parameters presented above.

The rest of the model parameters are the same as the reference set of parameters, with the exception of the parameter $\alpha_c$, which is not applicable and is replaced by parameter $b$. 
7.3.4.2 Results

- Pile loading

Figure 7-25 shows the load-displacement curves predicted with model 2 for the three different values of $b$. Unlike the footing case, where the parameter $b$ did not affect the predicted ultimate load, the ultimate pile load increases with decreasing value of $b$, and consequently increasing yield stress ratio $YSR$. In this case the suctions and stresses in the partially saturated zone are sufficiently high to produce $YSR$ values much higher than the equivalent $OCR$ value, therefore affecting significantly the size of the primary yield surface.

Plotted on the same figure is the load-displacement curve predicted with model 1 – option 2. It can be seen that this curve is very close to and slightly lower than that predicted with model 2 for $b = 0.226$ (approximately 3% lower). This is in accordance with Figure 7-24, which shows that the isotropic compression line...
for model 1 – option 2 is slightly lower than that corresponding to $b = 0.226$ for the confining stress level relevant to this analysis (higher than 200kPa).

![Figure 7-25: Load-settlement curves for different values of $b$](image)

- **Rise of groundwater table**

The progression of the vertical displacements of the pile with the rise of the groundwater table, at different constant loads, is shown in Figure 7-26, for model 1 – option 2 and model 2 with $b = 0.226$. It can be seen that the results are in good agreement, with model 1 giving slightly lower displacements at the higher loads of 32MN and 26MN, and identical results at the low load of 19MN. This is consistent with the results of the loading analyses discussed above, therefore confirming that the two models produce very similar results for the given $\alpha_c$ and $b$ values, and the stress and suction range under consideration.

The influence of the parameter $b$ on the pile response to wetting can be seen in Figure 7-27 which presents pile movements due to rise of the groundwater table for different values of $b$ (0.226 and 0.472) and initial pile settlement (14.2mm, 20mm and 30mm). It was chosen to make the comparison at the same values of
initial settlement instead of the same load levels because the latter would have been meaningless, especially at high loads, where the settlements increase rapidly (see Figure 7-25). The pile loads which correspond to initial displacements of 14.2mm, 20mm and 30mm are 19MN, 26MN and 32MN for $b = 0.226$, and 19MN, 25MN and 28MN for $b = 0.472$, respectively.

Figure 7-26: Progression of vertical displacements: comparison of two models

For the lowest initial settlement of 14.2mm the value of $b$ has no influence on the pile response; both analyses give purely elastic heave. For 20mm initial settlement the two analyses produce very close results, giving near-elastic heave. In contrast to them, for the large initial settlement of 30mm the predictions are very different. The analysis with $b = 0.226$ predicts much larger settlements than the $b = 0.472$ analysis indicating failure of the pile. It is acknowledged that the two analyses are not directly comparable, as they are not performed for the same load levels. However, close inspection of the load-displacement curves in Figure 7-25 shows that the $b = 0.472$ analysis starts closer to the ultimate pile load with the shaft friction fully mobilised and is therefore more likely to predict collapse. The difference between the analyses can therefore be attributed primarily to the value of the parameter $b$ and the shape of the isotropic compression line.
7.4 CONCLUSIONS

The two constitutive models presented in this thesis were used to perform a series of Finite Element analyses in order to investigate the behaviour of single piles in partially saturated soils. Two studies were carried out, a study of single piles in uniform soil and a case study of a single pile endbearing in partially saturated soil.

The following conclusions can be drawn from these studies:

a) First study: Single pile in uniform soil

- The ultimate pile load increases with the increase of the ground water table depth. The partially saturated FE analyses predicted larger ultimate load increase than the conventional FE analyses. The difference between the ultimate loads predicted by the two analyses increases with increasing depth of the groundwater table. For relatively small depths of 0 ~ 10m this
difference is relatively small, while the working loads (assuming factor of safety FOS > 2) are very similar. For large depths of the G.W.T., however, conventional analyses significantly underpredict the pile capacity even for FOS > 2.

- Model 1 – option 1 (linear isotropic compression line) proved incapable of predicting realistic soil behaviour for large depths of the groundwater table. This model should only be used at low stress and suction levels.

- Raising the water table was found to be important for the pile behaviour only when the pile tip was in partially saturated soil. In this case the collapse experienced by the soil under the tip led to excessive pile settlements. The conventional analyses could not reproduce this behaviour.

b) Second study: Canary Wharf piles

- The results of the finite element analyses of the pile loading were compared to the measured response during a pile load test. The partially saturated analysis was in good agreement with the field measurements, while the conventional analysis appeared to underpredict the pile capacity.

- The pile response to a rising groundwater table was found to strongly depend on the load at which the rise takes place. At low pile loads only a small amount of heave was predicted, while at high loads excessive settlements occurred. The results did not indicate any danger to the structure under consideration.

- The OCR of the soil was found to influence greatly the pile behaviour. The FE predictions were also found to be sensitive to the choice of the partially saturated model parameters $r$ and $\beta$.

- Model 2 analyses with different values of $b$ showed that the shape of the isotropic compression line has a significant effect on both the load-settlement curve and the response to a rising groundwater table.
• When the model parameters $\alpha_c$ (for model 1 – option 2) and $b$ (for model 2) are selected such as to give similar isotropic compression lines over the stress and suction range relevant to the problem analysed, the finite element predictions are also very similar.

• With regard to the pile response to rising G.W.T. it can generally be concluded that the shaft friction prevents the occurrence of excessive settlements. These are only predicted when the shaft friction is close to full mobilisation prior to the rise of the groundwater table.
Chapter 8

Conclusions and Recommendations

8.1 INTRODUCTION

The research presented in this thesis can be subdivided into two main parts. The first part involved the development of two generalised constitutive models, both describing the mechanical behaviour of partially and fully saturated soils. These models were implemented into the Imperial College Finite Element Program (ICFEP) and validated through a series of single element finite element analyses.

The second part of the research comprised the application of these models in some typical geotechnical engineering problems. Shallow and deep foundations were examined to investigate the influence of partial soil saturation on their behaviour. Both partially saturated and conventional (soil is completely dry for suctions higher than the air entry value) finite element analyses were performed with the two constitutive models developed and presented in this thesis. In this way the differences between the predictions for different types of analyses and different constitutive behaviour were highlighted.

The conclusions from this research are presented in more detail in the following sections.

8.2 TWO GENERALISED CONSTITUTIVE MODELS FOR FULLY AND PARTIALLY SATURATED SOILS

The mechanical behaviour of partially saturated soils is different to that of fully saturated soils and consequently cannot be described with conventional constitutive soil models. Two elastoplastic constitutive models for both fully and partially saturated soils were developed during this research and presented in this thesis. The two models are based on the Loading-Collapse yield surface concept,
introduced by Alonso et al. (1987) and first adopted in the Barcelona Basic model (Alonso et al. (1990)), upon which most existing constitutive models for partially saturated soils have been based.

The constitutive models presented in this thesis have the following advantages compared to most existing models:

- Use of a versatile expression for the yield and plastic potential surfaces. This allows the reproduction of many different shapes, including those assumed by some well known constitutive models.

- Non-linearity of the failure envelope in the $J$-$s$ plane can be modelled. This can be achieved by relating the increase of apparent cohesion with the increase of suction to the degree of saturation, $S_r$.

- Non-linearity of the partially saturated isotropic compression line can be modelled. In this way realistic amounts of wetting induced collapse and values of the yield stress ratio ($YSR$) can be predicted.

The fundamental difference between the two constitutive models lies in the shape of the isotropic compression line. Model 1 adopts a linear (option 1) or bilinear (option 2) expression, giving either constant increase of the amount of potential wetting induced collapse with the increase of the confining stress, or a constant amount of potential collapse beyond a certain value of the confining stress. Model 2 adopts an exponential expression, so that the amount of potential collapse increases with confining stress at low stresses, reaches a maximum value and then decreases to zero at very high confining stresses. Real soil behaviour is better represented by Model 2, but Model 1 is more straightforward, particularly at low confining stresses, and relatively easier to implement into a Finite Element program.

Other features of the two models include the following:

- The models are formulated in the four-dimensional net stress ($\sigma$)-equivalent suction ($s_{eq}$) stress space. For fully saturated conditions the stresses ($\sigma$) are effective stresses and $s_{eq} = 0$ kPa. The transition from fully
to partially saturated conditions takes place when the value of suction \( s \) becomes equal to the air entry value \( s_{\text{air}} \).

- A secondary yield surface is introduced (equivalent to the Suction-Increase yield surface of the Barcelona Basic model), which defines the limiting equivalent suction beyond which plastic strains take place.

- The plastic strains induced when the primary yield surface is activated are calculated from the change of the equivalent fully saturated yield stress, \( p_0^* \) (primary hardening/softening parameter). The plastic strains due to activation of the secondary yield surface are calculated from the change of the value of the yield suction, \( s_0 \), (secondary hardening/softening parameter). The two yield surfaces are coupled, therefore movement of either of them results in movement of the other.

- Within the elastic region, volumetric strains are calculated from the shape of the elastic loading/unloading and wetting/drying paths. These are controlled by the parameters \( \kappa \) and \( \kappa_s \). An ‘elastic’ surface is defined in \( \nu - \ln p - s_{\text{eq}} \) space for each value of the hardening/softening parameter. In order to avoid calculation of infinite strains at zero confining stress, a minimum bulk modulus, \( K_{\text{min}} \), is introduced. This leads to a warped shape of the ‘elastic’ surface in the low compressive and in the tensile confining stress region. Shear strains are calculated by adopting a constant shear modulus, \( G \), a constant Poisson’s ratio, \( \mu \), or a constant ratio \( G/p_0 \).

Twenty-two parameters are required for both models. A possible testing program from which these parameters can be obtained includes a minimum of three isotropic compression tests, two triaxial compression tests and one drying-wetting test.

The two constitutive models were implemented into ICFEP and validated through a series of single element axisymmetric finite element analyses. A variety of stress paths were simulated in order to examine and highlight all the model features, and the results in many cases were compared to theoretical calculations. Comparison was also made between finite element results and experimental data.
8.3 INFLUENCE OF PARTIAL SOIL SATURATION ON THE BEHAVIOUR OF SHALLOW AND DEEP FOUNDATIONS

The influence of partial soil saturation on the behaviour of foundations was investigated through three series of finite element analyses. The first study involved a surface strip footing, the second a single pile in uniform soil, and the third was based on a case history concerning the foundation piles of a high rise building in Canary Wharf, London.

The following general conclusions can be drawn:

- The footing bearing capacities, predicted by the conventional analyses (transition from fully saturated to completely dry conditions at the air entry value), were in good agreement with the analytically calculated values of the bearing capacity using the basic bearing capacity equation for dry above fully saturated soil.

- The bearing capacity of both footings and piles increases with increasing depth of the groundwater table. Partially saturated finite element analyses predict larger ultimate load increase than conventional analyses. The difference between the ultimate loads predicted by the two types of analysis increases with increasing depth of the groundwater table. This difference only becomes of significant importance at high depths of the groundwater table for the pile case. For footings, however, large differences are observed even at low depths of the groundwater table.

- A rise of the groundwater table may induce additional settlements. The behaviour during the rise of the groundwater table depends on the load at which this rise takes place. If the load is sufficiently high, excessive settlements and collapse take place. At low loads only small additional settlements take place in the footing case, while a small heave is observed in the pile case. The rise of the groundwater table was found to be important for the pile behaviour only when the pile tip was in partially saturated soil. In addition, it appears that the shaft friction prevents the occurrence of excessive settlements. Such settlements are only predicted
when the shaft friction is close to full mobilisation prior to the rise of the groundwater table.

- Conventional analyses always predict heave due to a rising groundwater table even at high values of the applied foundation load.

- The bearing capacity of a footing after a rise of the groundwater table seems to be independent of the load at which this rise took place.

- In the case of the Canary Wharf piles, the load-settlement curves predicted by the partially saturated analysis were in good agreement with the measured response during a pile load test, while the conventional analysis appeared to underestimate the pile capacity.

- The OCR of the soil was found to influence greatly the pile behaviour in the Canary Wharf pile case. The finite element predictions were also found to be sensitive to the choice of the parameters $r$ and $\beta$ of the partially saturated model.

Concerning the comparison between the predictions of the two models and the influence of the shape of the isotropic compression line, the main conclusions are:

- For the low suction and stress range involved in the strip footing analyses, the shape of the isotropic compression line does not affect the predicted load-settlement curve (models 1 and 2 give similar results). This indicates that in this suction and stress range it is the variation of apparent cohesion with suction that controls the shear strength of the soil. The Canary Wharf pile analyses, however, showed that for higher stress levels, the shape of the isotropic compression line has a significant effect on the predicted load-settlement curve.

- For the suction and stress ranges involved in the problems analysed (both footing and Canary Wharf pile analyses), the response to rising groundwater table significantly depended on the shape of the isotropic compression line.
• The Canary Wharf pile analyses showed that when the model parameters $\alpha_c$ (for model 1 – option 2) and $b$ (for model 2) are selected such as to give similar isotropic compression lines over the stress and suction range relevant to the problem analysed, the finite element predictions are also very similar.

8.4 RECOMMENDATIONS FOR FURTHER RESEARCH

8.4.1 Constitutive modelling

Constitutive models for partially saturated soils are generally much less advanced and lack the sophistication of many conventional fully saturated constitutive models.

Some advancements which would improve the models’ predictions are the following:

a) Most existing constitutive models are based on the behaviour of isotropically consolidated soils. However, in most cases in practice, soils are anisotropically consolidated. This implies rotated yield and plastic potential surfaces in the $J – p$ plane. Cui et al. (1995) pointed out this fact and proposed a constitutive model for anisotropically consolidated partially saturated soils. This model was, however, formulated in terms of conventional stress invariants ($q$, $p$) and was therefore only applicable for triaxial compression conditions. It would be useful to consider modification of the yield and plastic potential expression and re-formulation of the models presented in this thesis in stress tensor terms, in order to take account of this feature.

b) Modelling small strain elasticity has proven to be very important for fully saturated soils. Experimental investigation of the small strain behaviour of partially saturated soils and subsequent incorporation of this behaviour in the existing constitutive models would obviously prove very beneficiary.
c) There is need for experimental data, which unfortunately is not as readily available as for fully saturated soils. In addition, constitutive models for partially saturated soils have a high degree of complexity and obtaining the required parameters may not always be straightforward.

8.4.2 Numerical studies

The implementation of the two constitutive models presented in this thesis into ICFEP allowed the performance of finite element analyses of shallow and deep foundations in partially saturated soil. Further work could include undertaking more analyses to supplement these studies (e.g. different depths of the footing). It would also be very interesting to investigate the influence of partial soil saturation on other geotechnical structures, such as retaining walls and embankments.

In addition, the proposed new constitutive models can be used to analyse numerically geotechnical problems involving groundwater flow in partially saturated soil.
References


Appendix I

Flow chart for the calculation of the stress and plastic strain increments and the changes in hardening parameters

START

Read material properties
Set model option (linear or bilinear isotropic compression line) – only model 1
Set strength option (linear or non-linear increase of apparent cohesion) – both models
Set stresses (fully or partially saturated behaviour)

Illegal hardening parameters

ONLY PRIMARY YIELD SURFACE ACTIVE OR BOTH YIELD SURFACES ACTIVE

STOP PROGRAM

242
Calculate stress invariants

\[ p + J(s_{eq}) > 0 \]

**YES**

Model 1: Determine \( \lambda(s_{eq}) \) - Equation 4.25
Model 2: Determine \( \lambda_m \) - Equation 4.41

**Model 1**

Determine \( p_o \) - Equation 4.26 for model 1 - option 1
- Equation 4.27 for model 1 - option 2
- Equation 4.50 for model 2

Equation 4.50 is solved numerically using the Newton-Raphson algorithm

Evaluate yield function and plastic potential derivatives with respect to stresses – Equations 5.3 – 5.20

Calculate parameter \( A \)
- Equation 5.30 for model 1 - option 1
- Equation 5.31 for model 1 - option 2
- Equation 5.32 for model 2

Calculate elastic strains due to suction changes - Equation 5.33

STOP PROGRAM
Calculate plastic strains due to suction changes - Equation 5.34

Evaluate yield function derivative with respect to suction - Equations 5.36 – 5-42

Determine plastic strain multiplier \( \Lambda \) – Equation 3.27

Evaluate plastic strains due to stress changes – Equation 3.23

Evaluate stress increments - Equation 3.22

Evaluate total plastic strains: \( \{\Delta \varepsilon^p\} + \{\Delta \varepsilon^p_s\} \)

Determine change in hardening parameters – Equations 4.55 and 4.56
Illegal hardening parameters

\[ \Delta p_o^* = p_o^* \]
\[ \Delta s_o = s_o + s_{oir} \]

Secondary yield surface violated if not active from beginning

Calculate proportion of step to reach secondary yield surface

Both yield surfaces are active

YES

Use both surfaces for remainder of step

NO

Use secondary yield surface for remainder of step
Calculate elastic and plastic strains due to suction changes - Equations 5.33 - 5.35

Evaluate stress increments
Equation 3.22

Determine change in hardening parameters – Equations 4.55 and 4.56

Illegal hardening parameters

\[ dp_o^* = p_o^* \]
\[ ds_o = s_o + s_{air} \]

Primary yield surface violated

Use both surfaces for remainder of step

Calculate proportion of step to reach primary yield surface
Stress state is in the vicinity of the yield cone apex

Move stress point to a distance equal to TIPTOL from cone apex (TIPTOL is an input parameter)

Partially saturated behaviour

YES

Calculate effective stress changes:

\[ \{\Delta \sigma'\} = \{\Delta \sigma\} + \{\Delta s\} \]

END
Appendix II

Calculation of the angle of shearing resistance for plane strain conditions

For plane strain deformation, the out of plane strain increment, $\Delta \varepsilon_2$, is zero:

$$\Delta \varepsilon_2 = \Delta \varepsilon_2^e + \Delta \varepsilon_2^p = 0 \quad (II.1)$$

At failure the elastic component $\Delta \varepsilon_2^e$ reduces to zero as there is no change in strain. Consequently:

$$\Delta \varepsilon_2^p = 0 \quad (II.2)$$

The angle of dilation, $\nu$, can be expressed as follows:

$$\nu = \sin^{-1}\left(-\frac{\Delta \varepsilon_2^p + \Delta \varepsilon_2^p}{\Delta \varepsilon_2^p - \Delta \varepsilon_2^p}\right) \quad (II.3)$$

The Lode’s angle of the plastic strain increments is given by:

$$\theta_{\Delta \varepsilon^p} = \tan^{-1}\left[\frac{1}{\sqrt{3}} \left(2 \frac{\Delta \varepsilon_2^p - \Delta \varepsilon_3^p}{\Delta \varepsilon_2^p - \Delta \varepsilon_3^p} - 1\right)\right] \quad (II.4)$$

Combining Equations II.2, II.3 and II.4 gives the Lode’s angle of plastic strain increments at failure:

$$\theta_{\Delta \varepsilon^p} = \tan^{-1}\left(\frac{\sin \nu}{\sqrt{3}}\right) \quad (II.5)$$
The plastic strain increment direction is also normal to the projection of the plastic potential curve on the deviatoric plane, expressed as $\sqrt{J_{2ng}}$. Such a curve is shown schematically in Figure II-1, for one sixth of the deviatoric plane. From this figure:

$$
\begin{align*}
  x &= \sqrt{J_{2ng}} \sin(\theta) \\
  y &= \sqrt{J_{2ng}} \cos(\theta)
\end{align*}
$$

(II.6)

and therefore:

$$
\frac{d y}{d x} = \frac{\frac{\partial \sqrt{J_{2ng}}}{\partial \theta} \cos \theta - \sqrt{J_{2ng}} \sin \theta}{\frac{\partial \sqrt{J_{2ng}}}{\partial \theta} \sin \theta + \sqrt{J_{2ng}} \cos \theta}
$$

(II.7)

As the plastic strain increment vector is normal to the plastic potential:

$$
\theta_{\Delta e^p} = \tan^{-1} \left( -\frac{d y}{d x} \right)
$$

(II.8)
Combining Equations II.5 and II.8 gives:

\[
- \sin \psi = \frac{\partial \sqrt{J_{ng}}}{\partial \theta} \cos \theta - \sqrt{J_{ng}} \sin \theta
\]

\[
\frac{\partial \sqrt{J_{ng}}}{\partial \theta} \sin \theta + \sqrt{J_{ng}} \cos \theta
\]

At critical state conditions there is no volume change at failure, therefore \( \psi = 0 \), and Equation II.9 reduces to:

\[
\tan \theta = \frac{\partial \sqrt{J_{ng}}}{\partial \theta} \sqrt{J_{ng}}
\]

\( \sqrt{J_{ng}} \) can be obtained from the Matsuoka-Nakai criterion (Potts & Zdravkovic (1999)) which as presented in Chapter 4 is given by:

\[
\frac{2}{\sqrt{27}} C \cdot \sin(3\theta) \cdot J_{ng}^{3/2} + (C-3) \cdot J_{ng} - (C-9) = 0
\]

in which

\[
C = \frac{9 - M^2}{27} - \frac{M^2}{3} + 1
\]

The derivative \( \frac{\partial \sqrt{J_{ng}}}{\partial \theta} \) can be derived from Equation II.11 as follows:

\[
\frac{\partial \sqrt{J_{ng}}}{\partial \theta} = - \frac{3}{\sqrt{27}} C \cos(3\theta) J_{ng}
\]

\[
(C-3) + \frac{3}{\sqrt{27}} C \sin(3\theta) \sqrt{J_{ng}}
\]

Combining Equations II.10 and II.13 gives:

\[
\sqrt{J_{ng}} = - \frac{(C-3) \tan \theta}{\frac{3}{\sqrt{27}} C[\tan \theta \sin(3\theta) + \cos(3\theta)]}
\]
Combining Equations II.11 and II.14 gives the following equation from which the Lode’s angle, \( \theta \), can be calculated:

\[
\frac{\left[ \tan \theta \sin(3\theta) + \cos(3\theta) \right]^3}{\left[ \tan \theta \sin(3\theta) + 3 \cos(3\theta) \right] \tan^2 \theta} = \frac{(C - 3)^3}{C^2 (C - 9)} \tag{II.15}
\]

In order to calculate the value of the angle of shearing resistance, \( \phi \), for plane strain conditions, the Mohr-Coulomb hexagon must be defined which crosses the Matsuoka-Nakai surface at the point corresponding to the calculated plane strain value of \( \theta \). This can simply be done by substituting \( \theta \) and \( \sqrt{J_{\text{2ng}}} \) (which is calculated from Equation II.11 once the value of \( \theta \) is established) into the following equation:

\[
\sqrt{J_{\text{2ng}}} = \frac{\sin \phi}{\cos \theta + \frac{\sin \theta \sin \phi}{\sqrt{3}}} \tag{II.16}
\]

which defines the Mohr-Coulomb hexagon in the deviatoric plane.

For a value of \( M \) equal to 1.2 (selected in the analyses which were presented in Chapter 6), the following values were calculated:

\[
\theta = -22^\circ, \quad \sqrt{J_{\text{2ng}}} = 0.671 \quad \text{and} \quad \phi = 32.9^\circ
\]